MATH 301
TERM PAPER: "THE HAPPY ENDING PROBLEM"
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1. Introduction

During the winter of 1933 in Budapest, a group of students would meet regularly to discuss mathematics. Among this group was Paul Erdős, György (George) Szekeres and Esther Klein. At the time, Erdős was attending the University of Budapest, while Szekeres had recently graduated from the Technical University of Budapest with a degree in chemical engineering.[IP00]

At one of their meetings, Klein challenged the group to solve a complex problem in planar geometry. Klein proposed that the group consider five points on a flat surface, where no three of the five formed a straight line. It was obvious to all that when four of the points were joined, they formed a quadrilateral, but Klein also noticed that given five points, four of them always appeared to define a convex quadrilateral. After the group failed to prove her proposition, Klein offered an informal proof, as follows. She deduced three ways in which a convex polygon could enclose all five points. Her first, and simplest case, was when four points forming a quadrilateral enclosed the fifth point, thereby automatically satisfying the requirement. In the second case, Klein explained that if the convex polygon was a pentagon, any four of five points could be joined to form a quadrilateral and satisfy the requirement. Finally, if three of the points create a triangle, it was obvious that two remained inside the triangle. Klein reasoned that the two points left inside defined a line that split the triangle such that two of the triangle's points were on one side of the line. It followed that these two exterior points, plus the two interior points automatically formed a convex quadrilateral.[IP00]

The problem fascinated the group. In fact, Szekeres remarked in a memoir many years later that "[The group] realized that a simple-minded argment would not do and there was a feeling of excitement that a new type of geometrical problem [had] emerged...which we were only too eager to solve." [IP00] Further, Erdős, Szekeres and Klein sought to formalize the result and develop a proof for the case with more than 5 points. Szekeres and Erdős issued a paper titled *A Combinatorial Problem in Geometry* in 1935 formally proving the result. Specifically, they generalized that among randomly scattered points on a flat surface, there exists a set that forms a particular polygon if there are a sufficient number of points.

This result was dubbed the "Happy End Problem" by Erdős because it led to

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the marriage of Szekeres and Klein in 1935[CG97] [IP00] and it has been significant in the mathematical community over the last sixty years.[CG97] Although to date, no one has proved the precise number of points required to guarantee the presence of convex polygon for all cases, the problem continues to generate interesting results and related questions. The purpose of this paper is to clearly define "The Happy End Problem" and explore these related results and questions. In particular, as we will see, "The Happy End Problem" is still interesting to so many for two main reasons:

(1) *All of the exact values for the number of points in a general-position point set needed to form a convex n-gon have not yet been defined.*
(2) *The lower and upper bounds for the number of points in a general-position point set needed to form a convex polygon may not be sharp.*

The remainder of this essay will be broken up according to point 1 and point 2, above. Chapter 1 will explore the original proof as defined in Erdős and Szekeres' *A Combinatorial Problem in Geometry* (1935). Chapter 2 will consider those cases that are defined for "The Happy Ending Problem." In particular, Kalbfleisch, Kalbfleisch, and Stanton's *A Combinatorial Problem on Convex Regions* (1970) will be considered along with Peters and Szekeres' *Computer Solution to the 17-Point Erdős-Szekeres Problem* (2006). Chapter 3 will examine the proofs that have improved the bounds for the problem. Specifically, Chung and Graham's *Forced convex n-gons in the plane* (1997), Kleitman and Pachter's *Finding convex sets among points in the plane* (1998) and Tóth, and Valtr's *Note on the Erdős-Szekeres theorem* (1998) and *The Erdős-Szekeres theorem: upper bounds and related results.* Chapter 4 will briefly consider related problems and the Conclusion will provide recommedations for further research and a short aside.

2. Chapter 1: The Original Proof

In the introduction of this term paper, Klein's proposition was defined in the most general of terms. It is significant to note that from the general case (just by substituting values), Erdős and Szekeres formed the following conjecture (which Szekeres later amended to *"Probably"* be true [ES35] [MS00]:

Conjecture 1. $f(n) = 2^{n-2} + 1$ *for all* $n \geq 3$

Now, it remains to express Klein's informal proof, as Erdős and Szekeres did, in a more mathematical manner. Specifically,

Can we find for a given number $f(n) = N$ such that from any set containing at least N points, it is possible to select n points forming a convex polygon?[ES35].

Erdős and Szekeres argued that two important questions followed from the proposition[ES35]:

(1) *Does the number N corresponding to n exist?*
(2) *If so, how is $f(n) = N$ determined as a function of n?*

The body of their paper offers two proofs, which define that N corresponding to

n does exist, and offer a preliminary answer to the second question. In what follows, those proofs and their conclusions will be defined in the most general of terms.

2.1. **The First Proof.** Erdős and Szekeres employ a variation of Ramsey's Theorem in order to extrapolate the first proof in their paper. Let's define the first proof generally by beginning, as Erdős and Szekeres did, by defining the given number of points to be $1, 2, 3, ..., N$. Further, any arbitrary shape, or d-gon can be defined from any d-combination from the set. They conclude, from the general result, as defined in the introduction above, that among these d-combinations, there exists two classes, namely:

(1) *The concave class*
(2) *The convex class*

For the sake of clarity, recall the basic definitions of concave and convex polyons:

(1) *A Concave Polygon*: A polygon that has one or more interior angles greater than 180° with the result that all vertices point ‘inwards,’ towards the center.[COP]
(2) *A Convex Polygon*: A polygon that has all interior angles less than 180° with the result that all vertices point ‘outwards,’ away from the center.[CXP]

Next comes the crux of the proof, where Ramsey's Theorem is used. In its shortened form, Ramsey's Theorem relates to $k, l, i \in Z$, where $k \geq i$ and $l \geq i$. More specifically, let's assume two classes called a and b, of i-combinations, of m elements. It should be the case that each k-combination should contain at least one combination from class a and each l combination should contain at least one combination from class b. The theorem concludes by stating that it cannot be the case that $m < m_i(k, l)$ for sufficiently large m. Through induction, Erdős and Szekeres deduce that the opposite also must be true. Further, they apply the result to the d-combination, as related above, with the classes represented to be concave and convex, thereby concluding the proof. It should be noted that although the proof is logical, the numerical limits provided are not precise. The second proof issued by Erdős and Szekeres accounts for this omission [ES35].

2.2. **The Second Proof.** The second method is entirely based on geometric considerations [MS00]. Erdős and Szekeres consider the first quarter of the plane, whose points are clearly defined by (x, y). Then, choosing n points with monotonously increasing abscissae, the authors apply the following theorem [ES35]:

Theorem 1. *It is always possible to choose at least $\sqrt{n}$ points with increasing abscissae and either monotonously increasing or monotonously decreasing ordinates.*

Employing the original conjecture, replacement, and associating convexity with decreasing monotonously and concavity with increasing monotonously, Erdős and Szekeres obtain the following recurrence formula [ES35] [CG97]:

$$2^{n-2} + 1 \leq f(n) \leq \binom{2n-4}{n-2} + 1.$$

As discussed, Erdős and Szekeres proposed that the lower bound was precise for $n = 2, 3, 4, 5$, with the conjecture that it was so for all n. [TV] Further, after the initial proof, there were two parts that needed improvement. First, The equality held only for $n \leq 5$. Second, the upper bound was not sharp. Repeated attempts over the years have led to some general improvements on the equality and its specific values. These improvements, along with some related results form the basis for the remainder of the term paper.

3. Chapter 2: Defined $N(n)$ for "The Happy Ending Problem"

3.1. $f(5) = 9$**:** ***A combinatorial problem on convex regions*** **(1970).** In *A Combinatorial Problem in Geometry* (1935), Erdős and Szekeres claim that Mr. E. Makai proved that $f(5) = 9$, or in words, every set of nine points in the general position contains at least one convex pentagon [ES35]. The problem, however, was that Makai never published a formal proof for the result. Instead, the publication of the proof was issued by J.D. Kalbfleish, J.G. Kalbfleish and R.G. Stanton in *A combinatorial problem on convex regions* in 1970 [KS70]. Although yielding no new information, the formal proof was necessary to move forward with the problem.

3.2. $f(6) = 17$**:** ***Computer Solution to the 17-Point Erdős-Szekeres Problem*** **(2006).** In this essay, Szekeres and Peters provide a proof of the 17-point version of the original Klein conjecture. Specfically, the result is the computer proof of $f(6) = 17$, or equivalently 17 points in the plane always contain a convex 6-subset.[SP06] It remains to be explained, however, how Szekeres and Peters achieve this result. First, they define the Klein's original proposition, for $k > 1$, to be (P_k), as follows [SP06]:

Proposition 1. (P_k) *Every planar set of $n > 2^{k-2}$ points (in general position, no 3 points collinear) contains a subset of k points which form a convex k-gon (denoted a* convex k-subset*)*

For k=2,3,4, it should be obvious that the statement is trivially true. For $k = 5$, however, Szekeres and Peters identify that the problem already begins to present difficulties (this should be clear from the fact that the first formal proof was not issued until 1970!)[KS70]. Given the use of a computer, the initial issue is to define a suitable combinatorial definition of convexity. Szerekes and Peters achieve this by making use of the signature of triangles, or more specifically, every ordered triple in general position in the plane (they provide the example (p_i, p_j, p_k)) is oriented either clockwise or anticlockwise. The authors employ this by imparting a signature, defined $\sigma(i, j, k) = -$ or $+$, according to the orientation of the triples. More formally, Szekeres and Peters define T_n to be the set of ordered triples (i, j, k) of $S_n = \{1, 2, ..., m\}$. Precisely,

$$\sigma : T_n \to \{+, -\}$$

or equivalently, every ordered triple induces a signature. This notion, combined with the definition of "cups" (concavity) and "caps" (convexity) as defined in [CG97] (which will be explored further in the next chapter), form the defintion of a convex k-gon for Szekeres and Peters and their computer. More directly, the signature σ, when used with the "cups" and "caps" offers a unique identifier to recognize convexity in a given set. Perhaps Szekeres and Peters express it most clearly when they claim that their methodology allows them to ask whether the combinatorial version of (P_k) is valid for an arbitrary σ, or

Definition 1. ($P_{\sigma,k}$) *If $n > 2^{k-2}$, then relative to any $\sigma \in \sum_n$, there is a convex k-subset of S_n*

From this definition, Szekeres and Peters define a set of signatures (i.e. $\sigma_1, \sigma_2, ..., \sigma_n$) and define convex quadrilaterals and concave quadrilaterals based on equality between them. For example, if *abcd* is a concave quadrilateral, namely, a triangle *abd* with c inside the triangle or a triangle *acd* with b inside it, then the necessary relations are [SP06]:

$$\sigma_1 = -\sigma_4 \text{ and } \sigma_2 = \sigma_3$$

Now, having defined Szekeres and Peters' methodology and definitions, let's move to a high-level overview of their proof for $f(6) = 17$. Given what has been defined, it should be clear that the number of elements m, in the set defined A_m, required to represent a signature in $\sum_{17}$ over S_{17} is $\binom{17}{3} = 680$, where order does not matter.[SP06] Now, the goal is to select a set of six points, so let $abcdef$ be any such ordered set of S_{17}. It follows that they form a convex 6-subset if and only if its $\binom{6}{3} = 20$ triples satisfy a one of eight convex relations[SP06], defined according to the methodology previously described. From this, Szekeres and Peters are able to deduce the total number of convex and concave relations on S_{17}. The challenge then is to assign each of the concave signatures to the 6-subsets, which would prove $P_{\sigma,6}$. Because of the large number of signatures (for example, convex relations on $S_{17} = 99,008$ [SP06], Szekeres do not assign *every* signature. Instead, they select a subset, which satisfy certain geometric conditions, and claim that this is adequate. Indeed, a critique of the proof is that the sample that they examine is a tiny subset of the entire population.

This concludes the examination of defined $f(n)$ for "The Happy Ending Problem." Although the results described do not represent all of the attempts to define $f(n)$, they are generally regarded as the most significant over the course of the last sixty years. Next, modifications to the upper bound of the original proof will be considered.

4. Chapter 3: Improvements in the Upper Bound

4.1. **Improvement 1: *Forced convex n-gons in the plane* (1997).**

Recall the initial result obtained by Erdős and Szekeres:

$$2^{n-2} + 1 \leq f(n) \leq \binom{2n-4}{n-2} + 1.$$

From 1935-1997, the bounds listed remained largely unchanged. In 1997, however, Chung and Graham issued a seminal paper in which they removed the +1 from the upper bound for $n \geq 4$. Namely, their result yields [CG97]:

$$2^{n-2} + 1 \leq f(n) \leq \binom{2n-4}{n-2}, \text{ for } n \geq 4$$

In their proof, Chung and Graham define an m-cap to be a sequence of m points $x_1, x_2, ..., x_m$ such that the polygonal path connecting them is concave, or equivalently, the x_i have increasing x-coordinates, and the path from x_1 to x_m turns clockwise at each intermediate vertex. Similarly, an m-cup is the set of points $y_1, y_2, ..., y_m$ with increasing x-coordinates such that the polygonal path joining them is convex; namely, the path from y_1 to y_m always turns counterclockwise.

Chung and Graham first use induction on the original equality to demonstrate that in general position, if $X \subset E^2$ and $|X| > \binom{a+b-4}{a-2}$, then X must contain either a convex polygonal path or a concave polygonal path. As demonstrated in Erdős and Szekeres' original equality, the bound must be sharp and Chung and Graham define this as a Theorem central to their proof [CG97]:

Theorem 2. *If $X \subset E^2$ is in general position and $|X| \geq \binom{2n-4}{n-2}$ for $n \geq 4$, then X contains the vertices of a convex n-gon.*

Employing this Theorem, Chung and Graham present three Cases where proof by contradiction is used to demonstrate that it must be the case that

$$2^{n-2} + 1 \leq f(n) \leq \binom{2n-4}{n-2}, \text{ for } n \geq 4$$

Chung and Graham define their discovery to be "admittedly modest" but "hope that it might suggest methods which could give rise to more substantial reductions in the upper bound." [CG97]

4.2. **Improvement 2: *Finding convex sets among points in the plane* (1998).** Only a short period after Chung and Graham issued the paper described above, Kleitman and Pachter released *Finding convex sets among points in the plane*, in which they made another adjustment to the upper bound of Erdős and Szekeres intitial result [KP98] via [TV]:

Theorem 3.

$$2^{n-2} + 1 \leq f(n) \leq \binom{2n-4}{n-2} - 2n + 7, \textit{ for } n \geq 4$$

A point set is defined to be vertical if its two leftmost points have the same x-coordinate. Kleitman and Pachter explain that any point set can be made vertical and that any vertical point set also contains caps and cups, as defined previously by Chung and Graham.[CG97] Regarding the vertical point sets, however, Kleitman and Pachter offer the caveat that in these cases the vertical edge determined by the two leftmost points is to be the leftmost edge of a cup or a cap. [KP98] via [TV] Using the definition of vertical sets, Kleitman and Pachter use a series of contradictions to redefine the known values for the upper bound of $f(n)$. Their result, as related in the theorem above, is what follows and it is defined to be sharp.

4.3. **Improvement 3: *Note on the Erdős-Szekeres theorem* (1998) and *The Erdős-Szekeres theorem: upper bounds and related results* (No date).** Several months after Kleitman and Pachter issued their proof, Géza Tóth and Pavel Valtr proved that for the upper bound, $f(n) \leq \binom{2n-5}{n-2} + 2$, which represented an improvement roughly by a factor of 2. [TV] The proof for the theorem involves using one extreme point within a planar set and one point outside the convex elements of that set. Then, using a projective transformation, Tóth and Valtr deduce that the if the planar set, X contains $\binom{2n-5}{n-3} + 2$ points, the projective transformation of the planar set X determines either an $(n-1)$-cap or an n-cup. Further, the authors conclude that this implies that the planar set X contains n points in convex position [MS00] In a subsequent paper, the same authors employed the result issued by Chung and Graham, along with their earlier result (just described) to derive the following theorem, representing a further improvement by a factor of 1 [TV]:

Theorem 4. *For any $n \geq 5$, any set of $\binom{2n-5}{n-2}+1$ points in general position in the plane contains n points in convex position. That is, $f(n) \leq \binom{2n-5}{n-2}+1$.*

The theorem is proved by employing the same methodology used for their earlier result and extending the $(n-1)$-cap or an n-cup by one point. The result is a contradiction of their first proof, but a justification for $f(n) \leq \binom{2n-5}{n-2}+1$ for any $n \geq 5$. It should be noted that this is the result related by Chung and Graham, with the only difference being that in this case, $n \geq 5$, which is clearly significant.

Having considered when $f(n)$ is defined and the improvements made on the upper bound, it remains to discuss related problems. These related problems will be the topic of the next chapter.

5. Chapter 4: Related Problems (solved and unsolved)

Different mathematicians will undoubtedly define different related problems regarding "The Happy Ending Problem." As such, in the interest of brevity, this paper considers only a subset, deemed by the author to be most significant.

5.1. **Related Problem 1: The Erdős Problem on Empty Polygons.** In 1978, Paul Erdős posed a new problem on convex polygons[E179] [E279] [E80] that exhibited several similarities to the original Erdő-Szekeres Problem. In particular, the problem is limited to what can be proved by cups and caps. The problem, as stated by Erdős is as follows [E179] via [MS00]:

Problem 1. *For any positive integer $n \geq 3$, determine the smallest positive integer $H(n)$, if it exists, such that any set X of at least $H(n)$ points in general position in the plane contains n points which are the vertices of an empty convex polygon, i.e., a polygon whose interior does not contain any point of X.*

According to Morris and Soltan, it is trivial that $H(3) = 3$[MS00] and the case $H(4) = 5$ is easily deduced from a set of nine points that of course, contain no empty convex pentagon. In 1978, it was also demonstrated by Harborth in [H79] via [MS00] that $H(5) = 10$. He accomplished this using a direct geometric approach. Further, Morris and Soltan argue that $H(5) \geq 10$ follows immediately from the same set used to prove $H(4) = 5$ along with Harborth's proof.[MS00] Then, in 1983, Horton showed that $H(n)$ does not esist for all $n > 7$.[H83] via [MS00]

Having proved, $H(3) = 3$, $H(4) = 5$, $H(5) = 10$ and that $H(n)$ does not exist for $n > 7$, the most logical remaining proof is to determine whether the number $H(6)$ exists. Mathematicians have expressed the belief that $H(6)$ exists[H83], presented a conjecture that would imply the existence $H(6)$[V97] via [MS00], but have yet to precisely define its existence. Perhaps the closest proof is that of Overmars *et al.*[OSV89] via [MS00] In their proof, they constructed an algorithm of time complexity (time complexity represents how much time it will take an algorithm to execute given the parameters of its input [TC]), namely $O(n^2)$, through which they found a set of 26 points containing no empty convex 6-gon. The logical result, then, was that $H(6) > 27$, if it exists.[OSV89] via [MS00]

Further, this problem generates considerable interest as there remains a major proof, namely whether the number $H(6)$ exists, to be issued.

5.2. **Related Problem 2: Convex Bodies.** It has been demonstrated that the Erdős-Szekeres problem has a generalization for convex bodies of in the plane.([BT891] [BT892] [BT90]) via[MS00] Specifically, [BT891] via [MS00]

Theorem 5. *For any integer $n \geq 4$, there is a smallest positive integer $g(n)$ such that if F is a family of pairwise disjoint convex bodies in the plance, any three of which are in convex position and $|F| > g(n)$, then some n convex bodies of F are in convex position.*

Using Ramsey's Theorem, Bistztriczky and Tóth prove the existence of $g(n)$ and that $g(5) = 8$.[BT892] via [MS00] The remainder of work done on the problem is largely concerned with definining and redifining the appropriate bounds for $g(n)$ (as we saw was also the case for the Erdős-Szekeres problem). Further, the best known bounds for the problem are [MS00], [TV]:

$$2^{n-2} \leq g(3,n) \leq \binom{2n-4}{n-2}^2 \text{ [ES60] [PT98]}$$

$$\lfloor \frac{n+1}{4} \rfloor^2 \leq g(4,n) \leq n^3 \text{ [PT98]}$$

$$n-1+\lfloor \frac{n-1}{k-2} \rfloor \leq g(5,n) \leq 6n-12 \text{ [BT90] [T00]}$$

$$n-1+\lfloor \frac{n-1}{k-2} \rfloor \leq g(k,n) \leq n+\frac{1}{k-5}n \text{ for } k \geq 6 \text{ [BT90] [T00]}$$

This problem is very similar to the Erdős-Szekeres problem and continues to generate interest because of the opportunity to generate improvements for the lower and upper bounds. What's more, convex sets investigated in the Erdős-Szekeres problem have generally been defined to be disjoint (see Theorem above). More recently, however, Erdős-Szekeres type problems have been investigated where the points are replaced by convex sets that are not necessarily disjoint.[PT00] via [MS00],[TV] This discovery provides another niche for further research.

6. Conclusion: Recommendations for Further Research and An Aside

6.1. **Recommendations for Further Research.** The basis for the structure of this paper is the improvements made on the Erdős-Szekeres problem over the last sixty years. By this point, it should be very clear that the vast majority of those improvements take one of the following two forms (the topics of Chapters 1 and 2, respectively):

(1) *Defining the precise number of points in a general-position point set needed to form a convex polygon.*
(2) *Determining the bounds for the number of points in a general-position point set needed to form a convex polygon.*

As such, recommendations for further research should exist in some subcategory of either point 1 or point 2, or both, from above. From the research performed, it is clear that attempting to improve the bounds for the problem is useful, but has never generated precise results. Conversely, however, attempts to precisely define $f(n)$ (or equivalently point 1 above) have generated precise values, the most recent and significant being Peters and Szekeres' proof for $f(6) = 17$. This is a particularly significant result as it relies on more advanced modern technology to generate precise values for n. Moving forward, especially as technology improves, it should be the case that this technology can be used in more efficient ways to generate more meaningful results for the Erdős-Szekeres problem. As a result, my recommendations for further research include using computers, as Peters and Szekeres did, to solve the Erdős-Szekeres problem. While redefining the bounds is a meaningful intellectual exercise and undoubtedly has many benefits, my preference would be for reaching actual solutions to the problem. It should be the case that my recommendations for further research would generate actual solutions to the problem faster than would redefining the lower and upper bounds for the problem.

6.2. **An Aside.** For the author, the research and analysis required to construct this paper have transformed the “Happy Ending Problem” from an amorphous topic in planar geometry to a much more precise problem that generates definite interest. To this point, the particulars of the problem, related results and even recommendations for further research have been explored, yet there has been no comment on *why* the author chose this problem. In essence, my fascination with this problem stems from two main sources. First, the problem clearly has applications beyond the field of mathematics. Whether the problem will lead to more efficient algorithms in computer science (see [SP06]) or even help to generate new solutions for patients with atrial fibrilation and heart failure[CARD08], I am confident that this problem will impact the world in the future. Second, the problem makes us think. More specifically, as any good math problem should, the Erdős-Szekeres problem forces mathematicians to develop new strategies to formulate and solve problems. As Chung and Graham explain in [CG98], solving new problems “le[a]d[s] to new tools and techniques for...making advances in the area under investigation.” [CG98] Furthermore, the Erdős-Szekeres problem is so interesting, and also so important, because of its tendency to generate excitement among the mathematical community. As Chung and Graham relate, it is unquestionable that this excitement, along with the results described in the content of this paper, will lead to important advances in mathematics in the future.

Response: *Very well written, complete paper. I'm confused on one point. When you have a lower bound for $f(n)$, say $L \leq f(n)$, does that mean that there is a set with fewer than L points in which no concave n-gon exists?*

Problem statement 3/3 Current results 3/3 Important researchers 2/2 Why important and related problems 3/3 Organization 2/2 Use of language 4/4

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