Canonical Ramsey's Theorem (finite version)
Exposition by William Gasarch

1 Canonical Ramsey's Theorem for the Finite Complete Graphs

Notation 1.1 Let $n, a \in \mathsf{N}$ with $a < n$.

1. $[n]$ is the set $\{1, \ldots, n\}$.
2. If A is a set then $\binom{A}{a}$ is the set of all a-sized subsets of A.
3. K_n is the graph (V, E) where

$$V = [n]$$
$$E = \binom{[n]}{2}$$

   We will identify $\binom{[n]}{2}$ with K_n. We may refer to vertices and edges of $\binom{[n]}{2}$.

Def 1.2 Let COL be a coloring of $\binom{[n]}{2}$ (the edges of K_n). Let $V \subseteq [n]$. The set V is *homogenous* (henceforth *homog*) if there exists a color c such that every edge in $\binom{V}{2}$ is colored c.

The following is the standard Ramsey's theorem on graphs.

Theorem 1.3 *For all $k \in \mathsf{N}$, for all $c \in \mathsf{N}$, there exists n such that for all c-coloring of $\binom{[n]}{2}$ (the edges of K_n) there is a homog set of size k. (We can take $n = 2^{2k}$.)*

Note that Ramsey's Theorem uses only a fixed number of colors. What if we color $\binom{[n]}{2}$ with as many colors as we like? You may say *that's just stupid— color each edge a different color.* That is true— however, your coloring has a *rainbow set of size n*- every edge is different! This leads to the following conjecture:

Def 1.4 Let COL be a coloring of $\binom{[n]}{2}$. Let $V \subseteq [n]$. The set V is *rainbow* if every edge in $\binom{V}{2}$ is colored differently.

Conjecture: For all k there exists n such that for every coloring of $\binom{[n]}{2}$ there is either a homog set of size k or a rainbow set of size k.

This looks good. But alas its not true. Consider the following colorings:

$$COL(i,j) = i.$$

$$COL(i,j) = j.$$

We leave it to the reader to show that neither of these colorings has a homog set, nor a rainbow set. However both lead to a certain kind of homogeneity.

Def 1.5 Let COL be a coloring of $\binom{[n]}{2}$. Let $V \subseteq [n]$. The set V is *min-homogenous* (henceforth *min-homog*) if for all $a < b$ and $c < d$

$$COL(a,b) = COL(c,d) \text{ iff } a = c.$$

The set V is *max-homogenous* (henceforth *max-homog*) if for all $a < b$ and $c < d$

$$COL(a,b) = COL(c,d) \text{ iff } b = d.$$

We now state the Canonical Ramsey Theorem for finite graphs.

Theorem 1.6 *For all k there exists n such that, for all colorings of $\binom{[n]}{2}$ there is either a homog set of size k, a min-homog set of size k, a max-homog set of size k, or a rainbow set of size k. We denote the least value of n that works by $ER(k)$.*

We will give five proofs of Theorem 1.6 which give different bounds for $ER(k)$. Some of the proofs that give non-optimal bounds will have other nice features. We compare and contrast the proofs in the last section.

To even state the bounds we need to state the hypergraph Ramsey Theorem. and have a notation for hypergraph Ramsey numbers.

Def 1.7 Let COL be a coloring of $\binom{[n]}{a}$ (the edges of the complete a-hypergraph on n vertices). Let $V \subseteq [n]$. The set V is *homog* if there exists a color c such that every edge in $\binom{V}{a}$ is colored c.

Theorem 1.8 *For all a, for all c, for all k there exists n such that for all c-colorings of* $\binom{[n]}{a}$ *there exists a homog set of size k. We denote the least value of n by* $R_a(k;c)$.

Note 1.9 It is known that $R_2(k;c)$ is upper bounded by roughly $c^{O(ck)}$, $R_3(k;c)$ is upper bounded by roughly $c^{c^{O(ck)}}$ etc. The lower bound on $R_2(k;c)$ is roughly the same as the upper bound. For all $a \geq 3$ the upper bound on $R_a(k;c)$ is roughly one exponential lower than the upper bound.

In Section 2 we prove lemmas that we will need. In Section 3 we give a slight variants on the classical proof of the Canonical Ramsey Theorem due to Erdős and Rado [3]. Our version of the proof yields

$$ER(k) \leq R_4(k^2; 12).$$

In Section 4 we give a proof that uses the 3-hypergraph Ramsey Theorem. We believe this proof is new. It yields

$$ER(k) \leq R_3(k^3; 4).$$

BILL- CHECK THIS AGAINST RADO PROOF ALSO LEFMANN- DO WE USE HIS IDEAS AND DO WE CARE IF WE DO

In Section 5 we give a proof that uses the 2-hypergraph Ramsey Theorem (Ramsey Theorem on graphs). This proof, in the infinite version, is due to Mileti [5]. Our treatment is probably the first time its been written down for the finite case. In Section 6 we give a proof that does not use Ramsey's Theorem. It uses a strong version of the one-dimensional Canonical Ramsey Theorem. It yields

$$ER(k) \leq XXX.$$

In Section 7 we present a proof the yields the best upper bounds known, due to Lefmann and Rodl [4]. Our proof, which is a very slight variant of theirs, yields the bound

$$ER(k) \leq \left(\frac{9k^6}{16}\right)^{2(k-1)^2+1} \leq 2^{12k^2 \log k}.$$

2 Needed Lemmas

2.1 One Dimensional Canonical Ramsey Theorem

We need the following lemma which could be called the 1-dimensional Canonical Ramsey Theorem. We leave the proof to the reader.

Def 2.1 If COL is a coloring of $[m]$ then (1) a *homog subset of* $[m]$ *relative to* COL is a set that is all the same color, and (2) a *rainbow subset of* $[m]$ *relative to* COL is a set where every element has a different color.

Lemma 2.2 *Let* COL *be any coloring of* $[(m-1)^2+1]$*. Then there exists either a homog set of size* m *or a rainbow set of size* m*.*

2.2 Premises that Yield Homog or Min-Homog or Max-Homog

Lemma 2.3 *Let* COL *be a coloring of* $\binom{m}{2}$*.*

1. *Let*

$$X = \{x_1 < x_2 < \cdots < x_L\}$$

   such that the following occurs:

$$(\forall a < b < c)[COL(x_a, x_b) = COL(x_a, x_c)].$$

   Then either there exists a homog set of size at least $\sqrt{L}$ *or there exists a min-homog set of size at least* $\sqrt{L}$*.*

2. *Let*

$$X = \{x_1 < x_2 < \cdots < x_L\}$$

   such that the following occurs:

$$(\forall a < b < c)[COL(x_a, x_c) = COL(x_b, x_c)].$$

   Then either there exists a homog set of size at least $\sqrt{L}$ *or there exists a max-homog set of size at least* $\sqrt{L}$*.*

3. *Let*

$$X = \{x_1 < x_2 < \cdots < x_L\}$$

such that the following occurs:

$$(\forall a < b < c)[COL(x_a, x_b) = COL(x_b, x_c)].$$

Then there exists a homog set of size at least $L - 1$.

4. *Let*

$$X = \{x_1 < x_2 < \cdots < x_L\}$$

such that the following occurs:

$$(\forall a < b < c < d)[COL(x_a, x_b) = COL(x_c, x_d)].$$

Then there exists a homog set of size at least $L - 2$.

5. *Let*

$$X = \{x_1 < x_2 < \cdots < x_L\}$$

such that the following occurs:

$$(\forall a < b < c < d)[COL(x_a, x_c) = COL(x_b, x_d)].$$

Then there exists a homog set of size at least $\frac{L}{2} - 1$.

Proof:

We assume $\sqrt{L}$ is an integer.

1) Let $COL'(x_a) = COL(x_a, x_{a+1})$. Note that, for all $b > a$, $COL'(x_a) = COL(x_a, x_b)$. Apply Lemma 2.2 to COL' to obtain either a homog (relative to COL') set of size $\sqrt{L}$, which is a homog set relative to COL, or a rainbow set (relative to COL') which is a min-homog set relative to COL.

2) This part is similar to part 1 so we omit it.

3) Note that $COL(x_1, x_2) = COL(x_2, x_3) = \cdots = COL(x_{L-1}, x_L)$. We call this color RED. Let

$$H = \{x_1, x_2, x_3, \ldots, x_{L-1}\}.$$

Since for $1 \le a < b \le L - 1$, $COL(x_a, x_b) = COL(x_b, x_{b+1}) = RED$, H is homog.

4) Note that, for $3 \leq a < b$, $COL(x_1, x_2) = COL(x_a, x_b)$. We call this color RED. Let

$$H = \{x_3, \ldots, x_L\}.$$

Since for all $3 \leq a < b$, $COL(x_a, x_b) = COL(x_1, x_2) = RED$, H is homog.
5) Note that, for $1 \leq a < b \leq L$,

$$COL(x_a, x_{a+2}) = COL(x_{a+1}, x_{a+3}) = COL(x_{a+2}, x_{a+4}).$$

Assume L is even. Let

$$H' = \{x_2, x_4, x_6, \ldots, x_L\}.$$

This set H is of size $\frac{L}{2}$ and satisfies the premise of part 3. Hence there is a homog set of size $\frac{L}{2} - 1$. ∎

2.3 A Premise that Yields a Rainbow Set

The next definition and lemmas gives a way to get a rainbow set under some conditions.

Def 2.4 Let COL be a coloring of $\binom{[m]}{2}$. If c is a color and $v \in [m]$ then $\deg_c(v)$ is the number of c-colored edges with an endpoint in v.

The following theorem is due to Babai [2]. We include the proof since the paper is not available on-line and will eventually be lost to history.

Lemma 2.5 *Let $m \geq 3$. Let COL be a coloring of $\binom{[m]}{2}$. If for all $v \in [m]$ and all colors c $\deg_c(v) \leq 1$ then there exists a rainbow set of size $\geq (2m)^{1/3}$.*

Proof:
Let X be a maximal rainbow set. This means that,

$$(\forall y \in [m] - X)[X \cup \{y\} \text{ is not a rainbow set}].$$

Let $y \in [m] - X$. Why is $y \notin X$? One of the following must occur:

1. There exists $u, u_1, u_2 \in X$ such that $u_1 \neq u_2$ and $COL(y, u) = COL(u_1, u_2)$. (It is possible for $u = u_1$ or $u = u_2$.)

2. There exists $u_1 \neq u_2 \in X$ such that $COL(y,u_1) = COL(y,u_2)$. This cannot happen since then y has some color degree ≥ 2.

We map $[m]-X$ to $X \times \binom{X}{2}$ by mapping $y \in [m]-X$ to $(u,\{u_1,u_2\})$ as indicated in item 1 above. This map is injective since if y_1 and y_2 both map to $(u,\{u_1,u_2\})$ then $COL(y_1,u) = COL(y_2,u)$.

This map has domain of size $n-|X|$ and co-domain of size $|X|\binom{|X|}{2}$. Hence

$$m-|X| \leq |X|\binom{|X|}{2} = |X|^2(|X|-1)/2 = \frac{|X|^3-|X|^2}{2} \leq \frac{|X^3}{2} - |X|$$

$$m \leq \frac{|X|^3}{2}.$$

$$|X| \geq (2m)^{1/3}.$$

∎

Alon, Lefmann, and Rodl [1] have obtained a slight improvement and also showed that it cannot be improved past that.

Lemma 2.6 *Let $m \geq 3$.*

1. *Let COL be a coloring of $\binom{[m]}{2}$. If for all $v \in [m]$ and all colors c, $\deg_c(v) \leq 2$ then there exists a rainbow set of size $\geq \Omega((m\log m)^{1/3})$.*

2. *There exists a coloring of $\binom{[m]}{2}$. such that for all $v \in [m]$ and all colors c, $\deg_c(v) \leq 1$ and all rainbow sets are of size $\leq O((m\log m)^{1/3})$.*

3 A Proof that Uses the 4-Hypergraph Ramsey Theorem

We give a slight variant of the original proof of the Canonical Ramsey Theorem due to Erdős -Rado [3].

Theorem 3.1 $ER(k) \leq R_4(k^2, 12)$.

Proof:
Let $n = R_c^4(m)$ where we determine c and m later.

We are given $COL\colon \binom{[n]}{2} \rightarrow \mathsf{N}$. We use COL to obtain a FINITE coloring of $\binom{[n]}{4}$ which we denote COL'.

What is $COL'(x_1 < x_2 < x_3 < x_4)$. We look at COL on all $\binom{4}{2}$ pairs of $\{x_1, x_2, x_3, x_4\}$ and see how they compare to each other.

For each case we assume the negation of all the prior cases. Also, in each case, we indicate what happens if this is the color of the homog set of size m.

1. If $COL(x_1, x_2) = COL(x_1, x_3)$ then color (x_1, x_2, x_3, x_4) with $(12, 13)$. If this is the color of the homog set for COL' then, by Lemma 2.3.1, there is either a homog set or a min-homog set of size $\sqrt{m}$ for COL.

2. If $COL(x_1, x_2) = COL(x_1, x_4)$ then color (x_1, x_2, x_3, x_4) with $(12, 14)$. If this is the color of the homog set for COL' then,

3. If $COL(x_1, x_3) = COL(x_1, x_4)$ then color (x_1, x_2, x_3, x_4) with $(13, 14)$. If this is the color of the homog set for COL' then, by Lemma 2.3.1, there is either a homog set or a min-homog set of size $\sqrt{m}$ for COL.

4. If $COL(x_2, x_3) = COL(x_2, x_4)$ then color (x_1, x_2, x_3, x_4) with $(23, 24)$. If this is the color of the homog set for COL' then, by Lemma 2.3.1, there is either a homog set or a min-homog set of size $\sqrt{m}$ for COL.

5. If $COL(x_1, x_3) = COL(x_2, x_3)$ then color (x_1, x_2, x_3, x_4) with $(13, 23)$. If this is the color of the homog set for COL' then, by Lemma 2.3.2, there is either a homog set or a max-homog set of size $\sqrt{m}$ for COL.

6. If $COL(x_1, x_4) = COL(x_2, x_4)$ then color (x_1, x_2, x_3, x_4) with $(14, 24)$. If this is the color of the homog set for COL' then,

7. If $COL(x_1, x_4) = COL(x_3, x_4)$ then color (x_1, x_2, x_3, x_4) with $(14, 24)$. If this is the color of the homog set for COL' then, by Lemma 2.3.2, there is either a homog set or a max-homog set of size $\sqrt{m}$ for COL.

8. If $COL(x_2, x_4) = COL(x_3, x_4)$ then color (x_1, x_2, x_3, x_4) with $(24, 34)$. If this is the color of the homog set for COL' then, by Lemma 2.3.2, there is either a homog set or a max-homog set of size $\sqrt{m}$ for COL.

9. If $COL(x_1, x_2) = COL(x_2, x_3)$ then color (x_1, x_2, x_3, x_4) with $(12, 23)$. If this is the color of the homog set for COL' then, by Lemma 2.3.3, there is a homog set of size $m - 1$ for COL.

10. If $COL(x_1, x_2) = COL(x_2, x_4)$ then color (x_1, x_2, x_3, x_4) with $(12, 24)$. If this is the color of the homog set for COL' then, by Lemma 2.3.3, there is a homog set of size $m - 1$ for COL.

11. If $COL(x_1, x_3) = COL(x_3, x_4)$ then color (x_1, x_2, x_3, x_4) with $(13, 34)$. If this is the color of the homog set for COL' then, by Lemma 2.3.3, there is a homog set of size $m - 1$ for COL.

12. If $COL(x_2, x_3) = COL(x_3, x_4)$ then color (x_1, x_2, x_3, x_4) with $(23, 34)$. If this is the color of the homog set for COL' then, by Lemma 2.3.3, there is a homog set of size $m - 1$ for COL.

13. If $COL(x_1, x_2) = COL(x_3, x_4)$ then color (x_1, x_2, x_3, x_4) with $(12, 34)$. If this is the color of the homog set for COL' then, By Lemma 2.3.4 there is a homog set of size $m - 2$ for COL.

14. If $COL(x_1, x_3) = COL(x_2, x_4)$ then color (x_1, x_2, x_3, x_4) with $(13, 24)$. If this is the color of the homog set for COL' then, By Lemma 2.3.5 there is a homog set of size $\frac{m}{2} - 2$ for COL.

15. If none of the above occur then color (x_1, x_2, x_3, x_4) with $NONE$. If this is the color of the homog set for COL' then this is clearly a rainbow set of size m for COL.

We used 12 colors and obtained a homog set of size at least $\sqrt{k}$. Hence we need to take $m = k^2$. ∎

4 A Proof that Uses the 3-Hypergraph Ramsey Theorem

Theorem 4.1 $ER(k) \le R_3(k^3, 4)$.

Proof:

Let $n = R_c^3(m)$ where we determine c and m later.

We are given $COL\colon \binom{[n]}{2} \rightarrow \mathsf{N}$. We use COL to obtain a FINITE coloring of $\binom{[n]}{3}$ which we denote COL'.

What is $COL'(x_1 < x_2 < x_3)$. We look at COL of all $\binom{3}{2}$ pairs of $\{x_1, x_2, x_3\}$ and how they compare to each other to color $x_1 < x_2 < x_3$.

For each case we assume the negation of all the prior cases. Also, in each case, we indicate what happens if this is the color of the homog set of size m.

1. If $COL(x_1, x_2) = COL(x_1, x_3)$ then color (x_1, x_2, x_3) with $(12, 13)$. If this is the color of the homog set for COL' then, by Lemma 2.3.1, there is either a homog set or a min-homog set of size $\sqrt{m}$ for COL.

2. If $COL(x_1, x_3) = COL(x_2, x_3)$ then color (x_1, x_2, x_3) with $(13, 23)$. If this is the color of the homog set for COL' then, by Lemma 2.3.2, there is either a homog set or a max-homog set of size $\sqrt{m}$ for COL.

3. If $COL(x_1, x_2) = COL(x_2, x_3)$ then color (x_1, x_2, x_3) with $(12, 23)$. If this is the color of the homog set for COL' then, by Lemma 2.3.3, there is a homog set of size $m - 1$ for COL.

4. If none of the above occur then color (x_1, x_2, x_3) with $NONE$. Let H' be the homog set. Note that for all $v \in H'$ and for all colors c, restricted to H', $\deg_c(v) \leq 1$. Hence, by Lemma 2.5, there is a rainbow set of size $(2m)^{1/3}$.

We used 4 colors and obtained a homog set of size at least $(2m)^{1/3}$ Hence we need $m \leq (k/2)^3 = k^3/8 \leq k^3$. We take $m = k^3$. ∎

5 A Proof that Uses the 2-Hypergraph Ramsey Theorem

Theorem 5.1

$$ER(k) \leq 2^{2^{2^{2^{\Omega(k^4)}}}}.$$

Proof: This proof is more in the spirit of the original proof of Ramsey's theorem.

Intuition: In the usual proofs of Ramsey's Theorem we take a vertex v and see which of such that $\deg^R_{RED}(v)$ or $\deg^R_{BLUE}$ is "large." One of them must be at least half of the size of the vertices still in play. Here we change this up:

- If there is a color c such that $\deg_c(x)$ is "large" we will indeed delete all nodes $y > x$ with $COL(x, y) \neq c$, and label x with c.
- If this does not happen we will delete nodes so that, for all $y, z > x$, $COL(x, y) \neq COL(x, z)$ and label x with $*$.
- We will call the labels themselves colorings and we may use Lemma 2.2.
- At the end of the construction we may color edges between vertices that were labeled * (and survived). If we label x and y $(x < y)$ with $*$ then we will make sure that x and y either agree everywhere or nowhere. That is, either $(\forall z > y)[COL(x, z) = COL(y, z)]$ or $(\forall z > y)[COL(x, z) \neq COL(y, z)]$. We will color the edge between x and y either AGREE or DISAGREE. We will then use Ramsey's Theorem.

We will determine n, the initial number of nodes we start with, and m a parameter, later.
CONSTRUCTION
Phase 1:
Stage 0:

1. $N_0 = [n]$. COL' is not defined on any points.

Stage i: If $i = m+1$ then goto Phase Two. Assume that N_i, $x_0 < \cdots < x_{i-1}$, COL' is defined on $x_0, \ldots, x_{i-1}$. Let x_i be the least element in N_i. For $y \in N_i$ let $TCOL(y) = COL(x_i, y)$ ($TCOL$ stands for temporary coloring.) By Lemma 2.2 we have two cases.

1. There is a $\sqrt{|N_i|}$ sized homog set H' relative to $TCOL$. Let $COL(x_i)$ be its color of the homog set and let N_{i+1} be H'.
2. There is a $\sqrt{|N_i|}$ sized rainbow set H' relative to $TCOL$. Let $COL(x_i)$ be $*$ and let N_{i+1} be H'.

Phase 2: We introduce two new parameters: m_1 and m_2. We will need $m_1 + m_2 = m$. There are two cases.

1. There are m_1 vertices labeled with a color in N (so not labeled with $*$). Restrict COL' to those m_1 vertices and apply Lemma 2.2 to obtain either (1) a homog set relative to COL' of size $\sqrt{m_1}$, in which case there is a homog set relative to COL of size $\sqrt{m_1}$, or (2) a rainbow set relative to COL' of size $\sqrt{m_1}$, in which case there is a min-homog set relative to COL of size $\sqrt{m_1}$. We will need to make $\sqrt{m_1} \ge k$. Hence we will need $m_1 \ge k^2$. We will take $m_1 = k^2$.

2. There are at least m_2 vertices labeled with $*$. Call the set of such vertices H_0. (We will be thinning out so we will get $H_1, H_2, \ldots$).

   We will need a new parameter m_3.

   CONSTRUCTION

   Stage 0: H_0 is as above. Note that $|H_0| \ge m_2$.

   Stage i: If $i = m_3 + 1$ then goto Phase 3. Otherwise assume that $z_1, \ldots, z_{i-1}$ and H_i are defined. Let z_i be the least element of H_{i-1}. We thin out H_i without changing its index. For all $a < i$ we use the ordered pair (z_a, z_i) to both thin out H_i and color (z_a, z_i).

   Let

   $$AGREE(z_a, z_i) = \{v \in H_i : COL(z_a, v) = COL(z_i, v)\}.$$

   Case 1: If $|AGREE(z_a, z_i)| \ge \frac{|H_i|}{2}$ then Let H_i be $AGREE(z_a, z_i)$. Let $COL''(z_a, z_i)$ AGREE.

   Case 2: If $|AGREE(z_a, z_i)| \le \frac{|H_i|}{2}$ then Let H_i be $H_i - AGREE(z_a, z_i)$. Let $COL''(z_a, z_i)$ DISAGREE.

In Stage i we divide H_i by 2^i. Hence H_{m_3} is of size $\Omega(\frac{m^2}{2^{m_3^2}})$. In order to carry out this construction we need this quantity to be ≥ 1. Hence we need $m_2 \ge \Omega(2^{m_3^2})$.

Phase 3: We now have $z_1, \ldots, z_{m_3}$ such that (1) for all $1 \le i \le m_3$ all of the edges coming out of z_i to the right are different colors, and (2) for all $1 \le i < j \le m_3$ either z_i and z_j agree on all $z > z_j$ or disagree on all $z > z_j$. We view the agree/disagree distinction as a 2-coloring of $\binom{[m_3]}{2}$. Apply Ramsey's theorem to get a homogenous set H' (relative to the agree-disagree coloring) of size $\Omega(\log m_3)$).

1. If the color is AGREE then it is easy to see that H' is homogenous for COL.

2. If the color is DISAGREE then we have some work to do. Restrict COL to H'.

   Claim: Let $v \in H'$ and c be a color. Then $\deg_c(v) \le 2$.

   Proof of Claim: Assume, by way of contradiction, that $\deg_c(v) \ge 3$. There are two cases:

   - There exists $v_1, v_2 \in H'$ such that $v < v_1, v_2$ and $COL(v, v_1) = COL(v, v_2) = c$. This violates that v was colored $*$.
   - There exists $v_1, v_2 \in H'$ such that $v_1, v_2 < v$ and $COL(v_1, v) = COL(v_2, v) = c$. This violates that v_2 and v_2 disagree.

   End of Proof of Claim

   By Lemma **??** there is a rainbow set $H \subseteq H'$ of size $\Omega(|H'|^{1/4}) = \Omega((\log m_3)^{1/4})$.

END OF CONSTRUCTION

We need to set the parameters. We have already set $m_1 = k^2$. To get

$$\Omega((\log m_3)^{1/4}) \ge k$$

we need to set

$$m_3 \ge \Omega(2^{k^4}).$$

Hence

$$m_2 \ge \Omega(2^{m_3^2}) \ge 2^{2^{\Omega(k^4)}}.$$

Hence

$$m = m_1 + m_2 = k^2 + 2^{2^{\Omega(k^4)}} = 2^{2^{\Omega(k^4)}}.$$

The value of n has to be such that $n \ge 2$ and its possible to take its square root m times. So $n^{1/2^m} \ge 2$. Hence

$$n \ge 2^{2^m} = 2^{2^{2^{2^{\Omega(k^4)}}}}.$$

▮

6 A Proof that Does Not Use Ramsey's Theorem-I

FILL IN LATER

7 A Proof that Does Not use Ramsey's Theorem-II

We remind the reader of the definition of $\deg_c$ and also add the definitions of $\deg_c^L$ and $\deg_c^R$.

Def 7.1 Let COL be a coloring of $\binom{[m]}{2}$. Let c be a color and let $v \in [m]$

1. $\deg_c^R(v)$ is the number of c-colored edges (v, u) with $v < u$.
2. $\deg_c^R(v)$ is the number of c-colored edges (v, u) with $u < v$.
3. A *bad triple* is a triple a, b, c such that a, b, c does not form a rainbow K_3.

The next two lemmas show us how to, in some cases, reduce the number of bad triples.

Lemma 7.2 *Let COL be a coloring of $\binom{[m]}{2}$ such that, for every color c and vertex v, $\deg_c(v) \le d$. Then the number of bad triples is less than $\frac{dm^2}{2}$.*

Proof: Let b be the number of bad triples. We upper bound b by summing over all v that are the point of the triple with two same-colored edges coming out of it.

$$b \le \sum_{v\in[m]} \sum_{c\in\mathsf{N}} \text{Num of bad triples } \{v, u_1, u_2\} \text{ with } COL(v, u_1) = COL(v, u_2) = c\ .$$

(Note that we are not assuming $v < u_1, u_2$.)

We bound the inner summation. Since v is of degree $m - 1$ we can renumber the colors as $1, 2, \ldots, m-1$ (some of the $\deg_c(v)$ may be 0). Hence

$$b \le \sum_{v\in[m]} \sum_{c=1}^{m-1} \binom{\deg_c(v)}{2}.$$

Note that $\sum_{c=1}^{m} \deg_c(v) = m-1 \le m$ and $(\forall c)[\deg_c(v) \le d]$. The inner sum is maximized when $d = \deg_1(v) = \deg_2(v) = \cdots = \deg_{m/d}(v)$ and the rest of the $\deg_c(v)$'s are 0. Hence we have

$$b \le \sum_{v\in[m]} \sum_{c=1}^{m} \binom{\deg_c(v)}{2} \le \sum_{v\in[m]} (m/d)\binom{d}{2} < m\frac{m}{d}\frac{d^2}{2} = \frac{dm^2}{2}.$$

∎

Lemma 7.3 *Let COL be a coloring of* $\binom{[m]}{2}$ *that has* b *bad triples. Let* $1 \le m' \le m$. *There exists an* m'*-sized set of vertices with* $\le b\left(\frac{m'}{m}\right)^3$ *bad triples.*

Proof: Pick a set X of size m' at random. Let E be the expected number of bad triples. Note that

$$E = \sum_{\{v_1,v_2,v_3\}\ \text{bad}} \text{Prob that } \{v_1, v_2, v_3\} \subseteq X\ .$$

Let $\{v_1, v_2, v_3\}$ be a bad triple. The probability that all three nodes are in X is bounded by

$$\frac{\binom{m-3}{m'-3}}{\binom{m}{m'}} = \frac{m'(m'-1)(m'-2)}{m(m-1)(m-2)} \le \left(\frac{m'}{m}\right)^3.$$

Hence the expected number of bad triples is $\le b(\frac{m'}{m})^3$. Therefore there must exist some X that has $\le b(\frac{m'}{m})^3$ bad triples.

∎

Note 7.4 The above theorem presents the user with an interesting tradeoff. She wants a large set with few bad triples. If m' is large then you get a large set, but it will have many bad triples. If m' is small then you won't have many bad triples, but m' is small. We will need a Goldilocks-m' that is just right.

Now we can prove the theorem!

Theorem 7.5 *For all k the following hold.*

1.
$$ER(k) \le \left(\frac{k^9}{16}\right)^{2(k-2)^2+1} \le 2^{18k^2 \lg(k)}.$$

2.
$$ER(k) \le \left(\frac{9k^6}{16}\right)^{2(k-2)^2+1} \le 2^{12k^2 \lg(k)}.$$

Proof:

We will determine n later. We will have parameters m, m', m'', δ, s, t which we will choose later.

Intuition: In the usual proofs of Ramsey's Theorem we take a vertex v and see which of such that $\deg^R_{RED}(v)$ or $\deg^R_{BLUE}$ is large. One of them must be at least half of the size of the vertices still in play. Here we change this up:

- Instead of taking a particular vertex v we ask if there is *any* v and any color c such that either $\deg^L_c(v)$ or $\deg^R_c(v)$ is large. We hope to do this until either we have $(k-2)^2+1$ elements that have a large $\deg^L_c(v)$ for some c, or $(k-2)^2+1$ elements that have a large $\deg^R_c(v)$ for some c. We will then apply Lemma 2.2. (We take care of the extra point we need a different way.) We will need to iterate this process at most $2(k-2)^2+1$ times.

- What is large? Similar to the proof of Ramsey's theorem it will be a fraction of what is left, a fraction δ which we will pick later. Unlike the proof of Ramsey's theorem δ will depend on k.

- In the proof of Ramsey's theorem we were guaranteed that one of $\deg_{RED}(v)$ or $\deg_{BLUE}(v)$ was large. Here we have no such guarantee. We may fail. In that case something else happens and leads to a rainbow set!

Formally the construction will only use the points $\{2, \ldots, n-1\}$ so that we will have available a point bigger than all the points we finally have or smaller than. We ignore this in the construction and the analysis but we will point it out when we need it.

CONSTRUCTION
Phase 1:
Stage 0:

1. $V_0^L = V_0^R = \emptyset$. The set V_0^L will be vertices such that the edges from them to all vertices to their Left are the same color. Similar for V_0^R.

2. $N_0 = [n]$. COL' is not defined on any points.

Stage i: Assume that V_{i-1}^L, V_{i-1}^R, and N_{i-1} are already defined.

1. If there exists $x \in N_{i-1}$ and c a color such that $\deg_c^R(x) \geq \delta N_{i-1}$ then do the following:

$$\begin{aligned} V_i^R &= V_{i-1}^R \cup \{x\} \\ V_i^L &= V_{i-1}^L \\ N_i &= \{v \in N_{i-1} : x < v \wedge COL(x,v) = c\} \\ x_i &= x \\ COL'(x_i) &= c \end{aligned}$$

   Note that $|N_i| \geq \delta |N_{i-1}|$. If $|V_i^R| = (k-2)^2 + 1$ then goto Phase $2a$.

2. If there exists $x \in N_{i-1}$ and c a color such that $\deg_c^L(x) \geq \delta N_{i-1}$ then do the following:

$$\begin{aligned} V_i^R &= V_{i-1}^R \\ V_i^L &= V_{i-1}^L \cup \{x\} \\ N_i &= \{v \in N_{i-1} : v < x \wedge COL(x,v) = c\} \\ x_i &= x \\ COL'(x_i) &= c \end{aligned}$$

   Note that $|N_i| \geq \delta |N_{i-1}|$. If $|V_i^L| = (k-2)^2 + 1$ then goto Phase $2b$.

3. If neither case 1 or case 2 holds then goto Phase $2c$.

End of Phase 1

Since we goto Phase 2 if either $|V_i^R| = (k-2)^2 + 1$ or $|V_i^L| = (k-2)^2 + 1$ we iterate the above process at most $2(k-2)^2 + 1$ times. Let $s = 2(k-2)^2 + 1$.
Phase 2a: Restrict COL' to V_i^R and apply Lemma 2.2 to obtain that one of the following occurs.

1. There is a a set $H' \subseteq V_i^R$, homog relative to COL', of size $k-1$. Recall that n has not been used at all. It is easy to see that $H = H' \cup \{n\}$ is homog relative to COL.

2. There is a a set $H' \subseteq V_i^R$, rainbow relative to COL', of size $k-1$. Recall that n has not been used at all. It is easy to see that $H = H' \cup \{n\}$ is min-homog relative to COL.

We need to be able to carry out the construction for s stages. Note that after s stages $|N_s| \geq \delta^s n$. We need this to be ≥ 1. Hence we need

FIRST CONSTRAINT:

$$\delta \geq \left(\frac{1}{n}\right)^{1/s}.$$

If you got to Phase 2a you do NOT need to goto Phase 2b or 2c.

End of Phase 2a:

Phase $2b$**:** You got here because $|V_i^L| = (k-2)^2+1$. This is similar to Phase $2a$ so we omit it. We note that in this case you obtain either a homog set or a max-homog set.

End of Phase 2b:

Phase $2c$**:**

Assume that when you got here $N = N_i$ was of size m. The largest stage this could happen at was $s-1$. Hence we need

SECOND CONSTRAINT:

$$m \leq \delta^{s-1} n.$$

We take

$$n = \frac{1}{\delta^{s-1}} n.$$

This will also satisfy FIRST CONSTRAINT.

You got here because for all $v \in N$, for all colors c, $\deg_c^L(v) \leq \delta m$ and $\deg_c^R(v) \leq \delta m$. Hence $\deg_c(v) \leq 2\delta m$. By renumbering we assume that $N = \{1, \ldots, m\}$ and that the colors are $\{1, \ldots, m\}$. Let COL be the coloring restricted to $\binom{[m]}{2}$. Note that, for all vertices $v \in [m]$, for all colors c, $\deg_c(v) \leq 2\delta m$.

Note also that, for any vertex $v \in [m]$,

$$m - 1 < m = \sum_{c=1}^{m} \deg_c(v) \leq \sum_{c=1}^{m} \delta m = m^2\delta.$$

Hence
THIRD CONSTRAINT:

$$\delta \geq \frac{1}{m}.$$

Note that COL is a coloring of $\binom{[m]}{2}$ such that for every v and c, $\deg_c(v) \leq 2\delta m$. Hence, by Lemma 7.2, there are at most

$$\frac{2\delta m \times m^2}{2} = \delta m^3$$

bad triples.

By Lemma 7.3 there exists a subset X of size m' that has at most

$$\delta m^3 \times \left(\frac{m'}{m}\right)^3 = \delta(m')^3$$

bad triples.

We have two options for setting m' which lead to the different upper bounds. The first option gives a simpler proof and one less parameter; however, the second option gives a better bound. We admit here that the improvement of the upper bound is marginal.

Option 1: Set m' and δ so that there are *no* bad triples. Hence we need

$$\delta(m')^3 < 1$$

We now have a set X of size m' with no bad triples. We will use Lemma 2.5 on this set, hence we take

$$m' = \frac{k^3}{2}.$$

Hence

$$\delta = \frac{2}{m'3} = \frac{16}{k^9}.$$

By THIRD CONSTRAINT we need

$$\delta \geq \frac{1}{m}.$$

We take

$$m = \frac{1}{\delta} = \frac{k^9}{16}.$$

By the SECOND CONSTRAINT

$$m \le \delta^{s-1} n.$$

$$n = \frac{m}{\delta^{s-1}}.$$

Since $\delta = \frac{1}{m}$ we can express n in terms of m.

$$n = m^s = \left(\frac{k^9}{16}\right)^s = \left(\frac{k^9}{16}\right)^{2(k-2)^2+1}.$$

And we are DONE— with Option 1.
Option 2. We set m' such that the number of bad triples is so small that we can just remove one point from each. This will lead to a better value of n. Recall that the number of bad triples is $\delta(m')^3$.

We want the number of bad triples to be so small that if we just toss out one vertex from each we still have many (that is, m'') vertices.

FOURTH CONSTRAINT:

$$m' - \frac{\delta(m')^3}{3} \ge m''.$$

By renumbering we can assume the m'' vertices are $\{1, \ldots, m''\}$. Let COL be the coloring restricted to $\binom{[m'']}{2}$. Note that there are NO bad triples. By Lemma 2.5 there exists a rainbow set of size $(2m'')^{1/3}$. Since we want this to be $\ge k$ we have our

FIFTH CONSTRAINT:

$$m'' \ge \frac{k^3}{2}.$$

End of Phase $2c$

We now collect up all the constraints and see how to satisfy them in a way that minimizes n.

List of Constraints

1.

$$\delta \geq \left(\frac{1}{n}\right)^{1/s}$$

This constraint is implied by the next one so we do nothing.

2.

$$\delta \geq \left(\frac{m}{n}\right)^{1/(s-1)}.$$

We satisfy this by taking

$$n = \frac{m}{\delta^{s-1}}$$

This constraint is now satisfied; however, we need to know what m and δ are.

3.

$$\delta \geq \frac{1}{m}.$$

We will take

$$m = \frac{1}{\delta}.$$

This constraint is now satisfied; however, we need to know what δ is.

4.

$$m' - \frac{\delta}{3}(m')^3 \geq m''.$$

$$\delta \leq \frac{3m' - 3m''}{(m')^3}$$

Since we want δ as large as possible we will take δ to equal this upper bound. This constraint is now satisfied; however, we need to know what m', m'' are.

5.

$$m'' \geq \frac{k^3}{2}.$$

We take m'' equal to this lower bound. This constraint is now satisfied.

End of List of Constraints

$$m'' = \frac{k^3}{2}.$$

What should m' and δ be? We want to maximize δ. Recall that

$$\delta = \frac{3m' - 3m''}{(m')^3}.$$

We pick the value of $1 \le m' \le m$ that maximizes δ. Simple calculus reveals that this value is $m' = 1.5m''$. Hence

$$m' = 1.5m'' = \frac{1.5k^3}{2}.$$

$$\delta = \frac{3m' - 3m''}{(m')^3} = \frac{4.5m'' - 3m''}{(1.5m'')^3} = \frac{1.5m''}{(1.5m'')^3} = \frac{1}{(1.5m'')^2}$$

Note that

$$(1.5m'')^2 = \left(\frac{1.5k^3}{2}\right)^2 = \left(\frac{3k^3}{4}\right)^2 = \frac{9k^6}{16}.$$

Hence

$$\delta = \frac{16}{9k^6}.$$

Hence

$$m = \frac{1}{\delta} = \frac{9k^6}{16}.$$

We now know m and δ so we can find n. Since $m = \frac{1}{\delta}$ we express n in terms of m and then k.

$$n = \frac{m}{\delta^{s-1}} = m^s = \left(\frac{9k^6}{16}\right)^s = \left(\frac{9k^6}{16}\right)^{2(k-2)^2+1}.$$

∎

Note 7.6 Lemma 2.6 can be used to very slightly improve Theorem 7.5. We leave this to the reader.

Note 7.7 Our proof is an exposition of the proof of Lefmann and Rodl [4]. There is one minor difference. They end up with a constant of $\frac{27}{16}$ instead of $\frac{9}{16}$. They, like us, claim they will take out one point from each bad triple. However, they, unlike us, remove every point of every bad triple. This results in the worse constant.

8 Comparing the Proofs

In the table below we compare the proofs with the following criteria:

1. How powerful a Ramsey Theorem does the proof use. We list the proofs in this order. The more powerful a Ramsey Theorem you use (generally) the worse your bounds will be. Mileti [5] is concerned with the logical complexity of the Canonical Ramsey Theorem. For him, the more powerful a Ramsey Theorem you are using, the higher (which is bad) the logical complexity of the proof.

2. What is the bound? the lower the bound the better. In this case the 2-ary-II proof is clearly the best by far.

3. Does the proof extend to the a-ary case? This is one drawback of 2-ary-II: It does not extend to a-ary. All of the others, except for the 3-ary proof, do extend.

4. Does the proof extend to the infinite case? This is another drawback of 2-ary-II: It does not extend to the infinite graph case while all of the others do. Incidentally, all of the others except the 3-ary extend to the infinite a-ary case.

5. How complicated is the construction. We are asking this subjectively; however, we justify our choices. The 4-ary proof just applies the 4-ary hypergraph Ramsey Theorem and PUFF, done!. This is clearly the easiest construction. The 3-ary proof applies the 3-ary hypergraph Ramsey Theorem but then still has to apply Lemma 2.5. So that is just a bit harder. The 2-ary, NONE-I, and NONE-II are significantly more difficult so we discuss them in the next XXX.

6. LATER

Proof	bound	extends to a-ary?	extends to ∞?	How Complicated
4-ary	$R_4(k^2; 12)$	YES	YES	easy
3-ary	$R_3(k^3; 4)$	BILL?	YES	easy+ Lemma
2-ary	BILL?	YES	YES	BILL
NONE-I	BILL?	YES	YES	BILL
NONE-II	$2^{12k^2 \log k}$	NO	NO	YES

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