A Variant on $R(3) = 6$

Exposition by William Gasarch

December 20, 2024

Credit Where Credit Was Due

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The questions raised in these slides are due to Paul Erdös.

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The Theorem in these slides is due to Irving .

Reminder of Terminology

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We will mostly be studying $\mathrm{RAM}(G, 2, 3)$.

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Vote: YES or NO or UNKNOWN TO SCIENCE.

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Answer on next slide.

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We show the graph G and prove $\mathrm{RAM}(G)$.

Detour:
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We could also look at coloring **vertices**.

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We will use a result in Vertex-Ramsey to help Graph Ramsey.

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That will be a HW.

Back to
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If the K_5 does have v_0 then remove v_0 and you have that K_4 is a subgraph of H, contradiciton.

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See next slide for pictures and grand finale!

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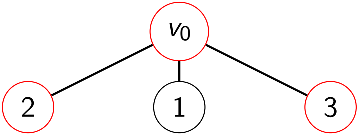

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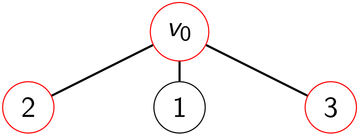

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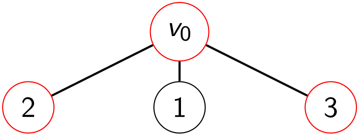

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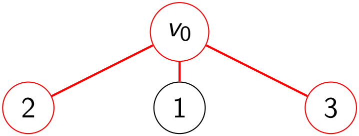
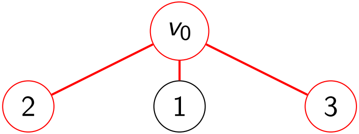


Focus on the Three R Edges



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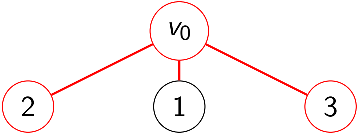

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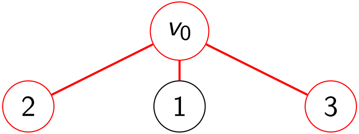

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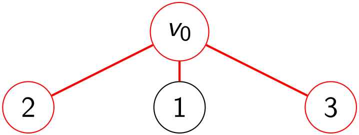

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Done!