Infinite Can Ramsey's Theorem
Three Proofs
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1 Introduction

In this document we proof the Infinite Can Ramsey Theorem on Graphs three ways.

Recall: **Theorem:** For every $\mathrm{COL}\colon \binom{\mathsf{N}}{2} \to [c]$ there is an infinite homogenous set.

What if the number of colors was *infinite?*

Do not necessarily get a homog set since could color EVERY edge differently. But then get infinite *rainbow set*.

So maybe:

Theorem: For every $\mathrm{COL}\colon \binom{\mathsf{N}}{2} \to \omega$ there is an infinite homogenous set OR an infinite rainb set.

FALSE:

- $\mathrm{COL}(i,j) = \min\{i,j\}$.
- $\mathrm{COL}(i,j) = \max\{i,j\}$.

Definition: Let $\mathrm{COL}\colon \binom{\mathsf{N}}{2} \to \omega$. Let $V \subseteq \mathsf{N}$.

- V is *homogenous* if $\mathrm{COL}(a,b) = \mathrm{COL}(c,d)$ iff $TRUE$.
- V is *min-homogenous* if $\mathrm{COL}(a,b) = \mathrm{COL}(c,d)$ iff $a = c$.
- V is *max-homogenous* if $\mathrm{COL}(a,b) = \mathrm{COL}(c,d)$ iff $b = d$.
- V is *rainbow* if $\mathrm{COL}(a,b) = \mathrm{COL}(c,d)$ iff $a = c$ and $b = d$.

2 One-Dim Can Ramsey Theorem

Theorem 2.1 *Let V be an countable set. Let $\mathrm{COL}\colon V \to \omega$. Then there exists either an infinite homog set (all the same color) or an infinite rainb set (all diff colors).*

Proof:
Case 1: There exists some color that appears infinity often. Call it c. Then

$$\{v\colon \mathrm{COL}(v) = c\}$$

is an infinite homog set.
Case 2: Every color appears finitely often. Then put a vertex in the set if it is the first vertex of its color. Formallly

$$\{v\colon (\forall v' < v)[\mathrm{COL}(v') \neq \mathrm{COL}(v)\}$$

is an infinite rainbow set. ∎

3 Proof of Can Ramsey Theorem on Graphs that uses 4-Hypergraph Ramsey

The following proof is due to Erdos and Rado [1]. It was the first proof of the theorem.

Theorem 3.1 *Let* $\mathrm{COL}\colon \binom{\mathsf{N}}{2}V \rightarrow \omega$. *Then one of the following occurs.*

- *There exists an infinite homog set.*
- *There exists an infinite min-homog set.*
- *There exists an infinite max-homog set.*
- *There exists an infinite rainbow set.*

Proof:
We are given $\mathrm{COL}\colon \binom{\mathsf{N}}{2} \rightarrow \omega$.
Want to find infinite homog OR min-homog OR max-homog OR rainbow set.

We use COL to define $\mathrm{COL}'\colon\ \colon \binom{\mathsf{N}}{4} \rightarrow [16]$
We then apply *4-ary Ramsey theorem.* (an "Application!")
In the cases below $x_1 < x_2 < x_3 < x_4$.
All cases assume negation of prior cases. For each color we say how to go from an infinite homog set for COL′ to an infinite homog or min-homog, or max-homog, or rainbow set.

1. $\mathrm{COL}(x_1, x_2) = \mathrm{COL}(x_1, x_3) \rightarrow \mathrm{COL}'(x_1 < x_2 < x_3 < x_4) = 1$.
2. $\mathrm{COL}(x_1, x_2) = \mathrm{COL}(x_1, x_4) \rightarrow \mathrm{COL}'(x_1 < x_2 < x_3 < x_4) = 2$.
3. $\mathrm{COL}(x_1, x_3) = \mathrm{COL}(x_1, x_4) \rightarrow \mathrm{COL}'(x_1 < x_2 < x_3 < x_4) = 3$.
4. $\mathrm{COL}(x_2, x_3) = \mathrm{COL}(x_2, x_4) \rightarrow \mathrm{COL}'(x_1 < x_2 < x_3 < x_4) = 4$.

Let H be an infinite homog set of color 1 (rest similar) for COL'. Let $\mathrm{COL}'' \colon H \rightarrow \mathsf{N}$ is $\mathrm{COL}''(x) =$ color of all (x, y) with $x < y \in H$.
Use **1-dim Can Ramsey!**:

- *Case 1:* COL'' has homog set H' then H' homog for COL.
- *Case 2:* COL'' has rainb set H' then H' min-homog for COL.

1. $\mathrm{COL}(x_1, x_3) = \mathrm{COL}(x_2, x_3) \rightarrow \mathrm{COL}'(x_1 < x_2 < x_3 < x_4) = 5$.
2. $\mathrm{COL}(x_1, x_4) = \mathrm{COL}(x_2, x_4) \rightarrow \mathrm{COL}'(x_1 < x_2 < x_3 < x_4) = 6$.
3. $\mathrm{COL}(x_1, x_4) = \mathrm{COL}(x_3, x_4) \rightarrow \mathrm{COL}'(x_1 < x_2 < x_3 < x_4) = 7$.
4. $\mathrm{COL}(x_2, x_4) = \mathrm{COL}(x_3, x_4) \rightarrow \mathrm{COL}'(x_1 < x_2 < x_3 < x_4) = 8$.

Let H be an infinite homog set of color 5 (rest similar) for COL'. Let $\mathrm{COL}'' \colon H \rightarrow \mathsf{N}$ is $\mathrm{COL}''(y) =$ color of all (x, y) with $x < y \in H$.

Use **1-dim Can Ramsey!**:

- *Case 1:* COL'' has homog set H' then H' homog for COL.
- *Case 2:* COL'' has rainbow set H' then H' max-homog for COL.

1. $\mathrm{COL}(x_1, x_2) = \mathrm{COL}(x_2, x_3) \Rightarrow \mathrm{COL}(x_1, x_2, x_3, x_4) = 9$.
2. $\mathrm{COL}(x_1, x_2) = \mathrm{COL}(x_2, x_4) \Rightarrow \mathrm{COL}(x_1, x_2, x_3, x_4) = 10$.
3. $\mathrm{COL}(x_1, x_2) = \mathrm{COL}(x_3, x_4) \Rightarrow \mathrm{COL}(x_1, x_2, x_3, x_4) = 11$.
4. $\mathrm{COL}(x_1, x_3) = \mathrm{COL}(x_2, x_4) \Rightarrow \mathrm{COL}(x_1, x_2, x_3, x_4) = 12$.
5. $\mathrm{COL}(x_1, x_3) = \mathrm{COL}(x_3, x_4) \Rightarrow \mathrm{COL}(x_1, x_2, x_3, x_4) = 13$.

6. $\mathrm{COL}(x_2, x_3) = \mathrm{COL}(x_1, x_4) \Rightarrow \mathrm{COL}(x_1, x_2, x_3, x_4) = 14$.

7. $\mathrm{COL}(x_2, x_3) = \mathrm{COL}(x_3, x_4) \Rightarrow \mathrm{COL}(x_1, x_2, x_3, x_4) = 15$.

Let H be an infinite homog set of color 9 (rest similar) for COL'.
For all $w < x < y < z \in H$.

$$\mathrm{COL}(w, x) = \mathrm{COL}(x, y) = \mathrm{COL}(y, z).$$

So H is homog for COL.
LAST COLOR:
If NONE of the above cases hold then $\mathrm{COL}(x_1, x_2, x_3, x_4) = 16$.
Let H be an infinite homog set of color 16 for COL'.
All edges from H diff colors, so Rainbow Set.
We leave this for an exercise.

Note 3.2 PROS and CONS of this proof:

1. Each Case easy. Note that Rainbow case was easy.
2. Lots of Cases.
3. Use of 4-ary hypergraph Ramsey makes finite version have large bounds.

4 Proof of Can Ramsey Theorem on Graphs that uses 3-Hypergraph Ramsey

The proof in this section is due to Rado [3].

Def 4.1 Let $\mathrm{COL}\colon \binom{\mathsf{N}}{2} \to \omega$. If c is a color and $v \in \mathsf{N}$ then $\deg_c(v)$ is the number of c-colored edges with v as an endpoint.

Lemma 4.2 *Let X be infinite. Let* $\mathrm{COL}\colon \binom{X}{2} \to \omega$. *If for every $x \in X$ and $c \in \omega$, $\deg_c(x) \le 1$ then there is an infinite rainb set.*

Proof: Let R be a MAXIMAL rainbow set of X.

$$(\forall y \in X - R)[R \cup \{y\} \text{ is not a rainbow set}].$$

Let $y \in X - R$. Why is $y \notin R$?

1. There exists $u \in R$ and $\{a, b\} \in \binom{R}{2}$ such that $\mathrm{COL}(y, u) = \mathrm{COL}(a, b)$.

2. There exists $\{a, b\} \in \binom{R}{2}$ such that $\mathrm{COL}(y, a) = \mathrm{COL}(y, b)$. This cannot happen since then y has color degree ≤ 1.

Map $X - R$ to $R \times \binom{R}{2}$: map $y \in X - R$ to $(u, \{a, b\})$ (item 1).

Map is injective: if y_1 and y_2 both map to $(u, \{a, b\})$ then $\mathrm{COL}(y_1, u) = \mathrm{COL}(y_2, u)$ but $\deg_c(u) \le 1$.

Injection from $X - R$ to $R \times \binom{R}{2}$. If R finite then injection from an infinite set to a finite set Impossible! Hence R is infinite. ∎

Theorem 4.3 *Let* $\mathrm{COL}\colon \binom{\mathsf{N}}{2} V \to \omega$. *Then one of the following occurs.*

- *There exists an infinite homog set.*
- *There exists an infinite min-homog set.*
- *There exists an infinite max-homog set.*
- *There exists an infinite rainbow set.*

Proof: Given $\mathrm{COL}\colon \binom{\mathsf{N}}{2} \to \omega$. We use COL to obtain $\mathrm{COL}'\colon \binom{\mathsf{N}}{3} \to [4]$ We will use the 3-ary Ramsey theorem. In all of the below $x_1 < x_2 < x_3$.

1. If $\mathrm{COL}(x_1, x_2) = \mathrm{COL}(x_1, x_3)$ then $\mathrm{COL}'(x_1 < x_2 < x_3) = 1$.

2. If $\mathrm{COL}(x_1, x_3) = \mathrm{COL}(x_2, x_3)$ then $\mathrm{COL}'(x_1 < x_2 < x_3) = 2$.

3. If $\mathrm{COL}(x_1, x_2) = \mathrm{COL}(x_2, x_3)$ then $\mathrm{COL}'(x_1 < x_2 < x_3) = 3$.

4. If none of the above occur then $\mathrm{COL}'(x_1 < x_2 < x_3) = 4$.

If there is an infinite homog set for COL' of color 1,2, or 3 then the proof there is an infinite homog or max-homog or min-homog or rainbow set is similar to the proof in Theorem 3.1.

Assume there is an infinite homog set H for COL' of color 4:

Then for all colors c and $h \in H$, we have $\deg_c(v) \le 1$, so we are done by the lemma.

∎

5 A Proof that Does Not Use Any Hypergraph Ramsey

The proof in this section is due to Mileti [2].

We will proof the Infinite Can Ramsey theory (for graphs) but not use any hypergraph Ramsey Theorem.

It will be close in spirit to the proof of the infinite Ramsey Theorem.

We first restate how we used the infinite 1-hypergraph Ramsey Theorem to prove the 2-hypergraph Ramsey Theorem:

If $\binom{\mathsf{N}}{2}$ is 2-colored and there is an infinite sequence of vertices:

$$X = \{x_1, x_2, x_3, \ldots\}$$

Then either

- There exists infinite $Y_R \subseteq X$ such that

$$(\forall x \in Y_R)[\mathrm{COL}(x, y) = R].$$

- There exists infinite $Y_B \subseteq X$ such that

$$(\forall x \in Y_B)[\mathrm{COL}(x, y) = B].$$

We then replace X with Y_R or Y_B.

We now describe the analog of that process which we will be using to prove 2-hypergraph Can Ramsey from 1-hypergraph Can Ramsey.

If $\binom{\mathsf{N}}{2}$ is colored (note no bound on the number of colors) and there is an infinite sequence of vertices:

$$x_1, x_2, x_3, \ldots$$

Then either

- There exists color c and infinite $Y_c \subseteq X$ such that

$$(\forall x \in Y)[\mathrm{COL}(x, y) = c].$$

- There exists infinite $Y_\omega \subseteq X$ and an infinite set of colors C such that,

$$(\forall c \in C)(\exists! y \in Y)[\mathrm{COL}(x, y) = c].$$

(Notation -$(\exists! y)$ means there is ONE y.)

We then replace X with Y_1 or Y_2 or $\cdots$ or Y_ω.
Here is the intuition: Either

- There is some color c such that $\mathrm{COL}(1, x)$ is c infinitely often. Then restrict to that set and color 1 with (H, c).
- For every color c the set of x with $\mathrm{COL}(1, x) = c$ is finite. Then thin out the set so that $\mathrm{COL}(1, x_2)$, $\mathrm{COL}(1, x_3)$, etc are all different. (When dealing with x_2 or x_3 later instead of x_1 this will get more complicated.)

We now describe it formally.

CONSTRUCTION
PART ONE

$$\begin{aligned} V_0 &= \mathsf{N} \\ x_1 &= 1 \end{aligned}$$

If $(\exists c)|\{v \in V_0 \mid \mathrm{COL}(x_1, v) = c\}| = \omega$ then:

- $c_1 = (H, c)$
- $V_1 = \{v \in V_0 \mid \mathrm{COL}(x_1, v) = c\}$. (Note that V_1 is infinite)

If $(\forall c)|\{v \in V_0 \mid \mathrm{COL}(x_1, v) = c\}| < \omega$ then:

- $V_1 = \{v \in V_0 \mid (\exists c)[\mathrm{COL}(x_1, v) = c \wedge (\forall x_1 < u < v)[\mathrm{COL}(x_1, u) \neq c]]\}$ (so v is the first first with $\mathrm{COL}(x_1, v) = c$. Hence there will only be ONE v with $\mathrm{COL}(x_1, v) = c$.) (Note that V_1 is infinite)
- $c_1 = (RB, 1)$. (The 1 only marks that this is the first rainbow-color assigned.)

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i:
x_i gets the least element of V_{i-1}.
If there exists c such that Y_c is infinite then

$$\begin{aligned} c_i = & \ (H, c) \\ V_i = & \ Y_c \end{aligned}$$

If no such c exist then there exists Y_ω. We initially take
$V_i = Y_\omega$
But we may thin it out. And we haven't colored x_i yet.
Do the following:
For all $1 \le j \le i-1$ such that $\mathrm{COL}(x_j) = (RB, k)$ for some k then:

1. If $|\{y \in Y_\omega : \mathrm{COL}(x_j, y) = \mathrm{COL}(x_i, y)\}| = \omega$ then let V_i be this set and let $c_i = c_j$. (So $\mathrm{COL}(x_i)$ will be of the form (RB, k) for some k). You are done and do not go to the next j.

2. If $|\{y \in Y_\omega : \mathrm{COL}(x_j, y) = \mathrm{COL}(x_i, y)\}| < \omega$ then let V_i be the Y_ω minus those vertices.

If Case 1 ever happens then we are done. If Case 2 always happens then note that x_i disagrees with every x_j on every element $> x_i$. We c_i with (RB, k) where k is the least number not used for a rainbow color yet.
END OF PART ONE
PART TWO
Consider the infinite sequence

$$c_1, c_2, \ldots$$

There are several cases:

- There is a c such that (H, c) appears infinitely often. Let

  $$H = \{x_i : c_i = (H, c)\}.$$

  This set is infinite homog.

- There is an infinite number of vertices colored H. Let

  $$H' = \{x_i : (\exists c)[c_i = (H, c)]\}$$

  By the 1-hypergraph Can Ramsey applied to the coloring $\mathrm{COL}(x_i) = c$, and the premise, we get a set H which we renumber so that

  $$H = \{y_1 < y_2 < y_3 < \cdots\}$$

  and $\mathrm{COL}(y_i) = (H, i)$. H is infinite min-homog.

- There is an k such that (RB, k) appears infinitely often. Let

$$H = \{x_i : c_i = (RB, k)\}.$$

This set is infinite max-homog.

- There is an infinite number of vertices colored RB. Let

$$H' = \{x_i : (\exists k)[c_i = (RB, k)]\}$$

By the 1-hypergraph Can Ramsey applied to the coloring $\mathrm{COL}(x_i) = k$, and the premise, we get a set H which we renumber so that

$$H = \{y_1 < y_2 < y_3 < \cdots\}$$

and $\mathrm{COL}(y_i) = (RB, i)$.

H need not be rainbow! We take a subset of H as follows

$H_0 =$

$z_1 = y_1$

Assume that $z_1, \ldots, z_n$ have been chosen and that all of the edges between them are different colors. Let $SET\mathrm{COL}_n$ be the set of colors of edges (there are $\binom{n}{2}$ of them). Also assume there is an infinite set H_n of vertices that have not been killed. All of the elements of H_n are $> z_n$. Find the least element z of H_n such that,

$$(\forall 1 \le i \le n)[\mathrm{COL}(z_i, z) \notin SET\mathrm{COL}_n].$$

AND

$$(\forall 1 \le i < j \le n)[\mathrm{COL}(z_i, z) \ne \mathrm{COL}(z_j, z)].$$

FIRST KEY: The second clause holds for all z since (after renumbering) $\mathrm{COL}'(z_i) = (RB, i)$ and $\mathrm{COL}'(z_j) = (RB, j)$, hence for all z $\mathrm{COL}(z_i, z) \ne \mathrm{COL}(z_j, z)$. (This is the place we use that they are all differeng RB-type colorings.)

SECOND KEY: we need to show that there exists a z satsifying the first clause. Assume, by way of contradiction, that no such z exists.

We map each $z \in H_n$ to the REASON it does not work. Map H_n to $\{1, \ldots, n\} \times SET\mathrm{COL}_n$ as follows:

$z \in H_n$. z DID NOT get to be z_{n+1}. Hence there is some i (take the least one) such that $\mathrm{COL}(z_i, z) = c \in SET\mathrm{COL}_n$. Let i be the least such i. Map H_n to (i, c).

This mapping maps an infinite set to a finite set. Hence some element of the co-domain is mapped to infinitely often. We just need twice. There is some (i, c) such that there is $z, z' \in H_n$ such that

$\mathrm{COL}(z_i, z) = c$

and

$\mathrm{COL}(z_i, z') = c$

This violated $\mathrm{COL}(z_i) = (RB, i)$.

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References

[1] P. Erdős and R. Rado. A combinatorial theorem. *Journal of the London Mathematical Society*, 25:249–255, 1950. `http://jlms.oxfordjournals.org/`.

[2] J. Mileti. The canonial Ramsey theorem and computability theory. *Transactions of the American Math Society*, 360, 2008. `www.math.grinnell.edu/~miletijo/research/compcanon.pdf`.

[3] R. Rado. Note on canonical partitions. *Bulletin of the London Mathematical Society*, 18:123–126, 1986. `http://blms.oxfordjournals.org/content/by/year`.