0.0.1 If ... then $(b_1, \ldots, b_n)$ is distinct-regular

We will prove the following theorem due to Rado [**?**, **?**].

Theorem 0.0.1 *If $(b_1, b_2, \ldots, b_n)$ is regular and there exists $\lambda_1, \ldots, \lambda_n$ distinct such that $\sum_{i=1}^{n} \lambda_i b_i = 0$ then $(b_1, \ldots, b_n)$ is distinct-regular.*

To prove this we need a Key Lemma:

Key lemma

The lemma is in three parts. The first one we use to characterize which vectors are distinct-regular. The second and third are used in a later section when we prove the Full Rado Theorem.

The following definitions are used in the third part of the lemma.

Def 0.0.2 Let $n \in \mathbb{N}$.

1. A set $G \subseteq \mathbb{N}^n$ is *homogeneous* if, for all $\alpha \in \mathbb{N}$,

$$(e_1, \ldots, e_n) \in G \implies (\alpha e_1, \ldots, \alpha e_n) \in G.$$

2. A set $G \subseteq \mathbb{N}^n$ is *regular* if, for all c, there exists $R = R(G; c)$ such that the following holds: For all c-colorings $\chi : [R] \to [c]$ there exists $\vec{e} = (e_1, \ldots, e_n) \in G$ such that all of the e_i's are colored the same.

Example 0.0.3

1. Let $G = \{(a, a + d, \ldots, a + (k-1)d) \mid a, d \in \mathbb{N}\}$ be the set of k-APs in $\mathbb{N}$. G is homogeneous. By VDW, G is also regular.

2. Let $b_1, \ldots, b_n \in \mathbb{Z}$. Let $G = \{(e_1, \ldots, e_n) \mid \sum_{i=1}^{n} b_i e_i = 0\}$. G is homogeneous. G is regular if and only if $(b_1, \ldots, b_n)$ is.

3. Let A be an $m \times n$ matrix. Let $G = \{\vec{e} \mid A\vec{e} = \vec{0}\}$. G is homogeneous. G is regular if and only if M is.

Lemma 0.0.4

1. *For all* $(b_1, \ldots, b_n) \in \mathbb{Z}^n$ *regular, for all* $c, M \in \mathbb{N}$*, there exists* $L = L(b_1, \ldots, b_n; c, M)$ *with the following property. For any c-coloring* $\chi\colon [L] \to [c]$ *there exists* $e_1, \ldots, e_n, d \in [L]$ *such that the following hold.*

   (a) $b_1e_1 + \cdots + b_ne_n = 0$.

   (b) All of these numbers have the same color:

$$\begin{array}{ccccccc} e_1 - Md, & \ldots, & e_1 - d, & e_1, & e_1 + d, & \ldots, & e_1 + Md \\ e_2 - Md, & \ldots, & e_2 - d, & e_2, & e_2 + d, & \ldots, & e_2 + Md \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ e_n - Md, & \ldots, & e_n - d, & e_n, & e_n + d, & \ldots, & e_n + Md. \end{array}$$

2. *For all* $(b_1, \ldots, b_n) \in \mathbb{Z}^n$ *regular, for all* $c, M, s \in \mathbb{N}$*, there exists* $L_2 = L_2(b_1, \ldots, b_n; c, M, s)$ *with the following property. For any c-coloring* $\chi\colon [L_2] \to [c]$ *there exists* $e_1, \ldots, e_n, d \in [L_2]$ *such that the following hold.*

   (a) $b_1e_1 + \cdots + b_ne_n = 0$.

   (b) All of these numbers have the same color:

$$\begin{array}{ccccccc} e_1 - Md, & \ldots, & e_1 - d, & e_1, & e_1 + d, & \ldots, & e_1 + Md \\ e_2 - Md, & \ldots, & e_2 - d, & e_2, & e_2 + d, & \ldots, & e_2 + Md \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ e_n - Md, & \ldots, & e_n - d, & e_n, & e_n + d, & \ldots, & e_n + Md \\ & & & sd. & & & \end{array}$$

3. *For all* $n \in \mathbb{N}$*, for all* $G \subseteq \mathbb{N}^n$*, G regular and homogeneous, for all* $c, M, s \in \mathbb{N}$ *there exists* $L_3 = L_3(G; c, M, s)$ *with the following property. For any c-coloring* $\chi\colon [L_3] \to [c]$ *there exists* $e_1, \ldots, e_n, d \in [L_3]$ *such that the following hold.*

   (a) $(e_1, \ldots, e_n) \in G$.

   (b) All of these numbers have the same color:

$$\begin{array}{ccccccc}
e_1 - Md, & \dots, & e_1 - d, & e_1, & e_1 + d, & \dots, & e_1 + Md \\
e_2 - Md, & \dots, & e_2 - d, & e_2, & e_2 + d, & \dots, & e_2 + Md \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
e_n - Md, & \dots, & e_n - d, & e_n, & e_n + d, & \dots, & e_n + Md \\
& & & sd. & & &
\end{array}$$

Proof: (Part 1)
Since $b_1, \dots, b_n$ is regular, by Definition **??** there exists $R = R(b_1, \dots, b_n; c)$ such that for any c-coloring of $[R]$ there exists $e_1, \dots, e_n$ such that
(1) all of the e_i's are the same color, and
(2) $\sum_{i=1}^{n} b_i e_i = 0$.

We will choose the desired number L later. Throughout the proof we will add conditions to L. The first one is that R divides L.

Let χ:$[L] \rightarrow [c]$ be a coloring.

We want to show that the conclusion of the theorem holds for χ.

We define a new coloring χ^*:$[L/R] \rightarrow [c]^R$ as follows:

$$\chi^*(n) = (\chi(n), \chi(2n), \chi(3n), \dots, \chi(Rn))\,.$$

In order to find an arithmetic progression, we will pick L so that $L/R \geq W(2X+1, c^R)$. We will determine X later.

Apply (a slight variant of) VDW to the c^R-coloring χ to obtain the following: There exists a, D (but not our desired d) such that

$$\chi^*(a - XD) = \chi^*(a - (X-1)D) = \cdots = \chi^*(a) = \cdots = \chi^*(a + XD).$$

Since we know

$$\chi^*(n) = (\chi(n), \chi(2n), \dots, \chi(Rn))\,,$$

this gives us

$$\begin{array}{ccccccccccc}
\chi(a - XD) & = & \chi(a - (X-1)D) & = & \cdots & = & \chi(a) & = & \cdots & = & \chi(a + XD) \\
\chi(2(a - XD)) & = & \chi(2(a - (X-1)D)) & = & \cdots & = & \chi(2a) & = & \cdots & = & \chi(2(a + XD)) \\
\chi(3(a - XD)) & = & \chi(3(a - (X-1)D)) & = & \cdots & = & \chi(3a) & = & \cdots & = & \chi(3(a + XD)) \\
\vdots & = & \vdots & = & \cdots & = & \vdots & = & \cdots & = & \vdots \\
\chi(R(a - XD)) & = & \chi(R(a - (X-1)D)) & = & \cdots & = & \chi(Ra) & = & \cdots & = & \chi(R(a + XD)).
\end{array}$$

We need a subset of these that are all the same color. Consider the coloring $\chi^{**}:[R] \rightarrow [c]$ defined by

$$\chi^{**}(n) = \chi(na).$$

By the definition of R there exists $f_1, \ldots, f_n$ such that

1. $\sum_{i=1}^{n} b_i f_i = 0$. Hence $\sum_{i=1}^{n} b_i(af_i) = a\sum_{i=1}^{n} b_i f_i = 0$.
2. $\chi^{**}(f_1) = \chi^{**}(f_2) = \cdots = \chi^{**}(f_n)$.

   By the definition of χ^{**} we have

$$\chi(af_1) = \chi(af_2) = \cdots = \chi(af_n).$$

Note that we have that the following are *all* the same color:

$$\begin{array}{cccccc}
(a - XD)f_1, & (a-(X-1)D)f_1, & \cdots, & af_1, & \cdots, & (a+XD)f_1 \\
(a - XD)f_2, & (a-(X-1)D)f_2, & \cdots, & af_2, & \cdots, & (a+XD)f_2 \\
(a - XD)f_3, & (a-(X-1)D)f_3, & \cdots, & af_3, & \cdots, & (a+XD)f_3 \\
\vdots & \vdots & & \vdots & & \vdots \\
(a - XD)f_n, & (a-(X-1)D)f_n, & \cdots, & af_n, & \cdots, & (a+XD)f_n.
\end{array}$$

For all i, $1 \le i \le n$ let $e_i = af_i$. We rewrite the above:

$$\begin{array}{cccccc}
e_1 - f_1XD, & e_1 - f_1(X-1)D, & \cdots, & e_1, & \cdots, & e_1 + f_1XD \\
e_2 - f_2XD, & e_2 - f_2(X-1)D, & \cdots, & e_2, & \cdots, & e_2 + f_2XD \\
e_3 - f_3XD, & e_3 - f_3(X-1)D, & \cdots, & e_3, & \cdots, & e_3 + f_3XD \\
\vdots & \vdots & & \vdots & & \vdots \\
e_n - f_nXD, & e_n - f_n(X-1)D, & \cdots, & e_n, & \cdots, & e_n + f_nXD.
\end{array}$$

We are almost there — we have our $e_1, \ldots, e_n$ that are the same color, and lots of additive terms from them are also that color. We just need a value of d such that

$$\{d, 2d, 3d, \ldots, Md\} \subseteq \{f_1D, 2f_1D, 3f_1D, \ldots, Xf_1D\},$$
$$\{d, 2d, 3d, \ldots, Md\} \subseteq \{f_2D, 2f_2D, 3f_2D, \ldots, Xf_2D\},$$
$$\vdots$$

$$\{d, 2d, 3d, \ldots, Md\} \subseteq \{f_nD, 2f_nD, 3f_nD, \ldots, Xf_nD\}.$$

We have no control over D, but we haven't chosen X or d yet. We know that, for all i, $f_i \le R$. Clearly $d = f_1f_2\cdots f_nD \le R^nD$ is a sensible choice, so we use that.

We need, for every $1 \le i \le n$,

$$\left\{\left(\prod_{j=1}^{n} f_i\right) D, 2\left(\prod_{j=1}^{n} f_i\right) D, \ldots, M\left(\prod_{j=1}^{n} f_i\right) D\right\} \subseteq \{f_iD, 2f_iD, \ldots, Xf_iD\}.$$

Equivalently, we need

$$\left\{\left(\prod_{j=1}^{n} f_i\right), 2\left(\prod_{j=1}^{n} f_i\right), \ldots, M\left(\prod_{j=1}^{n} f_i\right)\right\} \subseteq \{f_i, 2f_i, \ldots, Xf_i\}.$$

Taking $X = MR^{n-1}$ will suffice.

Since we have $X = R^{n-1}M$, we now know our bound for L:

$$L = R \cdot W(2R^{n-1}M + 1, c^R), \text{ where } R = R(b_1, \ldots, b_n; c).$$

(Part 2)
We prove this by induction on c.
Base Case: For $c = 1$ this is easy; however, we find the actual bound anyway. The only issue here is to make sure that the objects we want to color are actually in $[L(b_1, \ldots, b_n; 1, M, s)]$. Let $(e_1, \ldots, e_n) \in \mathbb{N}^n$ be a solution to $\sum_{i=1}^{n} b_ie_i = 0$ such that $e_{\min} = \min\{e_1, \ldots, e_n\} > M$. Let $e_{\max} = \max\{e_1, \ldots, e_n\} > M$. Let $L_2 = L_2(b_1, \ldots, b_n; 1, M, s) = \max\{e_{\max} + M, s\}$. Let χ:$[L_2] \to [1]$. We claim that $e_1, \ldots, e_n, 1$ work. Note that, for all $i \in [n]$ and $j \in \{-M, \ldots, M\}$, we have $e_i + j \times 1 \in [L_2]$. Also note that $s \times 1 \in [L_2]$. Thus, taking $d = 1$, we have our solution.

Induction Hypothesis: We assume the theorem is true for $c-1$ colors. In particular, for any M', $L_2(b_1, \ldots, b_n; c-1, M', s)$ exists. This proof will be similar to the proof of Lemma **??**.

Induction Step: We want to show that $L_2(b_1, \ldots, b_n; c, M, s)$ exists. We show that there is M' so that, if you c-color $[L]$ (where $L = L(b_1, \ldots, b_n; c, M')$ from part 1), then there exists the required $e_1, \ldots, e_n, d$. The M' will depend on L_2 for $c-1$ colors. Let χ be a c-coloring of $[L]$. By part 1 there exists $E_1, \ldots, E_n, D$ such that $\sum_{i=1}^{n} b_i E_i = 0$ and the following are all the same color, which we will call RED.

$$\begin{array}{ccccccc}
E_1 - M'D, & \ldots, & E_1 - D, & E_1, & E_1 + D, & \ldots, & E_1 + M'D \\
E_2 - M'D, & \ldots, & E_2 - D, & E_2, & E_2 + D, & \ldots, & E_2 + M'D \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
E_n - M'D, & \ldots, & E_n - D, & E_n, & E_n + D, & \ldots, & E_n + M'D.
\end{array}$$

There are now several cases.

Case 1: If sD is RED then we are done so long as $M' \geq M$. Use $d = D$.

Case 2: If $2sD$ is RED then we are done so long as $M' \geq 2M$. Use $d = 2D$.

$\vdots$

Case X: If XsD is RED then so long as $M' \geq MX$ we are done. Use $d = XD$.

Case X+1: None of the above cases hold. Hence

$$sD, 2sD, \ldots, XsD$$

are all *not* RED. Hence the coloring restricted to this set is a $c-1$ coloring. Let $X = L_2(b_1, \ldots, b_n; c-1, M, s)$, and $M' = MX$. Consider the $(c-1)$-coloring χ^* of $[M']$ defined by

$$\chi^*(x) = \chi(xsD).$$

By the induction hypothesis and the definition of M' there exists $e_1, \ldots, e_n, d$ such that $\sum_{i=1}^{n} b_i e_i = 0$ and all of the following are the same color under χ^*:

$$\begin{array}{ccccccc}
e_1 - Md, & e_1 - (M-1)d, & \ldots, & e_1, & \ldots, & e_1 + Md \\
e_2 - Md, & e_2 - (M-1)d, & \ldots, & e_2, & \ldots, & e_2 + Md \\
\vdots & \vdots & & \vdots & & \vdots \\
e_n - Md, & e_n - (M-1)d, & \ldots, & e_n, & \ldots, & e_n + Md
\end{array}$$

$$sd.$$

By the definition of χ^*, the following have the same color via χ:

$$\begin{array}{ccccccc}
(e_1 - Md)sD, & (e_1 - (M-1)d)sD, & \ldots, & e_1 sD, & \ldots, & (e_1 + Md)sD \\
(e_2 - Md)sD, & (e_2 - (M-1)d)sD, & \ldots, & e_2 sD, & \ldots, & (e_2 + Md)sD \\
\vdots & \vdots & & \vdots & & \vdots \\
(e_n - Md)sD, & (e_n - (M-1)d)sD, & \ldots, & e_n sD, & \ldots, & (e_n + Md)sD
\end{array}$$

$$sdsD.$$

By taking the vector $(e_1 sD, \ldots, e_n sD)$ and common difference sdD, we obtain the result.

(Part 3)
In both of the above parts, the only property of the set

$$\left\{ (x_1, \ldots, x_n) \;\middle|\; \sum_{i=1}^{n} b_i x_i = 0 \right\}$$

that we used is that it was homogeneous and regular. Hence all of the proofs go through without any change and we obtain this part of the lemma. ∎

Back to our Story

Theorem 0.0.5 *If $(b_1, \ldots, b_n)$ is regular and there exists $(\lambda_1, \ldots, \lambda_n)$ such that $\sum_{i=1}^{n} \lambda_i b_i = 0$ and all of the λ_i are distinct, then $(b_1, \ldots, b_n)$ is distinct-regular.*

Proof: Let M be a parameter to be picked later. Let $L = L(b_1, \ldots, b_n; c, M)$ from part 1 of Lemma 0.0.4. Let χ be a c-coloring of $[L]$. We know that there exists $e_1, \ldots, e_n, d \in [L]$ such that the following occur.

1. $b_1 e_1 + \cdots + b_n e_n = 0$.

2. The following are the same color:

$$\begin{array}{ccccccc}
e_1 - Md, & \ldots, & e_1 - d, & e_1, & e_1 + d, & \ldots, & e_1 + Md \\
e_2 - Md, & \ldots, & e_2 - d, & e_2, & e_2 + d, & \ldots, & e_2 + Md \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
e_n - Md, & \ldots, & e_n - d, & e_n, & e_n + d, & \ldots, & e_n + Md.
\end{array}$$

Let $A \in \mathbb{Z}$ be a constant to be picked later. Note that

$$\sum_{i=1}^{n} b_i(e_i + Ad\lambda_i) = \left(\sum_{i=1}^{n} b_i e_i\right) + \left(Ad\sum_{i=1}^{n} b_i\lambda_i\right) = 0.$$

Thus $(e_1+Ad\lambda_1, \ldots, e_n+Ad\lambda_n)$ is a solution. For it to be monochromatic, we need M to be such that there exists an A with

1. $e_1 + Ad\lambda_1$, ..., $e_n + Ad\lambda_n$ are all distinct, and
2. For all i, $|A\lambda_i| \leq M$.

Since $\lambda_i \neq \lambda_j$, there is at most 1 value of A which makes $e_i + Ad\lambda_i = e_j + Ad\lambda_j$ — viewing this condition as a linear equation in A. Therefore, there are at most $\binom{n}{2}$ values of A which make item 1 false.

In order to satisfy item 2 we need, for all i, $|A| \leq M/|\lambda_i|$. Let $\lambda = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$. We let $M = \binom{n}{2}\lambda$. Any choice of A with $|A| \leq \binom{n}{2}$ will satisfy condition 2. There are more than $\binom{n}{2}$ values of A that satisfy this, hence we can find a value of A one that satisfies items 1 and 2. ▮

Exercise 1 (Open-ended)

a) Consider the equation $10x_1 + 13x_2 - 40x_3 = 0$. By Theorem **??** there is a 40-coloring of $\mathbb{N}$ such that there is no monochromatic solution. Exercise **??** gives a 6-coloring with the same property, but we do not know whether it is best. Find the value of c such that

   - There is a c-coloring of $\mathbb{N}$ such that $10x_1 + 13x_2 - 40x_3 = 0$ has no monochromatic solution.
   - For every $c-1$-coloring of $\mathbb{N}$ there is a monochromatic solution to $10x_1 + 13x_2 - 40x_3 = 0$.

b) We define $(b_1, \ldots, b_n)$ be be *c-regular* if, for every c-coloring of $\mathbb{N}$, there is a monochromatic solution to $\sum_{i=1}^{n} b_i x_i = 0$. Find some condition X such that, for all $(b_1, \ldots, b_n)$ and c, $(b_1, \ldots, b_n)$ is c-regular iff X.

c) Define c-distinct-regular in the analogous way. Repeat the problem above with that notion of c-distinct regular.