Computability Theory and Ramsey Theory

An Exposition by William Gasarch

All of the results in this document are due to Jockusch [**?**]. For more results in computable combinatorics see the survey by Gasarch [**?**].

1 A Crash Course in Computability Theory

Notation 1.1

1. $M_1, M_2, \ldots$ is a standard list of Turing Machines (TMs). You can think of them as all Java programs.

2. We assume that from e we can extract the code for M_e.

3. $M_{e,s}(x)$ means that we run M_e for s steps.

4. $M(x) \downarrow = a$ means that $M(x)$ halts and outputs a.

5. $M(x) = a$ means that $M(x)$ halts and outputs a (we use the $\downarrow$ when we want to emphasize that $M(x)$ halts).

6. $M(x) \uparrow$ means that $M(x)$ does not halt.

7. A set A is *computable* if there is a TM M such that

$$x \in A \implies M(x) \downarrow = 1$$
$$x \notin A \implies M(x) \downarrow = 0$$

   (Older books use the term *recursive* instead of *computable*.)

8. If M is a TM such that on every input x, $M(x) \downarrow \in \{0, 1\}$ (so M computes some set) then $L(M) = \{x \colon M(x) = 1\}$ (so $L(M)$ is the set that M computes).

9. A set A is *computably enumerable (c.e.)* if there is a TM M such that

$$x \in A \implies M(x)\downarrow$$
$$x \notin A \implies M(x)\uparrow$$

   (Older books use the term *recursively enumerable (r.e.)* instead of *computably enumerable (c.e.)*.)

10. W_e is the domain of M_e, that is, $W_e = \{x\colon (\exists s)[M_{e,s}(x)\downarrow]$.

11. $W_{e,s} = \{x\colon M_{e,s}(x)\downarrow\}$.

12. A function$f$ is *computable* if there is a TM M such that, for all x, $M(x)\downarrow = f(x)$. (Older books use the term *recursive* instead of *computable*.)

Sets are classified in the Arithmetic hierarchy.

Notation 1.2

1. $A \in \Sigma_0$ if A is computable.

2. $A \in \Pi_0$ if A is computable.

3. $A \in \Sigma_1$ is there exists $B \in \Pi_0$ such that $A = \{x\colon (\exists y)[(x,y) \in B]\}$.

4. $A \in \Pi_1$ is there exists $B \in \Sigma_0$ such that $A = \{x\colon (\forall y)[(x,y) \in B]\}$.

5. Alternative definition: $A \in \Pi_1$ if $\overline{A} \in \Sigma_1$.

6. For $i \ge 1$ $A \in \Sigma_i$ is there exists $B \in \Pi_{i-1}$ such that $A = \{x\colon (\exists y)[(x,y) \in B]\}$

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Examples and Facts

1. $\text{HALT} = \{(e, x) \colon (\exists s)[M_{e,s}(x) \downarrow\} \in \Sigma_1 - \Sigma_0$

2. $W_0, W_1, \ldots$ is a list of all Σ_1 sets.

3. FIN is the set of all e such that W_e is finite.

$$\text{FIN} = \{e \colon (\exists x)(\forall y, s)[y > x \implies y \notin W_{e,s}\} \in \Sigma_2 - \Pi_2.$$

   (The proof that $\text{FIN} \notin \Pi_2$ is not easy.)

4. INF is the set of all e such that W_e is infinite. $\text{INF} \in \Pi_2 - \Sigma_2$. (The proof that $\text{INF} \notin \Sigma_2$ is not easy.)

5. COF is the set of all e such that W_e is co-finite. We leave it to you to show that $\text{COF} \in \Sigma_3$. (The proof that $\text{COF} \notin \Pi_3$ is not easy.)

6. $\Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \subset \cdots$.

7. $\Pi_0 \subset \Pi_1 \subset \Pi_2 \subset \cdots$.

8. For all $i \geq 1$, Σ_i and Π_i are incomparable.

2 A Computable Coloring With No Infinite Σ_1 Homog Set

Theorem 2.1 *There exists a computable* $\text{COL} \colon \binom{\mathsf{N}}{2} \to [2]$ *such that there is no infinite* Σ_1 *homog set.*

Proof: We use that $W_0, W_1, \ldots$ is a list of all Σ_1 sets.

We construct computable $\text{COL} \colon \binom{\mathsf{N}}{2} \to [2]$ to satisfy the following requirements (NOTE- *requirements* is the most important word in computability theory.)

$$R_e : W_e \text{ infinite } \implies W_e \text{ NOT a homog set }.$$

How can we achieve this? If $x, y, s \in W_e$ and $\text{COL}(x, s) \neq \text{COL}(y, x)$ then W_e is not a homog set. Note that if a set is homog then *every* pair is the same color, so just having *two pairs* be different colors is enough to make the set not homog.

We restate the requirement.

$$R_e : W_e \text{ infinite } \implies (\exists x, y, s \in W_e)[\text{COL}(x, s) \neq \text{COL}(y, s)].$$

Requirement will either be *activated* or *not activated*. Here are some notes about that before doing the formal construction.

1. If R_e is activated then we will associate a set $D_e \subseteq W_e$ with $|D_e| = 2^{e+1}$ to it. When a requirement is in this state we will be working on satisfying it. We won't actually know when it is satisfied; however, we will later prove that it was.

2. If R_e is not activated then we are probably waiting 2^{e+1} elements of W_e to appear. If this never happens then W_e is finite so R_e is satisfied. If it does happen then R_e is activated.

3. You might think that we make all of the D_e's disjoint. Alas, this is not possible. But note the following:

   $|D_0| = 2$

   $|D_1| = 4$. Hence $|D_1 - D_0| \geq 2$.

   $|D_2| = 8$. Hence $|D_2 - (D_1 \cup D_0)| \geq 2$.

   and more generally:

   $|D_e| = 2^{e+1}$. Hence $|D_e - \cup_{i=1}^{e-1} D_i| \geq 2^{e+1} - \sum_{i=0}^{e-1} 2^{i+1} = 2$.

CONSTRUCTION OF COLORING

Stage 0: COL is not defined on anything. For all e, R_e is not activated.

Stage s: We will define $\mathrm{COL}(0,s),\ldots,\mathrm{COL}(s-1,s)$. We will may also activate some requirement and make some progress on requirements that are already activated.

For $e = 0, 1, \ldots, s$:

1. If R_e is not activated then check if there exists $D_e \subseteq W_{e,s} \cap \{0,\ldots,s\}$ such that $|D_e| = 2^{e+1}$. If YES then activated R_e and associate D_e to it.

2. If R_e is activated then let $x, y \in D_e$ be the least numbers that are not in $D_0 \cup \cdots \cup D_{e-1}$. Hence $\mathrm{COL}(x,s)$ and $\mathrm{COL}(y,s)$ have not yet been satisfied. Assume $x < y$. Let:

   - $\mathrm{COL}(x,s) =$ RED
   - $\mathrm{COL}(y,s) =$ BLUE .

After you to through all all of the $0 \le e \le s$, define all other $\mathrm{COL}(x,s)$ where $0 \le x \le s-1$ that have not been defined by $\mathrm{COL}(x,y) =$ RED . This is arbitrary. The important things is that ALL $\mathrm{COL}(x,s)$ where $0 \le x \le s-1$ are now defined. This is why COL is computable— at stage s we have defined all $\mathrm{COL}(x,y)$ with $0 \le x < y \le s$.

END OF CONSTRUCTION

We show that each requirement is eventually satisfied.

For pedagogue we first look at R_0.

If W_0 is finite then R_0 is satisfied.

Assume W_0 is infinite. We show that R_0 is satisfied. Let $x < y$ be the first two elements that show up in W_0. Let s_0 be the least number such that $x, y \in W_{0,s_0}$. At state s_0, R_0 will be activated with $D_0 = \{x, y\}$. Note that, for ALL $s \ge s_0 - 1$:

$\mathrm{COL}(x,s) =$ RED

$\mathrm{COL}(y,s) =$ BLUE

Since W_0 is infinite there is SOME $s \geq s_0 + 1$ with $s \in W_e$. Hence $x, y, s \in W_0$ and show that W_0 is NOT homogenous.

Can we show R_1 is satisfied the same way? Yes but with a caveat- we won't use the first two elements that show up on W_1. We'll use the first two elements that show up on W_1 that are not in D_0. But there is a further caveat which we illustrate with an example.

1. At Stage 100 R_1 is activated with $D_1 = \{10, 11, 19, 22\}$. R_0 has still not been activated.
2. For $101 \leq s \leq 999$ R_0 has still not been activated. Hence when R_1 is processed we get:
   (a) $\mathrm{COL}(10, s) =$ RED
   (b) $\mathrm{COL}(11, s) =$ BLUE
3. Stage 1000: R_0 gets activated with $D_0 = \{11, 111, 299, 788\}$?
4. Let $s \geq 1000$.

   When R_0 is processed we get:
   (a) $\mathrm{COL}(11, s) =$ RED
   (b) $\mathrm{COL}(111, s) =$ BLUE .

   When R_1 is processed we get:
   (a) $\mathrm{COL}(10, s) =$ RED
   (b) $\mathrm{COL}(19, s) =$ BLUE .

Lets just look at R_1. If W_1 is infinite then there exists an $s \geq 100$ such that $s \in W_1$. If $100 \leq s \leq 999$ then we R_1 is satisfied by using $\{10, 11, s\}$. If $s \geq 1000$ then we R_1 is satisfied by using $\{10, 19, s\}$. Note that all that matters is that once R_1 is activated, R_1 will be satisfied and it does not matter what R_0 is doing.

We now prove that the all requirements are satisfied.

Claim: Let $e \in \mathsf{N}$. Then W_e is satisfied.

Proof of Claim:

If W_e is finite then R_e is satisfied. So we assume W_e is infinite. Let s_e be the least number such that

- $e \le s_e$.
- $|W_{e,s_e} \cap \{0, \ldots, s_e\}| \ge 2^{e+1}$.

Then R_e will be activated at stage s_e and D_e will be created.

For every stage $s \ge s_e$, when R_e is processed there will be $x < y \in D_e$ such that

- $\mathrm{COL}(x, s) =$ RED
- $\mathrm{COL}(y, s) =$ BLUE .

We know that $x, y \in W_e$ but we know nothing about s. However, W_e is infinite. Let s be the least element of

$$\{s_e, s_e + 1, s_e + 2, \ldots\}$$

that is in W_e. At stage s we will set $\mathrm{COL}(x, s) =$ RED and $\mathrm{COL}(y, s) =$ BLUE . Since $x, y, s \in W_e$, requirement R_e is satisfied.

End of Proof of Claim

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3 Every Computable Coloring has an Infinite Σ_3 Homog set

Take the standard proof of the infinite 2-ary Ramsey Theorem. Let COL be the given coloring of $\binom{\mathsf{N}}{2}$. Assume COL is computable.

The function COL' from N to $\{R, B\}$ can be computed by asking Π_2 questions. Hence we say informally $\mathrm{COL}' \le_{\mathrm{T}} \Pi_2$. One can show that using this all three sets: R, B, and $DEAD$ are Σ_3.

We now have a subtle point. If all we want to know is the complexity of a homog set we can say that ONE OF R or B is infinite, hence there IS a Σ_3-homog set. And this is the answer we will give. But notice that we do not know which of R or B is the homog set. That would require a Σ_4-question.

Can we do better? YES! See the next section.

4 Every Computable Coloring has an Infinite Π_2 Homog set

We obtain this with a modification of the usual proof of Ramsey's theorem. the key is that we don't really toss things out- we guess on what the colors are and change our mind.

Theorem 4.1 *For every computable coloring* $\mathrm{COL}\colon \binom{\mathsf{N}}{2} \to [2]$ *there is an infinite* Π_2 *homog set.*

Proof:

We are given computable $\mathrm{COL}\colon \binom{\mathsf{N}}{2} \to [2]$.

CONSTRUCTION of $x_1, x_2, \ldots$ and $c_1, c_2, \ldots$.

NOTE: at the end of stage s we might have $x_1, \ldots, x_i$ defined where $i < s$. We will not try to keep track of how big i is. Also, we may have at stage (say) 1000 a sequence of length 50, and then at stage 1001 have a sequence of length only 25. The sequence will grow eventually but do so in fits and starts.

$$x_1 = 1$$

$$c_1 = \text{ RED We are guessing. We might change our mind later}$$

Let $s \ge 2$, and assume that $x_1, \ldots, x_{s-1}$ and $c_1, \ldots, c_{s-1}$ are defined.

1. Ask HALT

Does there exists $x \geq x_{s-1}$ *such that, for all* $1 \leq i \leq s-1$, $\mathrm{COL}(x_i, x) = c_i$*?*

2. If YES then (using that COL is computable) find the least such x.

$$x_i = x$$

$$c_i = \text{RED} \quad \text{We are guessing. We might change our mind later}$$

We have implicitly tossed out all of the numbers between x_{i-1} and x_i.

3. If NO then we ask HALT how far back we can go. More rigorously we ask the following sequence of questions until we get a YES.

   - *Does there exists* $x \geq x_{s-1}$ *such that, for all* $1 \leq i \leq s-2$, $\mathrm{COL}(x_i, x) = c_i$*)?*
   - *Does there exists* $x \geq x_{s-1}$ *such that, for all* $1 \leq i \leq s-3$, $\mathrm{COL}(x_i, x) = c_i$*)?*
   - $\vdots$
   - *Does there exists* $x \geq x_{s-1}$ *such that, for all* $1 \leq i \leq 2$, $\mathrm{COL}(x_i, x) = c_i$*)?*
   - *Does there exists* $x \geq x_{s-1}$ *such that, for all* $1 \leq i \leq 1$, $\mathrm{COL}(x_i, x) = c_i$*)?*

   (One of these must be a YES since (1) if $c_1 = RED$ and there are NO red edges coming out of x_1 then there must be an infinite number of BLUE edges, and (2) if c_1=BLUE its because there are only a finite number of RED edges coming out of x_1 so there are an infinite number of BLUE edges. Let i_0 be such that *There exists* $x \geq x_{s-1}$ *such that, for all* $1 \leq i \leq i_0$, $\mathrm{COL}(x_i, x) = c_i$) Do the following:

   (a) Change the color of c_{i+1}. (We will later see that this change must have been from RED to BLUE .

   (b) Wipe out $x_{i+2}, \ldots, x_{s-1}$.

   (c) Search for the $x \geq x_{s-1}$ that the question asked says exist.

(d) x_{i+2} is now x.

(e) c_{i+2} is now RED .

END OF CONSTRUCTION of $x_1, x_2 \ldots$ and $c_1, c_2, \ldots$.

We need to show that there is a Π_2 homog set.

Let X be the set of x_i that are put on the board and stay on the board.

Let R be the set of $x_i \in X$ whose final color is RED .

Claim 1: Once a number turns from RED to BLUE it can't go back to RED again.

Proof:

If a number is turned BLUE its because there are only a finite number of RED edges coming out of it. Hence there must be an infinite number of BLUE edges coming out of it. Hence it will never change color (though it may be tossed out).

End of Proof

Claim 1: $X, R \in \Pi_2$.

Proof:

We show that $\overline{X} \in \Sigma_2$. In order to NOT be in X you must have, at some point in the construction, been tossed out.

$$\overline{X} = \{x : (\exists x)[\text{ at stage } s \text{ of the construction } x \text{ was tossed out }]\}.$$

Note that the condition is computable-in-HALT. Hence $\overline{X}$ is c.e.-in-HALT. It is known that if a set is c.e.-in-HALT then it is in Σ_2. Hence $\overline{X} \in \Sigma_2$.

We show that $\overline{R} \in \Sigma_2$. In order to NOT be in R you must have to either NOT be in X or have been turned blue. Note that once you turn at some point in the construction, been tossed out.

$$\overline{R} = \overline{X} \cup \{x : (\exists x)[\text{ at stage } s \text{ of the construction } x \text{ was turned BLUE}]\}.$$

Note that the condition is computable-in-HALT. Hence $\overline{R}$ is c.e.-in-HALT. so $\overline{R} \in \Sigma_2$.

End of Proof

We have shown X, R are Π_2 but have not shown that B is- and in fact B might not be. But we show that B is Π_2 when we need it to be.

There are two cases:

1. If R is infinite then R is an infinite homog set that is Π_2.
2. If R is finite then B is X minus a finite number of elements. Since X is Π_2, B is Π_2.

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