

# Rectangle Free Coloring of Grids

Stephen Fenner- U of SC    William Gasarch- U of MD  
Charles Glover- U of MD    Semmy Purewal- Col. of Charleston

# Credit Where Credit is Due

This Work Grew Out of a Project In the UMCP SPIRAL (Summer Program in Research and Learning) Program for College Math Majors at HBCU's.

One of the students, **Brett Jefferson** has his own paper on this subject.

**ALSO:** Zarankiewics [7] asked similar questions.

# Square Theorem:

## Theorem

*For all  $c$ , there exists  $G$  such that  
for every  $c$ -coloring of  $G \times G$  there exists a monochromatic square.*

...	...	...	...	...
...	$R$	...	$R$	...
...	$\vdots$	...	$\vdots$	...
...	$R$	...	$R$	...
...	...	...	...	...

# Proving the Square Theorem and Bounding $G(c)$

How to prove Square Theorem?

1. Corollary of Hales-Jewitt Theorem [1]. Bounds on  $G$  HUGE!
2. Corollary of Gallai's theorem [3,4,6]. Bounds on  $G$  HUGE!
3. From VDW directly (folklore). Bounds on  $G$  HUGE!
4. Directly (folklore?). Bounds on  $G$  HUGE!
5. Graham and Solymosi [2].  $G \leq 2^{2^{81}}$ . Better but still HUGE.

Best known upper and lower bounds:

1.  $G(2) \leq 2^{2^{81}}$ .
2.  $\Omega(c^{4/3}) \leq G(c)$ . (Upper bound not writable-downable.)

# What If We Only Care About Rectangles?

## Definition

$G_{n,m}$  is the grid  $[n] \times [m]$ .

1.  $G_{n,m}$  is  **$c$ -colorable** if there is a  $c$ -colorings of  $G_{n,m}$  such that no rectangle has all four corners the same color.
2.  $\chi(G_{n,m})$  is the least  $c$  such that  $G_{n,m}$  is  $c$ -colorable.

# Our Main Question

Fix  $c$

**Exactly which  $G_{n,m}$  are  $c$ -colorable?**

# Two Motivations!

1. Relaxed version of Square Theorem- hope for better bounds.
2. Coloring  $G_{n,m}$  without rectangles is equivalent to coloring edges of  $K_{n,m}$  without getting monochromatic  $K_{2,2}$ .

Our results yield **Bipartite Ramsey Numbers!**

# Obstruction Sets

## Definition

$G_{n,m}$  contains  $G_{a,b}$  if  $a \leq n$  and  $b \leq m$ .

## Theorem

*For all  $c$  there exists a unique finite set of grids  $\text{OBS}_c$  such that*

*$G_{n,m}$  is  $c$ -colorable iff*

*$G_{n,m}$  does not contain any element of  $\text{OBS}_c$ .*

1. Can prove using well-quasi-orderings. No bound on  $|\text{OBS}_c|$ .
2. Our tools yield alternative proof and show

$$2\sqrt{c}(1 - o(1)) \leq |\text{OBS}_c| \leq 2c^2.$$



# Rephrase Main Question

Fix c

**What is  $OBS_c$**

# Rectangle Free Sets and Density

## Definition

$G_{n,m}$  is the grid  $[n] \times [m]$ .

1.  $X \subseteq G_{n,m}$  is **Rectangle Free** if there are NOT four vertices in  $X$  that form a rectangle.
2.  $\text{rfree}(G_{n,m})$  is the size of the largest Rect Free subset of  $G_{n,m}$ .

# Rectangle Free subset of $G_{21,12}$ of size $63 = \left\lceil \frac{21 \cdot 12}{4} \right\rceil$

	01	02	03	04	05	06	07	08	09	10	11	12
1	•	•										
2	•		•									
3		•	•									
4			•	•	•							
5		•		•		•						
6	•				•	•						
7						•	•	•				
8					•		•		•			
9				•				•	•			
10						•				•	•	
11					•					•		•
12				•							•	•
13			•			•			•			•
14			•					•		•		
15			•				•				•	
16		•							•	•		
17		•			•			•			•	
18		•					•					•
19	•								•		•	
20	•							•				•
21	•			•			•			•		

# Colorings Imply Rectangle Free Sets

## Lemma

Let  $n, m, c \in \mathbb{N}$ . If  $\chi(G_{n,m}) \leq c$  then  $\text{rfree}(G_{n,m}) \geq \lceil mn/c \rceil$ .

**Note:** We use to get non-col results as density results!!

# PART I: 2-COLORABILITY

We will EXACTLY Characterize which  $G_{n,m}$  are 2-colorable!

# $G_{5,5}$ IS NOT 2-Colorable!

## Theorem

$G_{5,5}$  *is not* 2-Colorable.

## Proof:

$$\begin{aligned}\chi(G_{5,5}) = 2 &\implies \text{rfree}(G_{5,5}) \geq \lceil 25/2 \rceil = 13 \\ &\implies \text{there exists a column with } \geq \lceil 13/5 \rceil = 3 \text{ } R\text{'s}\end{aligned}$$

## Case 1: There is a column with 5 $R$ 's

Case 1: There is a column with 5  $R$ 's.

$R$	○	○	○	○
$R$	○	○	○	○
$R$	○	○	○	○
$R$	○	○	○	○
$R$	○	○	○	○

Remaining columns have  $\leq 1$   $R$  so

$$\text{Number of } R\text{'s} \leq 5 + 1 + 1 + 1 + 1 = 9 < 13.$$

## Case 2: There is a column with 4 $R$ 's

Case 2: There is a column with 4  $R$ 's.

$R$	○	○	○	○
$R$	○	○	○	○
$R$	○	○	○	○
$R$	○	○	○	○
	○	○	○	○

Remaining columns have  $\leq 2$   $R$ 's

$$\text{Number of } R\text{'s} \leq 4 + 2 + 2 + 2 + 2 \leq 12 < 13$$



## Case 3: Max in a column is 3 $R$ 's

Case 3: Max in a column is 3  $R$ 's.

Case 3a: There are  $\leq 2$  columns with 3  $R$ 's.

Number of  $R$ 's  $\leq 3 + 3 + 2 + 2 + 2 \leq 12 < 13$ .

Case 3b: There are  $\geq 3$  columns with 3  $R$ 's.

$R$	○	○	○	○
$R$	○	○	○	○
$R$	$R$	○	○	○
○	$R$	○	○	○
○	$R$	○	○	○

Can't put in a third column with 3  $R$ 's!

## Case 4: Max in a column is $\leq 2R$ 's

Case 4: Max in a column is  $\leq 2R$ 's.

$$\text{Number of } R\text{'s} \leq 2 + 2 + 2 + 2 + 2 \leq 10 < 13.$$

No more cases. We are Done! Q.E.D.

# $G_{4,6}$ IS 2-Colorable

Theorem

$G_{4,6}$  *is* 2-Colorable.

Proof.

$R$	$R$	$R$	$B$	$B$	$B$
$R$	$B$	$B$	$R$	$R$	$B$
$B$	$R$	$B$	$R$	$B$	$R$
$B$	$B$	$R$	$B$	$R$	$R$



# $G_{3,7}$ IS NOT 2-Colorable

Theorem

$G_{3,7}$  *is not* 2-Colorable.

Proof.

$$\begin{aligned}\chi(G_{3,7}) = 2 &\implies \text{rfree}(G_{3,7}) \geq (\lceil 21/2 \rceil = 11 \\ &\implies \text{there is a row with } \geq \lceil 11/3 \rceil = 4 \text{ } R\text{'s}\end{aligned}$$

Proof similar to  $G_{5,5}$ — lots of cases.



# Complete Char of 2-Colorability

## Theorem

$$\text{OBS}_2 = \{G_{3,7}, G_{5,5}, G_{7,3}\}.$$

## Proof.

Follows from results  $G_{5,5}, G_{7,3}$  not 2-colorable and  $G_{4,6}$  is 2-colorable. □

## PART II: TOOLS TO SHOW $G_{n,m}$ NOT $c$ -COLORABLE

We show that if  $A$  is a Rectangle Free subset of  $G_{n,m}$  then there is a relation between  $|A|$  and  $n$  and  $m$ .

# Bound on Size of Rectangle Free Sets (new)

## Theorem

Let  $a, n, m \in \mathbb{N}$ . Let  $q, r$  be such that  $a = qn + r$  with  $0 \leq r \leq n$ . Assume that there exists  $A \subseteq G_{m,n}$  such that  $|A| = a$  and  $A$  is rectangle-free.

1. If  $q \geq 2$  then

$$n \leq \left\lfloor \frac{m(m-1) - 2rq}{q(q-1)} \right\rfloor.$$

2. If  $q = 1$  then

$$r \leq \frac{m(m-1)}{2}.$$

# Proof of Theorem

$A \subseteq G_{n,m}$ , rectangle free.

$x_i$  is number of points in  $i^{\text{th}}$  column.

	1	$\dots$	$m$
1		$\dots$	
$\vdots$		$\vdots$	
$n$		$\dots$	
	$x_1$ points $\binom{x_1}{2}$ pairs of points	$\dots$ $\dots$	$x_m$ points $\binom{x_m}{2}$ pairs of points



# Proof of Theorem

$A \subseteq G_{n,m}$ , rectangle free.

$x_i$  is number of points in  $i^{\text{th}}$  column.

	1	...	$m$
1		...	
$\vdots$		$\vdots$	
$n$		...	
	$x_1$ points $\binom{x_1}{2}$ pairs of points	...	$x_m$ points $\binom{x_m}{2}$ pairs of points

$$\sum_{i=1}^m \binom{x_i}{2} \leq \binom{n}{2}.$$

## Proof of Theorem (cont)

$$\sum_{i=1}^m \binom{x_i}{2} \leq \binom{n}{2}.$$

Sum minimized when  $x_1 = \dots = x_m = x$

$$m \binom{x}{2} \leq \binom{n}{2}.$$

$$x \leq \frac{m + \sqrt{m^2 + 4m(n^2 - n)}}{2m}$$

$$|A| \leq xm \leq \frac{m + \sqrt{m^2 + 4m(n^2 - n)}}{2}$$

# PART III: TOOLS TO SHOW $G_{n,m}$ IS $c$ -COLORABLE

We define and use **Strong  $c$ -Colorings** to get  $c$ -Colorings

# Strong $c$ -Colorings

## Definition

Let  $c, n, m \in \mathbb{N}$ .  $\chi : G_{n,m} \rightarrow [c]$ .  $\chi$  is a **strong  $c$ -coloring** if the following holds: CANNOT have a rectangle with the two right most corners are same color and the two left most corners the same color.

**Example:** A strong 3-coloring of  $G_{4,6}$ .

$R$	$R$	$G$	$R$	$G$	$G$
$B$	$G$	$R$	$G$	$R$	$G$
$G$	$B$	$B$	$G$	$G$	$R$
$G$	$G$	$G$	$B$	$B$	$B$

# Strong Coloring Lemma

Let  $c, n, m \in \mathbb{N}$ . If  $G_{n,m}$  is strongly  $c$ -colorable then  $G_{n,cm}$  is  $c$ -colorable.

Example:

R	R	G	R	G	G	B	B	R	B	R	R	G	G	B	G	B	B
B	G	R	G	R	G	G	R	B	R	B	R	R	B	G	B	G	B
G	B	B	G	G	R	R	G	G	R	R	B	B	R	R	B	B	G
G	G	G	B	B	B	R	R	R	G	G	G	B	B	B	R	R	R

# Combinatorial Coloring Theorem

Let  $c \geq 2$ .

1. There is a strong  $c$ -coloring of  $G_{c+1, \binom{c+1}{2}}$ .
2. There is a  $c$ -coloring of  $G_{c+1, m}$  where  $m = c \binom{c+1}{2}$ .

**Example:** Strong 5-coloring of  $G_{6,15}$ .

O	O	O	O	O	R	R	R	R	R	R	R	R	R	R
O	R	R	R	R	O	O	O	O	B	B	B	B	B	B
R	O	B	B	B	O	B	B	B	O	O	O	G	G	G
B	B	O	G	G	B	O	G	G	O	G	G	O	O	P
G	G	G	O	P	G	G	O	P	G	O	P	O	P	O
P	P	P	P	O	P	P	P	O	P	P	O	P	O	O

# Coloring Using Primes!

## Theorem

Let  $p$  be a prime.

1. There is a strong  $p$ -coloring of  $G_{p^2, p+1}$ .
2. There is a  $p$ -coloring of  $G_{p^2, p^2+p}$ .

## Proof.

Uses geometry over finite fields. □

**Note:** Have more general theorem.

# Tournament Graph Coloring Theorem

Let  $c \geq 2$ .

1. There is a strong  $c$ -coloring of  $G_{2c, 2c-1}$ .
2. There is a  $c$ -coloring of  $G_{2c, 2c^2-c}$ .

**Proof.**

Uses tournament graphs.





# PART IV: 3-COLORABILITY

We will EXACTLY Characterize which  $G_{n,m}$  are 3-colorable!

# Easy 3-Colorable Results

## Theorem

1. *The following grids **are not** 3-colorable.*

$G_{4,19}$ ,  $G_{19,4}$ ,  $G_{5,16}$ ,  $G_{16,5}$ ,  $G_{7,13}$ ,  $G_{13,7}$ ,  $G_{10,12}$ ,  $G_{12,10}$ ,  $G_{11,11}$ .

2. *The following grids **are** 3-colorable.*

$G_{3,19}$ ,  $G_{19,3}$ ,  $G_{4,18}$ ,  $G_{18,4}$ ,  $G_{6,15}$ ,  $G_{15,6}$ ,  $G_{9,12}$ ,  $G_{12,9}$ .

## Proof.

Follows from tools.



# $G_{10,10}$ is 3-colorable

Theorem

$G_{10,10}$  is 3-colorable.

Proof.

UGLY! TOOLS DID NOT HELP AT ALL!!

R	R	R	R	B	B	G	G	B	G
R	B	B	G	R	R	R	G	G	B
G	R	B	G	R	B	B	R	R	G
G	B	R	B	B	R	G	R	G	R
R	B	G	G	G	B	G	B	R	R
G	R	B	B	G	G	R	B	B	R
B	G	R	B	G	B	R	G	R	B
B	B	G	R	R	G	B	G	B	R
G	G	G	R	B	R	B	B	R	B
B	G	B	R	B	G	R	R	G	G

# $G_{10,11}$ is not 3-colorable

## Theorem

$G_{10,11}$  is not 3-colorable.

## Proof.

You don't want to see this. UGLY case hacking. □

# Complete Char of 3-colorability

## Theorem

$\text{OBS}_3 =$

$$\{G_{4,19}, G_{5,16}, G_{7,13}, G_{10,11}, G_{11,10}, G_{13,7}, G_{16,5}, G_{19,4}\}.$$

## Proof.

Follows from above results on grids being or not being 3-colorable. □

# PART V: 4-COLORABILITY

We will EXACTLY Characterize which  $G_{n,m}$  are 4-colorable!

# Easy NOT 4-Colorable Results

## Theorem

The following grids *are* NOT 4-colorable:

1.  $G_{5,41}$  and  $G_{41,5}$
2.  $G_{6,31}$  and  $G_{31,6}$
3.  $G_{7,29}$  and  $G_{29,7}$
4.  $G_{9,25}$  and  $G_{25,9}$
5.  $G_{10,23}$  and  $G_{23,10}$
6.  $G_{11,22}$  and  $G_{22,11}$
7.  $G_{13,21}$  and  $G_{21,13}$
8.  $G_{17,20}$  and  $G_{20,17}$
9.  $G_{18,19}$  and  $G_{19,18}$

Follows from tools for proving grids *are* NOT colorable.

# Easy IS 4-Colorable Results

## Theorem

The following grids *are* 4-colorable:

1.  $G_{4,41}$  and  $G_{41,4}$ .
2.  $G_{5,40}$  and  $G_{40,5}$ .
3.  $G_{6,30}$  and  $G_{30,6}$ .
4.  $G_{8,28}$  and  $G_{28,8}$ .
5.  $G_{16,20}$  and  $G_{20,16}$ .

Follows from tools for proving grids *are* colorable.



# Theorems with UGLY Proofs

## Theorem

1.  $G_{17,19}$  *is NOT 4-colorable: Used some tools.*
2.  $G_{24,9}$  *is 4-colorable: Used strong coloring of  $G_{9,6}$ .*

# Theorems with UGLY Proofs

## Theorem

1.  $G_{17,19}$  *is NOT* 4-colorable: Used some tools.
2.  $G_{24,9}$  *is* 4-colorable: Used strong coloring of  $G_{9,6}$ .

P	R	R	P	R	R
P	B	B	R	P	B
P	G	G	B	B	P
R	P	G	P	G	B
B	P	R	B	P	G
G	P	B	G	R	P
G	B	P	P	B	G
R	G	P	G	P	R
B	R	P	R	G	P

# 4-coloring of $G_{21,11}$ Due to Brad Loren

	1	2	3	4	5	6	7	8	9	10	11
1	G	B	B	G	R	P	R	G	P	B	P
2	B	G	G	P	B	G	P	R	R	B	R
3	R	R	B	P	B	P	B	P	G	G	R
4	P	R	P	G	B	B	R	P	R	G	B
5	R	P	G	B	B	P	P	B	R	G	G
6	B	R	P	R	G	P	B	R	G	P	B
7	P	G	B	R	G	B	R	G	P	P	R
8	P	P	G	B	R	B	G	R	G	B	P
9	R	B	R	B	G	G	R	P	P	G	B
10	R	P	P	R	G	R	B	B	P	B	G
11	B	P	R	R	P	B	G	G	R	P	G
12	R	B	P	P	P	B	B	R	G	R	G
13	G	G	B	B	R	R	P	P	R	P	G
14	G	B	R	P	B	G	G	R	B	P	P
15	G	P	G	P	G	R	R	R	B	B	B
16	B	B	R	G	P	G	P	B	P	R	G
17	P	G	B	G	P	P	R	B	G	R	B
18	B	P	B	G	G	R	G	P	B	R	R
19	P	G	R	P	R	B	G	B	B	G	R
20	B	R	P	B	R	G	P	G	G	R	P
21	G	R	R	B	P	R	B	P	B	G	P

# 4-coloring of $G_{22,10}$ Due to Brad Loren

	1	2	3	4	5	6	7	8	9	10
1	P	G	R	R	G	G	P	P	B	B
2	G	P	B	G	B	B	P	R	P	R
3	B	G	B	R	P	P	G	R	P	B
4	P	P	G	G	R	R	B	B	G	P
5	P	B	P	P	G	R	R	G	G	R
6	P	B	R	B	R	P	G	R	G	G
7	G	P	G	P	B	P	R	B	R	G
8	P	R	R	B	P	B	G	G	B	R
9	P	B	B	R	R	G	R	G	P	G
10	R	R	B	B	P	G	R	B	G	P
11	R	G	G	P	R	B	B	G	P	R
12	R	B	R	G	G	P	P	B	B	G
13	B	R	G	B	G	R	B	R	P	P
14	G	G	P	B	B	P	R	R	G	B
15	R	G	P	R	B	R	B	P	P	G
16	B	B	P	G	P	B	P	G	R	R
17	G	P	B	R	P	G	B	P	B	R
18	R	B	G	P	B	G	P	R	R	P
19	G	B	R	P	P	R	B	G	R	B
20	B	R	P	G	R	G	G	B	R	P
21	B	R	G	R	B	P	G	P	B	P
22	G	P	P	R	G	B	G	B	R	B

# Absolute Results

## Theorem

1. *The following grids are in  $\text{OBS}_4$ :  $G_{5,41}$ ,  $G_{6,31}$ ,  $G_{7,29}$ ,  $G_{9,25}$ ,  $G_{10,23}$ ,  $G_{11,22}$ ,  $G_{22,11}$ ,  $G_{23,10}$ ,  $G_{25,9}$ ,  $G_{29,7}$ ,  $G_{31,6}$ ,  $G_{41,5}$ .*
2. *For each of the following grids it is not known if it is 4-colorable. These are the only such.  $G_{17,17}$ ,  $G_{17,18}$ ,  $G_{18,17}$ ,  $G_{18,18}$ .  $G_{21,12}$ ,  $G_{22,10}$ .*
3. *Exactly one of these is in  $\text{OBS}_4$ :  $G_{21,11}$ ,  $G_{21,12}$ .*
4. *Exactly one of these is in  $\text{OBS}_4$ :  $G_{17,19}$ ,  $G_{17,18}$ ,  $G_{17,17}$ .*
5. *If  $G_{19,17} \in \text{OBS}_4$  then it is possible that  $G_{18,18} \in \text{OBS}_4$ .*

# Rectangle Free Conjecture

Recall the following lemma:

## Lemma

Let  $n, m, c \in \mathbb{N}$ . If  $\chi(G_{n,m}) \leq c$  then  $\text{rfree}(G_{n,m}) \geq \lceil nm/c \rceil$ .

# Rectangle Free Conjecture

Recall the following lemma:

## Lemma

Let  $n, m, c \in \mathbb{N}$ . If  $\chi(G_{n,m}) \leq c$  then  $\text{rfree}(G_{n,m}) \geq \lceil nm/c \rceil$ .

**Rectangle-Free Conjecture (RFC)** is the converse:

Let  $n, m, c \geq 2$ . If  $\text{rfree}(G_{n,m}) \geq \lceil nm/c \rceil$  then  $G_{n,m}$  is  $c$ -colorable.

# Rectangle Free Subset of $G_{22,10}$ of Size of size

$$55 = \left\lceil \frac{22 \cdot 10}{4} \right\rceil$$

	01	02	03	04	05	06	07	08	09	10
1	•						•			
2		•					•			
3			•				•			
4				•			•			
5					•		•			
6						•	•			
7	•	•						•		
8			•	•				•		
9					•	•		•		
10		•	•						•	
11				•	•				•	
12	•					•			•	
13	•			•						•
14		•				•				•
15			•		•					•
16		•			•					
17	•		•							
18				•		•				
19			•			•				
20		•		•						
21	•				•					
22							•	•	•	•

If RFC is true then  $G_{22,10}$  is 4-colorable.



# Rectangle Free subset of $G_{21,12}$ of size $63 = \left\lceil \frac{21 \cdot 12}{4} \right\rceil$

	01	02	03	04	05	06	07	08	09	10	11	12
1	•	•										
2	•		•									
3		•	•									
4			•	•	•							
5		•		•		•						
6	•				•	•						
7						•	•	•				
8					•		•		•			
9				•				•	•			
10						•				•	•	
11					•					•		•
12				•							•	•
13			•			•			•			•
14			•					•		•		
15			•				•				•	
16		•							•	•		
17		•			•			•			•	
18		•					•					•
19	•								•		•	
20	•							•				•
21	•			•			•			•		

If RFC is true then  $G_{21,12}$  is 4-colorable.

# Rectangle Free subset of $G_{18,18}$ of size $81 = \lceil \frac{18 \cdot 18}{4} \rceil$

	01	02	03	04	05	06	07	08	09	10	11	12	13	14	15	16	17	18
1		•		•										•		•	•	
2	•	•								•	•		•					
3	•								•						•	•		•
4						•			•			•	•	•				
5		•	•			•												•
6	•			•		•	•											
7							•	•		•				•				•
8			•				•		•		•						•	
9		•			•		•					•			•			
10				•							•	•						•
11	•		•		•									•				
12			•	•				•					•		•			
13				•	•			•			•					•		
14	•							•				•					•	
15				•	•				•	•								
16						•				•					•		•	
17			•							•		•				•		
18					•								•				•	•

If RFC is true then  $G_{18,18}$  is 4-colorable. NOTE: If delete 2nd column and 5th Row have 74-sized RFC of  $G_{17,17}$ .

# Assuming RFC...

## Theorem

*If RFC then*

$$\text{OBS}_4 = \{G_{41,5}, G_{31,6}, G_{29,7}, G_{25,9}, G_{23,10}, G_{22,11}, G_{21,13}, G_{19,17}\} \cup \\ \{G_{13,21}, G_{11,22}, G_{10,23}, G_{9,25}, G_{7,29}, G_{6,31}, G_{5,41}\}.$$

## Proof.

Follows from known 4-colorability, non-4-colorability results, and Rect Free Sets above. □

# CASH PRIZE!

The first person to email me both (1) plaintext, and (2) LaTeX, of a 4-coloring of the  $17 \times 17$  grid that has no monochromatic rectangles will receive \$289.00.

# CASH PRIZE!

The first person to email me both (1) plaintext, and (2) LaTeX, of a 4-coloring of the  $17 \times 17$  grid that has no monochromatic rectangles will receive \$289.00.

Also want to know about  $18 \times 18$ .

# CASH PRIZE!

The first person to email me both (1) plaintext, and (2) LaTeX, of a 4-coloring of the  $17 \times 17$  grid that has no monochromatic rectangles will receive \$289.00.

Also want to know about  $18 \times 18$ .

Bernd Steinbach and Christian Postoff solved it!

# CASH PRIZE!

The first person to email me both (1) plaintext, and (2) LaTeX, of a 4-coloring of the  $17 \times 17$  grid that has no monochromatic rectangles will receive \$289.00.

Also want to know about  $18 \times 18$ .

Bernd Steinbach and Christian Postoff solved it!

So  $OBS_4$  is known!

# PART VI: OPEN QUESTIONS

1. What is  $\text{OBS}_5$ ?
2. Prove or disprove **Rectangle Free Conjecture**.
3. Have  $\Omega(\sqrt{c}) \leq |\text{OBS}_c| \leq O(c^2)$ . Get better bounds!
4. Refine tools so can prove **ugly** results **cleanly**.



# Bibliography

- 1 R. Graham, B. Rothchild, and J. Spencer. *Ramsey Theory*. Wiley, 1990.
- 2 R. Graham and J. Solymosi. Monochromatic equilateral right triangles on the integer grid. *Topics in Discrete Mathematics, Algorithms and Combinatorics*, 2006.  
[www.math.ucsd.edu/~ron/06\\_03\\_righttriangles.pdf](http://www.math.ucsd.edu/~ron/06_03_righttriangles.pdf)  
or [www.cs.umd.edu/~vdw/graham-solymosi.pdf](http://www.cs.umd.edu/~vdw/graham-solymosi.pdf).
- 3 R. Rado. Studien zur kombinatorik. *Mathematische Zeitschrift*, pages 424–480, 1933.  
<http://www.cs.umd.edu/~gasarch/vdw/vdw.html>.
- 4 R. Rado. Notes on combinatorial analysis. *Proceedings of the London Mathematical Society*, pages 122–160, 1943.  
<http://www.cs.umd.edu/~gasarch/vdw/vdw.html>.

## Bibliography (continued)

- 5 I. Reiman. Über ein problem von k.zarankiewicz. *Acta. Math. Acad. Soc. Hung.*, 9:269–279, 1958.
- 6 Witt. Ein kombinatorischer satz de elementargeometrie. *Mathematische Nachrichten*, pages 261–262, 1951.  
<http://www.cs.umd.edu/~gasarch/vdw/vdw.html>.
- 7 K. Zarankiewicz. Problem P 101. *Colloq. Math.*