The Distinct Volumes Problem

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Bill thinks of one— next page.

Points and Distances

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Next Step: Finite version: Can use Finite Can Ramsey to prove the following: For every set of n points in the plane there is a subset of size $\Omega(\log n)$ where all distances are distinct. (Much better is known.)

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1. Dumped Ramsey**!** Added co-authors**!** Got **new** results!
2. What about **Area**? If there are n points in $\mathbb{R}^2$ want large subset so that all areas are distinct.
3. More general question: n points in $\mathbb{R}^d$ and looking for all a-volumes to be different. (This question seems to be new.)

EXAMPLES with DISTANCES

The following is an **EXAMPLE** of the kind of theorems we will be talking about.
If there are n points in $\mathbb{R}^2$ then there is a subset of size $\Omega(n^{1/3})$ with all distances between points **DIFF**.

EXAMPLES with AREAS

If there are n points in $\mathbb{R}^2$ *then there is a subset of size* $\Omega(n^{1/5})$ *with all triangle areas* **DIFF**.

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FALSE: Take n points on a LINE. All triangle areas are 0.

We state theorems in **no three collinear** form to get around this issue.

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Easy Lemma

Lemma If there is a MAP from X to Y that is $\leq c$-to-1 then $|Y| \geq |X|/c$.
We will call this LEMMA.

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Case 1: $|M| \geq n^{1/3}$ DONE!

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Case 1: $|M| \ge n^{1/3}$ DONE!

Case 2: $|M| \le n^{1/3}$. So $|X - M| = \Theta(|X|)$. By LEMMA

$$\begin{aligned} |\binom{M}{2} + M \times \binom{M}{2}| &\ge 0.5|X - M| = \Omega(|X|) = \Omega(n) \\ M &\ge \Omega(n^{1/3}) \end{aligned}$$

On Circle

Thm: For all $X \subseteq \mathbb{S}^1$ (the circle) of size n there exists a dist-rainbow subset of size $\Omega(n^{1/3})$.

Proof: Use **MAXIMAL DIST-RAINBOW SET**. Similar Proof.

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Thm: If $X = \{1, \ldots, n\}$ then the largest dist-rainbow subset is of size $\leq (1 + o(1))n^{1/2}$.

The $d = 2$ Case

Thm: For all $X \subseteq \mathbb{R}^2$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.

Proof: Let M be a **MAXIMAL DIST-RAINBOW SET.**

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All INVERSE IMG's lie on LINES or CIRCLES.

The $d = 2$ Case- Cont

$f : X - M \to \binom{M}{2} \cup M \times \binom{M}{2}$
All INVERSE IMG's lie on LINES or CIRCLES. δ TBD.
Cases 1 and 2 induct into line and circle case.
Case 1: $(\exists x_1, x_2)[(f^{-1}(\{x_1, x_2\})| \geq n^{\delta}]$.
$\geq n^{\delta}$ points on a line, so rainbow set size $\geq \Omega(n^{\delta/3})$.

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Set $\delta/3 = (1 - \delta)/3$. $\delta = 1/2$. Get $\Omega(n^{1/6})$.

On Sphere

Thm: For all $X \subseteq \mathbb{S}^2$ (surface of sphere) of size n there exists a dist-rainbow subset of size $\Omega(n^{1/6})$.
Proof: Use **MAXIMAL DIST-RAINBOW SET**. Similar Proof.

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Note: Better is known: Charalambides showed $\Omega(n^{1/3})$.

General d Case

Thm:
For all $X \subseteq \mathbb{R}^d$ of size n $\exists$ dist-rainbow subset of size $\Omega(n^{1/3d})$.
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Need result on $\mathbb{S}^d$ and $\mathbb{R}^d$ to get result for $\mathbb{S}^{d+1}$ and $\mathbb{R}^{d+1}$.

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Note: Better is known. In 1995 Thiele showed $\Omega(n^{1/(3d-2)})$. But WE improved that!

General d Case- Much Better

Thm: For all $d \geq 2$, for all $X \subseteq \mathbb{R}^d$ of size n there exists a dist-rainbow subset of size $\Omega(n^{1/(3d-3)}(\log n)^{\frac{1}{3}-\frac{2}{3d-3}})$.

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Proof: Use **VARIANT ON MAX DIST-RAINBOW SET**

d	$n^{1/3d}$	$n^{1/(3d-3)}(\log n)^{\frac{1}{3}-\frac{2}{3d-3}}$
1	$n^{1/3}$	$--$
2	$n^{1/6}$	$n^{1/3}(\log n)^{-1/3}$
3	$n^{1/9}$	$n^{1/6}(\log n)^{0}$
4	$n^{1/12}$	$n^{1/9}(\log n)^{1/12}$
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Can we do better? Best we can hope for is roughly $n^{1/d}$.

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What is $f^{-1}(\{x_1, x_2\}, \{x_1, x_3\})$? SEE NEXT SLIDE FOR GEOM LEMMA.

Lemma On Area

Lemma: Let L_1 and L_2 be lines in R^2.

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Sketch: $\mathrm{AREA}(L_1, p) = \mathrm{AREA}(L_2, p)$ iff
$|L_1| \times |L_1 - p| = |L_2| \times |L_2 - p|$ iff $\frac{|L_1-p|}{|L_2-p|} = \frac{|L_1|}{|L_2|}$. This is a line.

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$f : X - M \rightarrow \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$ FINITE-to-1.

Area $d = 2$ Case- Cont

$f : X - M \to \binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}$ is FINITE-to-1.
Case 1: $|M| \geq n^{1/5}$ DONE!

Case 2: $|M| \leq n^{1/5}$. Then $|X - M| = \Theta(|X|)$. Since MAP is finite-to-1, by LEMMA

$$
\begin{aligned}
|\binom{M}{2} \times \binom{M}{2} \cup \binom{M}{2} \times \binom{M}{3}| &\geq \Omega(|X - M|) = \Omega(|X|) = \Omega(n) \\
|M| &\geq \Omega(n^{1/5})
\end{aligned}
$$

Volume $d = 3$

Thm: For all $X \subseteq \mathbb{R}^3$ of size n, no four on a plane, there exists Vol-rainbow set of size $\Omega(n^\delta)$. (δ TBD)
Similar. Left for the reader.

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4. Either:
   All INVERSE IMG's are small, so use LEMMA.
   OR
   Some INVERSE IMG's are large subsets of $\mathbb{R}^d$ or $\mathbb{S}^d$, so induct.

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KEY: We cared about $X \subseteq \mathbb{R}^d$ but had to work with $\mathbb{S}^d$ as well.
NOW we will have to work with more complicated objects.

What Do Inverse Images Look Like?

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Def: (Informally) An **Algebraic Variety in $\mathbb{R}^d$** is a set of points in $\mathbb{R}^d$ that satisfy a polynomial equation in d variables.

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Thm Let $2 \leq a \leq d+1$. Let $r \in \mathbb{N}$. For all varieties V of dim d and degree r for all sets of n points on V there exists an a-rainbow set of size $\Omega(n^{1/(2a-1)d})$.

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