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ON A GENERALIZATION OF RAMSEY THEORY

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In [2, 3], Chung and Liu introduce the following generalization of Ramsey Theory for graphs. Choose c colors, and integers $d_1, d_2, \ldots, d_n$ satisfying $1 \leq d_1 < d_2 < \cdots < d_n < c$. Order the $\binom{c}{d_1}$ subsets of d_1 colors, $\binom{c}{d_2}$ subsets of d_2 colors, $\ldots$, $\binom{c}{d_n}$ subsets of d_n colors and let $t = \sum_i \binom{c}{d_i}$. For graphs $G_1, G_2, \ldots, G_t$, the $(d_1, d_2, \ldots, d_n)$-chromatic Ramsey number denoted by $R^c_{d_1, d_2, \ldots, d_n}(G_1, G_2, \ldots, G_t)$, is the smallest integer p such that if the edges of K_p are colored with c colors in any fashion, then for some i, the subgraph whose edges are colored with the ith subset of colors contains G_i. The numbers $R^2_1(G_1, G_2)$, simply denoted $R(G_1, G_2)$, have been surveyed in [1] and in particular if $G_1, G_2, \ldots, G_c$ are complete graphs, then $R^c_1(G_1, G_2, \ldots, G_c)$, denoted $R(G_1, G_2, \ldots, G_c)$, are the classical Ramsey numbers [7].

Chung and Liu have determined some numbers of the form $R^3_2(K_i, K_j, K_m)$, $R^3_{1,2}(K_{i_1}, K_{i_2}, \ldots, K_{i_6})$ and $R^4_{1,3}(K_{i_1}, \ldots, K_{i_8})$. In this paper we are mainly concerned with the numbers $R^4_{1,2}(K_{i_1}, K_{i_2}, \ldots, K_{i_{10}})$.

1. Introduction

In [2, 3], Chung and Liu introduce the following generalization of Ramsey Theory for graphs. Choose c colors, and integers $d_1, d_2, \ldots, d_n$ satisfying $1 \leq d_1 < d_2 < \cdots < d_n < c$. Order the $\binom{c}{d_1}$ subsets of d_1 colors, $\binom{c}{d_2}$ subsets of d_2 colors, $\ldots$, $\binom{c}{d_n}$ subsets of d_n colors and let $t = \sum_i \binom{c}{d_i}$. For graphs $G_1, G_2, \ldots, G_t$, the $(d_1, d_2, \ldots, d_n)$-chromatic Ramsey number denoted by $R^c_{d_1, d_2, \ldots, d_n}(G_1, G_2, \ldots, G_t)$, is the smallest integer p such that if the edges of K_p are colored with c colors in any fashion, then for some i, the subgraph whose edges are colored with the ith subset of colors contains G_i. The numbers $R^2_1(G_1, G_2)$, simply denoted $R(G_1, G_2)$, have been surveyed in [1] and in particular if $G_1, G_2, \ldots, G_c$ are complete graphs, then $R^c_1(G_1, G_2, \ldots, G_c)$, denoted $R(G_1, G_2, \ldots, G_c)$, are the classical Ramsey numbers [7].

Chung and Liu have determined some numbers of the form $R^3_2(K_i, K_j, K_m)$, $R^3_{1,2}(K_{i_1}, K_{i_2}, \ldots, K_{i_6})$ and $R^4_{1,3}(K_{i_1}, \ldots, K_{i_8})$. In this paper we are mainly concerned with the numbers $R^4_{1,2}(K_{i_1}, K_{i_2}, \ldots, K_{i_{10}})$.

2. General inequalities

For convenience, we designate colors by lower case Greek letters and establish the following notational convention. The number

$$R^4_{1,2}(G_1, G_2, G_3, G_4, G_{12}, G_{34}, G_{13}, G_{24}, G_{14}, G_{23})$$

is the smallest integer p so that if the edges of K_p are 4-colored with α, β, γ and δ, then K_p contains at least one of the following: an $\alpha-G_1$, $\beta-G_2$, $\gamma-G_3$, $\delta-G_4$, $\alpha\beta-G_{12}$, $\gamma\delta-G_{34}$, $\alpha\gamma-G_{13}$, $\beta\delta-G_{24}$, $\alpha\delta-G_{14}$ or a $\beta\gamma-G_{23}$. For positive integers s less than p, a coloring of K_s with α, β, γ and δ containing none of the prescribed subgraphs is called *proper*. Note that $p-1$ is the largest integer for which a proper 4-coloring exists. Also it will be convenient to replace K_n by n in argument lists.

Theorem 2.1. *$R^4_{1,2}(G_1, G_2, G_3, G_4, i_{12}, \ldots, i_{23}) \leqslant R(G_1, G_2, G_3, G_4)$. Furthermore if $i_{kl} \geqslant R(G_k, G_l)$ for all pairs k, l, then equality holds.*

Proof. If $s = R(G_1, G_2, G_3, G_4)$, every 4-coloring of K_s contains an $\alpha-G_1$, $\beta-G_2$, $\gamma-G_3$ or a $\delta-G_4$, so clearly $R^4_{1,2}(G_1, G_2, G_3, G_4, i_{12}, \ldots, i_{23}) \leqslant s$. We know there is a 4-coloring of K_{s-1} containing no $\alpha-G_1$, $\beta-G_2$, $\gamma-G_3$ nor $\delta-G_4$. Clearly, such a coloring can contain no $\alpha\beta-K_{i_{12}}$, $\gamma\delta-K_{i_{34}}$, etc., thus completing the proof. □

As special cases, Theorem 2.1 yields:

$$R^4_{1,2}(P_3, P_3, P_3, P_3, 3, 3, 3, 3, 3, 3) = R(P_3, P_3, P_3, P_3) = 5,$$

and

$$R^4_{1,2}(3, 3, 3, 3, 6, 6, 6, 6, 6, 6) = R(3, 3, 3, 3).$$

Theorem 2.2. *If $s = \min\{R(G_{12}, G_{34}), R(G_{13}, G_{24}), R(G_{14}, G_{23})\}$, then*

$$R^4_{1,2}(G_1, G_2, G_3, G_4, G_{12}, \ldots, G_{23}) \leqslant s.$$

Proof. Without loss of generality, let $s = R(G_{12}, G_{34})$. If we 4-color K_s, then by considering α and β as one color and γ and δ as a second color there must exist an $\alpha\beta-G_{12}$ or a $\gamma\delta-G_{34}$, by the definition of $R(G_{12}, G_{34})$. The result now follows immediately. □

3. Exact results

In this section we determine the numbers $R^4_{1,2}(i_1, i_2, i_3, i_4, i_5, \ldots, i_{10})$ for various special cases.

Theorem 3.1

$$R^4_{1,2}(3, 3, 3, \ldots, 3, n) = \begin{cases} 5 & \text{if } n = 3, \\ 6 & \text{if } n \geqslant 4. \end{cases}$$

Proof. If $n = 3$, then a proper 4-coloring of K_5 contains no monochromatic nor bichromatic triangles. Thus no point can be incident with more than one edge of any one color. But this implies that each point must be incident with exactly one edge of each color, which contradicts the fact that each color must have an even valence sum. Fig. 1a illustrates a proper coloring of K_4; thus $R^4_{1,2}(3,3,\ldots,3,3)=5$.

If $n\geq 4$, we know from Theorem 2.2 that $R^4_{1,2}(3,3,\ldots,3,n)\leq R(3,3)=6$, and equality follows from the proper coloring of K_5 in Fig. 1b. $\square$

Theorem 2.2 also yields the following simple extension:

$$R^4_{1,2}(3,3,3,3,3,3,t_1,t_2,t_3,t_4)=6,$$

if any one of t_1, t_2, t_3, or t_4 is greater than 3.

Theorem 3.2. $R^4_{1,2}(3,3,3,3,3,4,t_1,t_2,t_3,t_4)=9$ *when* $t_1\geq 3$ *and* t_2, t_3 *and* t_4 *are all greater than* 3.

Proof. By Theorem 2.2,

$$R^4_{1,2}(3,3,3,3,3,4,t_1,t_2,t_3,t_4)\leq R(3,4)=9$$

and equality follows from the proper coloring of K_8 in Fig. 2. $\square$

In [3] Chung and Liu show that $R^3_{1,2}(3,3,3,4,4,4)=8$. Clearly we can conclude, by omitting a color, that $R^4_{1,2}(3,3,3,3,3,4,3,4,3,4)\geq 8$. Since the proof to show that this number is less than 9 is a tedious case-by-case analysis using standard techniques, we merely state the result:

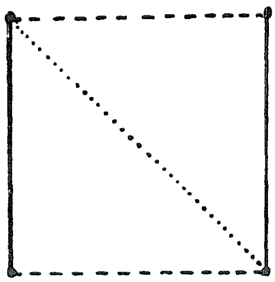
Theorem 3.3 [6]. $R^4_{1,2}(3,3,3,3,3,4,3,4,3,4)=8$.

From Theorem 2.2 we have the following three inequalities:

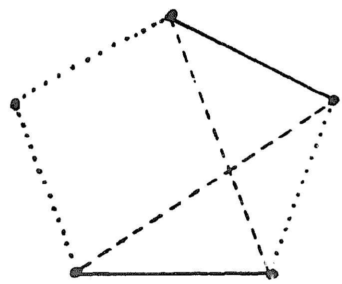
(a) $$R^4_{1,2}(3,3,3,3,3,5,w,x,y,z)\leq 14\begin{cases} w,x,y,z\geq 3,\\ w+x,\ y+z\geq 8,\end{cases}$$

(b) $$R^4_{1,2}(3,3,3,3,3,6,w,x,y,z)\leq 18\begin{cases} w,x,y,z\geq 3,\\ w+x,\ y+z\geq 9,\end{cases}$$

(c) $$R^4_{1,2}(3,3,3,3,4,4,w,x,y,z)\leq 18\begin{cases} w,x,y,z\geq 4,\\ w+x,\ y+z\geq 8.\end{cases}$$



a



b

Fig. 1. Solid lines are α-edges, dashed lines are β-edges, dotted lines are γ and all others are δ.

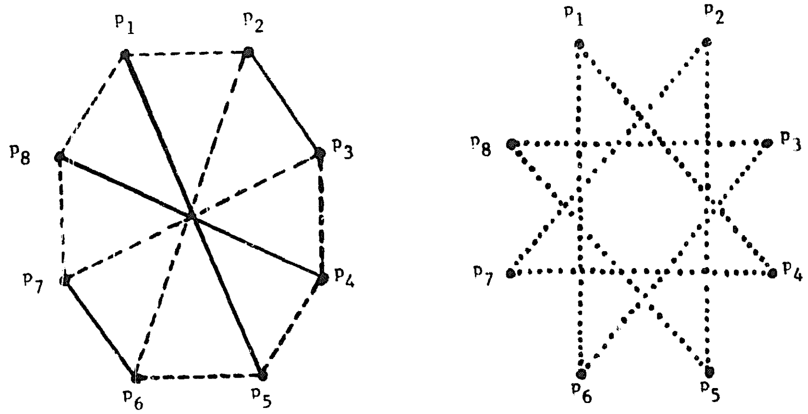


Fig. 2. Solid lines are α-edges, dashed lines are β-edges, dotted lines are γ and all others are δ.

In some cases, equality holds;

Theorem 3.4. $R^4_{1,2}(3,3,3,3,3,5,w,x,y,z) = 14$ *for* $w, y \geq 3$, $x, z \geq 5$.

Proof. Clearly whenever $w, y \geq 3$ and $x, z \geq 5$ we can conclude

$$R^4_{1,2}(3,3,3,3,3,5,w,x,y,z) \geq R^3_{1,2}(3,3,3,5,5,5) = 14. \quad \square$$

Definition (Sabidussi [8]). Let G be a group, and let H be a subset of G such that the identity element of G is not in H and $H = H^{-1}$. By the *Cayley graph*, denoted $X(G,H)$, of G with respect to H we mean the graph given by $V(X(G,H)) = G$, and $E(X(G,H)) = \{[g, gh] \mid g \in G, h \in H\}$.

Theorem 3.5. $R^4_{1,2}(3,3,3,3,3,5,w,x,y,z) = 14$ *for* $w, x, y, z \geq 4$.

Proof. Let $H_\alpha = \{\pm 2\}$, $H_\beta = \{\pm 3\}$, $H_\gamma = \{\pm 1, \pm 4\}$ and $H_\delta = \{\pm 5, \pm 6\}$. Color the edges of Cayley graphs $X(Z_{13}, H_\alpha)$, $X(Z_{13}, H_\beta)$, $X(Z_{13}, H_\gamma)$, $X(Z_{13}, H_\delta)$ in colors α, β, γ and δ respectively. Clearly this is a well defined 4-coloring of the edges of K_{13}, and it is easy to see it is proper. Theorem 2.2 implies equality.

Theorem 3.6. *If* $w, y \geq 3$ *and* $x, z, \geq 6$, *then*

$$R^4_{1,2}(3,3,3,3,3,6,w,x,y,z) = 17.$$

Proof. Since $R^3_{1,2}(3,3,3,6,6,6) = 17$ we can conclude that $R^4_{1,2}(3,3,3,3,3,6,w,x,y,z) \geq 17$ under the given hypothesis.

Suppose that there exists a proper coloring of the edges of K_{17}. By considering α and β as one color we produce a 3-coloring of K_{17}. Since $R(3,3,3) = 17$ we can conclude that there must exist either an $\alpha\beta - K_3$, $\gamma - K_3$ or $\delta - K_3$ in this coloring. In any case this contradicts the assumption of the existence of a proper 4-coloring of K_{17}. The theorem follows. $\square$

Theorem 3.7. $R^4_{1,2}(3, 3, 3, 3, 4, 4, w, x, y, z) = 18$ *for* $w, x, y, z \geq 5$.

Proof. Let $H_\alpha = \{\pm 1, \pm 4\}$, $H_\beta = \{\pm 2, \pm 8\}$ $H_\gamma = \{\pm 3, \pm 5\}$, and $H_\delta = \{\pm 6, \pm 7\}$. Construct Cayley graphs as before, the result follows. □

Conclusion

Almost untouched is the extension to the numbers of the form $R^n_{i,j}(G_1, G_2, \ldots)$ in which G_k is not necessarily complete. The numbers $R^3_2(K_{1,x}, K_{1,y}, K_{1,z})$ have been established [4], and the author has attained $R^3_{1,2}(3, 3, 3, K_{1,x}, K_{1,y}, K_{1,z})$ (see [6]).

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