Extended VDWs Theorem

Exposition by William Gasarch

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VDW and Extended VDW

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$A, A+D, \ldots, A+(k-1)D$ are color CCC. So $COL(D) \neq CCC$.

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$A, A+\frac{XD}{k-1}, A+\frac{2XD}{k-1}, \ldots, A+\frac{(k-1)XD}{k-1}$. So $COL(\frac{XD}{k-1}) \neq CCC$.

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$D, 2D, \ldots, \frac{X}{k-1}D$ not colored CCC, only use $c - 1$ colors.

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This is an exercise.