Finite Ramsey's Theorem
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1 Ramsey's Theorem for the Finite Complete Graphs

Theorem 1.1 *For all k there exists n such that for every* $\mathrm{COL}\colon \binom{[n]}{[2]} \to [2]$ *there is a homog set of size k. KEY: We can take $n = 2^{2k-1}$.*

Proof: Let $\mathrm{COL}\colon \binom{[n]}{[2]} \to [2]$. We define an finite sequence of vertices,

$$x_1, x_2, \ldots, x_{2k-1}$$

and an finite sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots, V_{2k-1}$$

that are based on COL.

Here is the intuition: Vertex $x_1 = 1$ has $n-1$ edges coming out of it. Some are RED, and some are BLUE. Hence there are either $\geq \frac{n-1}{2}$ RED edges coming out of x_1, or there are $\geq \frac{n-1}{2}$ BLUE edges coming out of x_1 (or both). Let c_1 be a color such that x_1 has $\frac{n-1}{2}$ edges coming out of it that are colored c_1. Let V_1 be the set of vertices v such that $\mathrm{COL}(v, x_1) = c_1$. Then keep iterating this process.

We now describe it formally.

$$\begin{aligned} V_0 &= [n] \\ x_1 &= 1 \end{aligned}$$

$$\begin{aligned} c_1 &= \text{RED if } |\{v \in V_0 \mid \mathrm{COL}(v, x_1) = \text{RED}\}| \geq \tfrac{|V_0|-1}{2} \\ &= \text{BLUE otherwise} \\ V_1 &= \{v \in V_0 \mid \mathrm{COL}(v, x_1) = c_1\} \text{ (note that } |V_1| \geq \tfrac{|V_0|-1}{2}) \end{aligned}$$

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i:

$$x_i = \text{the least number in } V_{i-1}$$

$$\begin{aligned} c_i &= \text{RED if } |\{v \in V_{i-1} \mid \mathrm{COL}(v, x_i) = \text{RED}\}| \geq \tfrac{|V_{i-1}|-1}{2} \\ &= \text{BLUE otherwise} \\ V_i &= \{v \in V_{i-1} \mid \mathrm{COL}(v, x_i) = c_i\} \text{ (note that } |V_i| \geq \tfrac{V_{i-1}}{2} \text{)} \end{aligned}$$

(NOTE- look at the step where we define c_i. We are using the fact that if you 2-color X you are guaranteed some color appears $|X|/2$ times. we are using the 1-hypergraph Ramsey Theorem. Later when we prove Ramsey on 3-hypergraphs we will use Ramsey on 2-hypergraphs.)

How long can this sequence go on for? Well, x_i can be defined if V_{i-1} is nonempty. We can show by induction that, for every i, $|V_i| \geq \frac{n}{2^i}$ (this is not quite right because of the -1 but we ignore this detail). Hence the sequence

$$x_1, x_2, \ldots$$

will go until V_i is empty. Since $|V_0| = n$ and at every stage the set is cut in half, this will go on for $\log_2(n)$ iterations. Hence we want $2k - 1 \geq \log_2(n)$ so we need $n = 2^{2k-1}$.

Consider the infinite sequence

$$c_1, c_2, \ldots, c_{2k-1}.$$

Each of the colors in this sequence is either RED or BLUE. Hence there must be a a subset of k of them that are the same color

$$c_{i_1} = c_{i_2} = \cdots = c_{i_k}$$

Denote this color by c, and consider the vertices

$$H = \{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\}$$

We leave it to the reader to show that H is homog. ∎