BILL, RECORD LECTURE!!!!

BILL RECORD LECTURE!!!

Coloring $\binom{Z}{2}$

Exposition by William Gasarch-U of MD

Not Quite Homogenous

Not Quite Homogenous

Def A coloring is **finite** if it only uses a finite number of colors.

Not Quite Homogenous

Def A coloring is **finite** if it only uses a finite number of colors.
Convention For concreteness we will use **1,000,000** (the Arushi number) when we want to say **any number**.

Not Quite Homogenous

Def A coloring is **finite** if it only uses a finite number of colors.
Convention For concreteness we will use **1,000,000** (the Arushi number) when we want to say **any number**.
Def Let $\mathrm{COL}\colon \binom{A}{2} \rightarrow [1,000,000]$. Let $c \in \mathbb{N}$. Let $H \subseteq A$.

Not Quite Homogenous

Def A coloring is **finite** if it only uses a finite number of colors.
Convention For concreteness we will use **1,000,000** (the Arushi number) when we want to say **any number**.
Def Let $\mathrm{COL}\colon \binom{A}{2} \rightarrow [1,000,000]$. Let $c \in \mathbb{N}$. Let $H \subseteq A$.
H is **c-homog** if COL restricted to $\binom{H}{2}$ takes on $\le c$ values.

Not Quite Homogenous

Def A coloring is **finite** if it only uses a finite number of colors.
Convention For concreteness we will use **1,000,000** (the Arushi number) when we want to say **any number**.
Def Let $\mathrm{COL}\colon \binom{A}{2} \rightarrow [1,000,000]$. Let $c \in \mathbb{N}$. Let $H \subseteq A$.
H is **c-homog** if COL restricted to $\binom{H}{2}$ takes on $\le c$ values.
Notation If L_1 and L_2 are both linearly ordered set then $L_1 \equiv L_2$ means that $\exists$ an order preserving bijection between L_1 and L_2.

Not Quite Homogenous

Def A coloring is **finite** if it only uses a finite number of colors.
Convention For concreteness we will use **1,000,000** (the Arushi number) when we want to say **any number**.
Def Let $\mathrm{COL}\colon \binom{A}{2} \rightarrow [1,000,000]$. Let $c \in \mathbb{N}$. Let $H \subseteq A$.
H is **c-homog** if COL restricted to $\binom{H}{2}$ takes on $\le c$ values.
Notation If L_1 and L_2 are both linearly ordered set then $L_1 \equiv L_2$ means that $\exists$ an order preserving bijection between L_1 and L_2.
Our Question Fill in c in the following:

Not Quite Homogenous

Def A coloring is **finite** if it only uses a finite number of colors.
Convention For concreteness we will use **1,000,000** (the Arushi number) when we want to say **any number**.
Def Let $\mathrm{COL}\colon \binom{A}{2} \rightarrow [1,000,000]$. Let $c \in \mathbb{N}$. Let $H \subseteq A$.
H is **c-homog** if COL restricted to $\binom{H}{2}$ takes on $\le c$ values.
Notation If L_1 and L_2 are both linearly ordered set then $L_1 \equiv L_2$ means that $\exists$ an order preserving bijection between L_1 and L_2.
Our Question Fill in c in the following:
$\forall \mathrm{COL}\colon \binom{\mathbb{Z}}{2} \rightarrow [1,000,000]$ $\exists$ *inf* $H \subseteq \mathbb{Z}$, $H \equiv \mathbb{Z}$, H *is c-homog.*

Not Quite Homogenous

Def A coloring is **finite** if it only uses a finite number of colors.
Convention For concreteness we will use **1,000,000** (the Arushi number) when we want to say **any number**.
Def Let $\mathrm{COL}\colon \binom{A}{2} \rightarrow [1,000,000]$. Let $c \in \mathbb{N}$. Let $H \subseteq A$.
H is ***c*-homog** if COL restricted to $\binom{H}{2}$ takes on $\leq c$ values.
Notation If L_1 and L_2 are both linearly ordered set then $L_1 \equiv L_2$ means that $\exists$ an order preserving bijection between L_1 and L_2.
Our Question Fill in c in the following:
$\forall \mathrm{COL}\colon \binom{\mathbb{Z}}{2} \rightarrow [1,000,000]\ \exists$ *inf* $H \subseteq \mathbb{Z}$, $H \equiv \mathbb{Z}$, H *is c-homog.*
$\exists \mathrm{COL}\colon \binom{\mathbb{Z}}{2} \rightarrow [c]$ *no* $(c-1)$*-homog set.*

$c \geq 4$

Thm $\exists\ \mathrm{COL}\colon \binom{Z}{2} \to [4]$ with no infinite 3-homog $H \equiv \mathbb{Z}$.

$c \geq 4$

Thm $\exists\ \mathrm{COL}\colon \binom{Z}{2} \to [4]$ with no infinite 3-homog $H \equiv \mathbb{Z}$.
Convention We will take $\mathbb{Z}$ to be the following set:
$\{\ldots, -6, -4, -2\} \cup \{1, 3, 5, \ldots\}$.

$c \geq 4$

Thm $\exists\ \mathrm{COL}\colon \binom{Z}{2} \rightarrow [4]$ with no infinite 3-homog $H \equiv \mathbb{Z}$.
Convention We will take $\mathbb{Z}$ to be the following set:
$\{\ldots, -6, -4, -2\} \cup \{1, 3, 5, \ldots\}$.
We define $\mathrm{COL}(x, y)\colon \binom{\mathbb{Z}}{2}$. We assume $|x| < |y|$

$c \geq 4$

Thm $\exists\ \mathrm{COL}\colon \binom{Z}{2} \rightarrow [4]$ with no infinite 3-homog $H \equiv \mathbb{Z}$.
Convention We will take $\mathbb{Z}$ to be the following set:
$\{\ldots, -6, -4, -2\} \cup \{1, 3, 5, \ldots\}$.
We define $\mathrm{COL}(x, y)\colon \binom{\mathbb{Z}}{2}$. We assume $|x| < |y|$

$$\mathrm{COL}(x,y) = \begin{cases} 1 & \text{if } x, y \geq 1 \\ 2 & \text{if } x, y \leq -1 \\ 3 & \text{if } x \leq -1,\ y \geq 1 \\ 4 & \text{if } y \leq -1,\ x \geq 1 \end{cases} \quad (1)$$

$c \geq 4$

Thm $\exists\ \mathrm{COL}\colon \binom{Z}{2} \to [4]$ with no infinite 3-homog $H \equiv \mathbb{Z}$.
Convention We will take $\mathbb{Z}$ to be the following set:
$\{\ldots, -6, -4, -2\} \cup \{1, 3, 5, \ldots\}$.
We define $\mathrm{COL}(x, y)\colon \binom{\mathbb{Z}}{2}$. We assume $|x| < |y|$

$$\mathrm{COL}(x,y) = \begin{cases} 1 & \text{if } x, y \geq 1 \\ 2 & \text{if } x, y \leq -1 \\ 3 & \text{if } x \leq -1,\ y \geq 1 \\ 4 & \text{if } y \leq -1,\ x \geq 1 \end{cases} \tag{1}$$

There is no 3-homog $H \equiv \mathbb{Z}$. Left to the reader.

$c \leq 4$

$\forall \ \mathrm{COL}\colon \binom{\mathbb{Z}}{2} \rightarrow [1,000,000] \ \exists \ H \equiv \mathbb{Z}$, H *is 4-homog.*

$c \leq 4$

$\forall\ \mathrm{COL}\colon \binom{\mathbb{Z}}{2} \rightarrow [1,000,000]\ \exists\ H \equiv \mathbb{Z}$, *$H$ is 4-homog.*

1) First look at the coloring restricted to $\binom{\mathbb{N}}{2}$. Use infinite Ramsey to get $H_1 \subseteq \mathbb{N}$ such that COL restricted to $\binom{\mathbb{N}}{2}$ is constant. Assume constant is c_1.

$c \le 4$

$\forall \ \mathrm{COL} \colon \binom{\mathbb{Z}}{2} \rightarrow [1,000,000] \ \exists \ H \equiv \mathbb{Z}$, *$H$ is 4-homog.*

1) First look at the coloring restricted to $\binom{\mathbb{N}}{2}$. Use infinite Ramsey to get $H_1 \subseteq \mathbb{N}$ such that COL restricted to $\binom{\mathbb{N}}{2}$ is constant. Assume constant is c_1.

2) Second look at the coloring restricted to $\binom{-\mathbb{N}}{2}$. Use infinite Ramsey to get $H_2 \subseteq -\mathbb{N}$ such that COL restricted to $\binom{\mathbb{N}}{2}$ is constant. Assume constant is c_2.

$c \le 4$

$\forall \ \mathrm{COL}\colon \binom{\mathbb{Z}}{2} \rightarrow [1,000,000] \ \exists \ H \equiv \mathbb{Z}$, *$H$ is 4-homog.*

1) First look at the coloring restricted to $\binom{\mathbb{N}}{2}$. Use infinite Ramsey to get $H_1 \subseteq \mathbb{N}$ such that COL restricted to $\binom{\mathbb{N}}{2}$ is constant. Assume constant is c_1.

2) Second look at the coloring restricted to $\binom{-\mathbb{N}}{2}$. Use infinite Ramsey to get $H_2 \subseteq -\mathbb{N}$ such that COL restricted to $\binom{\mathbb{N}}{2}$ is constant. Assume constant is c_2.

3) Now we must deal with the edges between $\mathbb{N}$ and $-\mathbb{N}$. This will be the bulk of the proof.

$c \le 4$

$\forall$ COL: $\binom{\mathbb{Z}}{2} \rightarrow [1,000,000]$ $\exists$ $H \equiv \mathbb{Z}$, *H is 4-homog.*

1) First look at the coloring restricted to $\binom{\mathbb{N}}{2}$. Use infinite Ramsey to get $H_1 \subseteq \mathbb{N}$ such that COL restricted to $\binom{\mathbb{N}}{2}$ is constant. Assume constant is c_1.

2) Second look at the coloring restricted to $\binom{-\mathbb{N}}{2}$. Use infinite Ramsey to get $H_2 \subseteq -\mathbb{N}$ such that COL restricted to $\binom{\mathbb{N}}{2}$ is constant. Assume constant is c_2.

3) Now we must deal with the edges between $\mathbb{N}$ and $-\mathbb{N}$. This will be the bulk of the proof.

4) We need a thm about bipartite graphs. This will be used for

$c \le 4$

$\forall$ COL: $\binom{\mathbb{Z}}{2} \rightarrow [1,000,000]$ $\exists$ $H \equiv \mathbb{Z}$, *H is 4-homog.*

1) First look at the coloring restricted to $\binom{\mathbb{N}}{2}$. Use infinite Ramsey to get $H_1 \subseteq \mathbb{N}$ such that COL restricted to $\binom{\mathbb{N}}{2}$ is constant. Assume constant is c_1.

2) Second look at the coloring restricted to $\binom{-\mathbb{N}}{2}$. Use infinite Ramsey to get $H_2 \subseteq -\mathbb{N}$ such that COL restricted to $\binom{\mathbb{N}}{2}$ is constant. Assume constant is c_2.

3) Now we must deal with the edges between $\mathbb{N}$ and $-\mathbb{N}$. This will be the bulk of the proof.

4) We need a thm about bipartite graphs. This will be used for $\binom{\mathbb{Z}}{2}$ and

$c \le 4$

$\forall$ COL: $\binom{\mathbb{Z}}{2} \rightarrow [1,000,000]$ $\exists$ $H \equiv \mathbb{Z}$, *H is 4-homog.*

1) First look at the coloring restricted to $\binom{\mathbb{N}}{2}$. Use infinite Ramsey to get $H_1 \subseteq \mathbb{N}$ such that COL restricted to $\binom{\mathbb{N}}{2}$ is constant. Assume constant is c_1.

2) Second look at the coloring restricted to $\binom{-\mathbb{N}}{2}$. Use infinite Ramsey to get $H_2 \subseteq -\mathbb{N}$ such that COL restricted to $\binom{\mathbb{N}}{2}$ is constant. Assume constant is c_2.

3) Now we must deal with the edges between $\mathbb{N}$ and $-\mathbb{N}$. This will be the bulk of the proof.

4) We need a thm about bipartite graphs. This will be used for $\binom{\mathbb{Z}}{2}$ and the **Genevieve** graphs $\binom{\alpha}{2}$ where α is a ctble ordinal.

Bipartite Graphs

Bipartite Graphs

Def A Bipartite Graph is (L, R, E) where the vertices are $L \cup R$ and $E \subseteq L \times R$ (so no edges within L or within R). L stands for Left, R stands for Right.

Bipartite Graphs

Def A Bipartite Graph is (L, R, E) where the vertices are $L \cup R$ and $E \subseteq L \times R$ (so no edges within L or within R). L stands for Left, R stands for Right.

Def Let $n, m \in \mathbb{N}$. The **Complete (n, m)-Bipartite Graph**, denoted $K_{n,m}$ is the bipartite graph $([n], [m], [n] \times [m])$.

Bipartite Graphs

Def A Bipartite Graph is (L, R, E) where the vertices are $L \cup R$ and $E \subseteq L \times R$ (so no edges within L or within R). L stands for Left, R stands for Right.

Def Let $n, m \in \mathbb{N}$. The **Complete (n, m)-Bipartite Graph**, denoted $K_{n,m}$ is the bipartite graph $([n], [m], [n] \times [m])$.

Def $K_{\mathbb{N},\mathbb{N}}$ is the bipartite graph $(\mathbb{N}, \mathbb{N}, \mathbb{N} \times \mathbb{N})$.

Bipartite Graphs

Def A Bipartite Graph is (L, R, E) where the vertices are $L \cup R$ and $E \subseteq L \times R$ (so no edges within L or within R). L stands for Left, R stands for Right.

Def Let $n, m \in \mathbb{N}$. The **Complete (n, m)-Bipartite Graph**, denoted $K_{n,m}$ is the bipartite graph $([n], [m], [n] \times [m])$.

Def $K_{\mathbb{N},\mathbb{N}}$ is the bipartite graph $(\mathbb{N}, \mathbb{N}, \mathbb{N} \times \mathbb{N})$.

Note A coloring of the edges of $K_{\mathbb{N},\mathbb{N}}$ is a coloring of $\mathbb{N} \times \mathbb{N}$.

Infinite Ramsey Theory for $K_{\mathbb{N},\mathbb{N}}$

Ramsey Theory for $\mathbb{N} \times \mathbb{N}$

Ramsey Theory for $\mathbb{N} \times \mathbb{N}$

Def Let $\mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \rightarrow [1,000,000]$. Let $c \in \mathbb{N}$.

Ramsey Theory for $\mathbb{N} \times \mathbb{N}$

Def Let $\mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \rightarrow [1,000,000]$. Let $c \in \mathbb{N}$.
$H_1 \times H_2 \subseteq \mathbb{N} \times \mathbb{N}$ is ***c*-homog** if

Ramsey Theory for $\mathbb{N} \times \mathbb{N}$

Def Let $\mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \rightarrow [1,000,000]$. Let $c \in \mathbb{N}$.
$H_1 \times H_2 \subseteq \mathbb{N} \times \mathbb{N}$ is ***c*-homog** if
COL restricted to $H_1 \times H_2$ takes on $\leq c$ values, and

Ramsey Theory for $\mathbb{N} \times \mathbb{N}$

Def Let $\mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [1,000,000]$. Let $c \in \mathbb{N}$.
$H_1 \times H_2 \subseteq \mathbb{N} \times \mathbb{N}$ is ***c*-homog** if
COL restricted to $H_1 \times H_2$ takes on $\le c$ values, and
H_1, H_2 both infinite.

Ramsey Theory for $\mathbb{N} \times \mathbb{N}$

Def Let $\mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [1,000,000]$. Let $c \in \mathbb{N}$.
$H_1 \times H_2 \subseteq \mathbb{N} \times \mathbb{N}$ is ***c*-homog** if
COL restricted to $H_1 \times H_2$ takes on $\le c$ values, and
H_1, H_2 both infinite.

We want a value of c such that the following is true:
Thm $\forall \mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [1,000,000]$ $\exists$ c-homog $H_1 \times H_2$.
Thm $\exists \mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [c]$ no $c-1$-homog $H_1 \times H_2$.

$c \geq 2$

Thm $\exists\ \mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [2]$ with no infinite 1-homog $H_1 \times H_2$.

$c \geq 2$

Thm $\exists\ \mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [2]$ with no infinite 1-homog $H_1 \times H_2$.
We use $\mathrm{EVEN}^+ \times \mathrm{ODD}^+$ instead of $\mathbb{N} \times \mathbb{N}$.

$c \geq 2$

Thm $\exists\ \mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [2]$ with no infinite 1-homog $H_1 \times H_2$.
We use $\mathrm{EVEN}^+ \times \mathrm{ODD}^+$ instead of $\mathbb{N} \times \mathbb{N}$.
We define $\mathrm{COL}(x, y)\colon \mathbb{N} \times \mathbb{N} \to [2]$.

$c \geq 2$

Thm $\exists$ $\mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \rightarrow [2]$ with no infinite 1-homog $H_1 \times H_2$.
We use $\mathrm{EVEN}^{+} \times \mathrm{ODD}^{+}$ instead of $\mathbb{N} \times \mathbb{N}$.
We define $\mathrm{COL}(x, y)\colon \mathbb{N} \times \mathbb{N} \rightarrow [2]$.

$$\mathrm{COL}(x, y) = \begin{cases} 1 & \text{if } x < y \\ 2 & \text{if } x > y \end{cases} \tag{2}$$

$c \geq 2$

Thm $\exists\ \mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \rightarrow [2]$ with no infinite 1-homog $H_1 \times H_2$.
We use $\mathrm{EVEN}^+ \times \mathrm{ODD}^+$ instead of $\mathbb{N} \times \mathbb{N}$.
We define $\mathrm{COL}(x, y)\colon \mathbb{N} \times \mathbb{N} \rightarrow [2]$.

$$\mathrm{COL}(x, y) = \begin{cases} 1 & \text{if } x < y \\ 2 & \text{if } x > y \end{cases} \tag{2}$$

There is no 1-homog $H_1 \times H_2 \subseteq \mathbb{N} \times \mathbb{N}$. Left to the reader.

$c \leq 2$

$\forall\ \mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [\mathbf{1,000,000}]\ \exists\ \mathbf{H_1} \times \mathbf{H_2}$**, 2-homog.**

$c \le 2$

$\forall \ \mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \rightarrow \mathbf{[1,000,000]} \ \exists \ \mathbf{H_1} \times \mathbf{H_2}$**, 2-homog.**

We do an example. The formal construction is left to the reader.

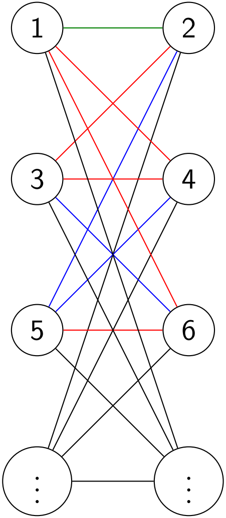
$c \leq 2$

$\forall\ \mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \rightarrow [1,000,000]\ \exists$ **$H_1 \times H_2$, 2-homog.**

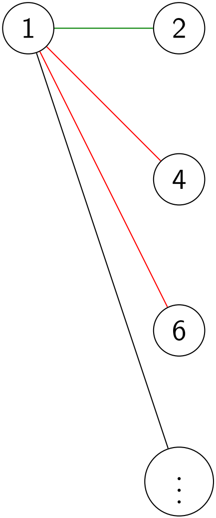
We do an example. The formal construction is left to the reader.

Initially we have $\mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \rightarrow [1,000,000]$.

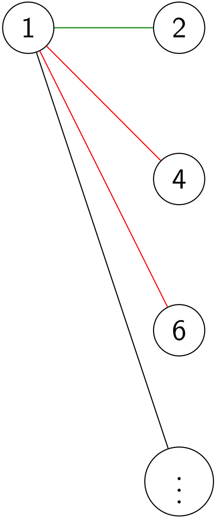
Example of finite coloring of $\mathbb{N} \times \mathbb{N}$



Focus on Vertex 1 On The Left

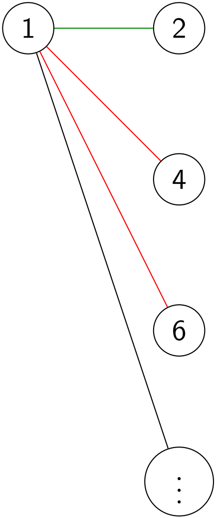


Focus on Vertex 1 On The Left



Let c be least color such that $\exists^\infty x, \mathrm{COL}(1, x) = c$. We assume R.

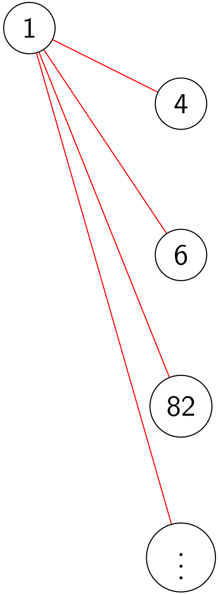
Focus on Vertex 1 On The Left



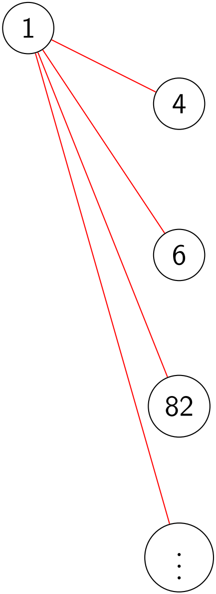
Let c be least color such that $\exists^\infty x, \mathrm{COL}(1, x) = c$. We assume R.

Kill All Those On The Right Who Disagree.

Focus on Vertex 1 On The Left After The Massacre

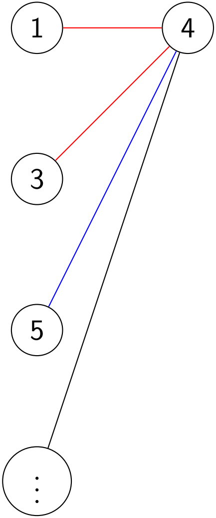


Focus on Vertex 1 On The Left After The Massacre

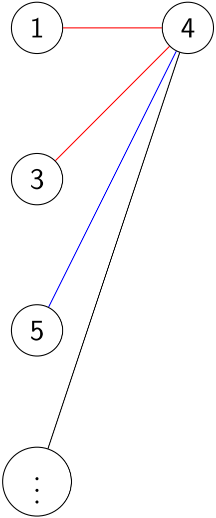


1 is immortal (for now). We focus on 4.

Focusing on 4 On The Right

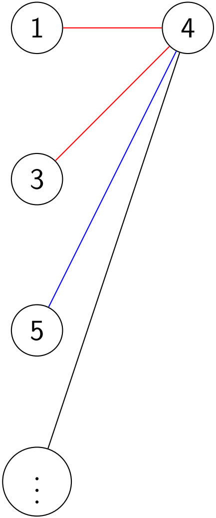


Focusing on 4 On The Right



Let c be least color such that $\exists^\infty x, \mathrm{COL}(x, 4) = c$. We assume B.

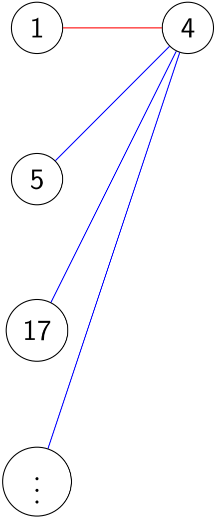
Focusing on 4 On The Right



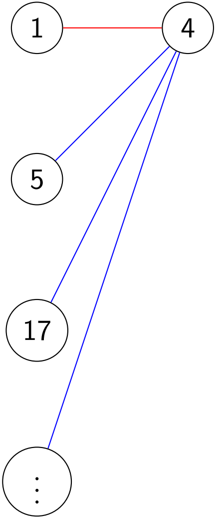
Let c be least color such that $\exists^{\infty} x, \mathrm{COL}(x, 4) = c$. We assume B.

Kill all those on the Left Who Disagree

After Processing 4

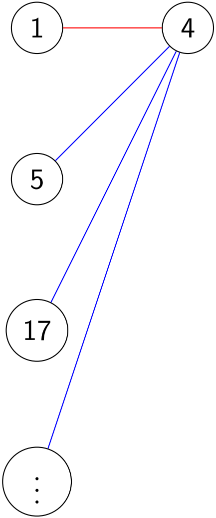


After Processing 4



Let c be least color such that $\exists^\infty x, \mathrm{COL}(x,4) = c$. We assume B.

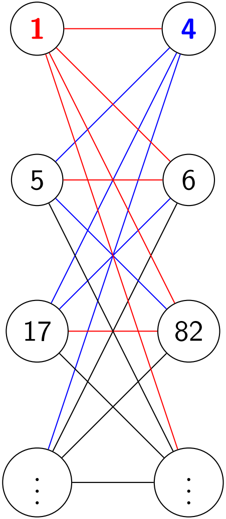
After Processing 4



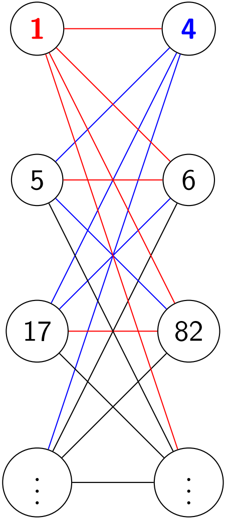
Let c be least color such that $\exists^\infty x, \mathrm{COL}(x,4) = c$. We assume *B*.

Kill all those on the Left Who Disagree

We Have Processed 1 and 4

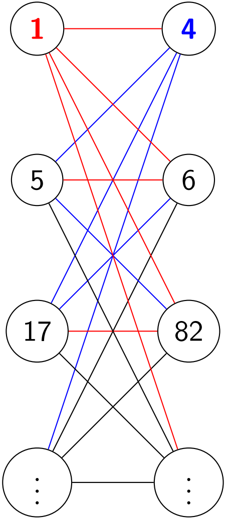


We Have Processed 1 and 4



1 is colored *R*. 4 is colored *B*. 1,4 immortal (for now).

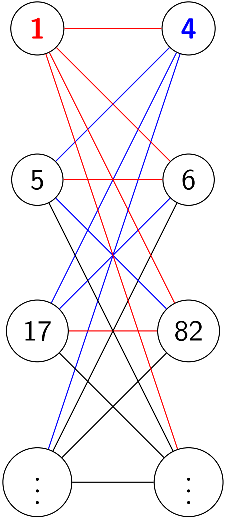
We Have Processed 1 and 4



1 is colored R. 4 is colored B. 1,4 immortal (for now).

4 is B even though one of the edges out of it is R.

We Have Processed 1 and 4

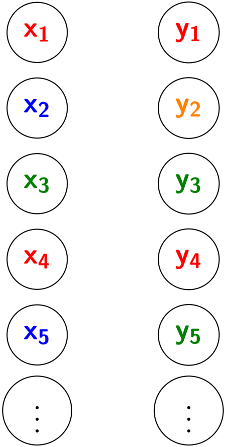


1 is colored *R*. 4 is colored *B*. 1,4 immortal (for now).

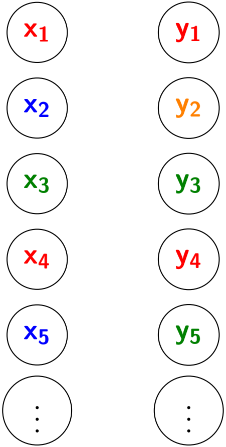
4 is *B* even though one of the edges out of it is *R*.

Key If $x > 4$ then $\mathrm{COL}(x, 4) = B$.

So You Thought You Were Immortal. HA!

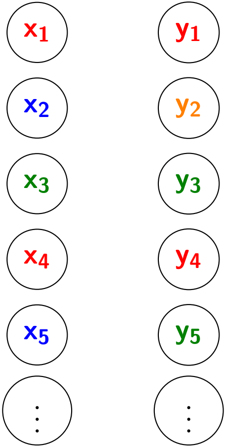


So You Thought You Were Immortal. HA!



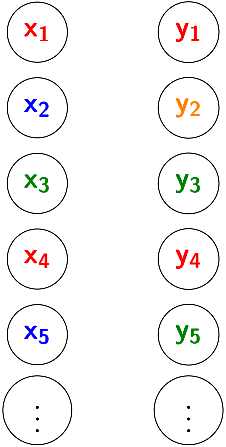
Finite number of colors (though could be large)

So You Thought You Were Immortal. HA!



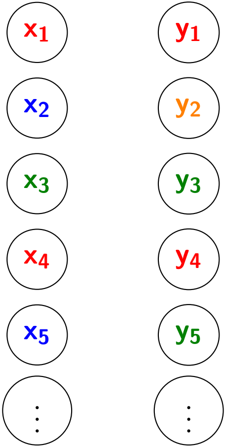
Finite number of colors (though could be large)
$(\exists c)(\exists^{\infty} x)[\mathrm{COL}'(x) = c]$. Assume R.

So You Thought You Were Immortal. HA!



Finite number of colors (though could be large)
$(\exists c)(\exists^{\infty} x)[\mathrm{COL}'(x) = c]$. Assume R.
$(\exists d)(\exists^{\infty} x)[\mathrm{COL}'(y) = d]$. Assume G.

So You Thought You Were Immortal. HA!



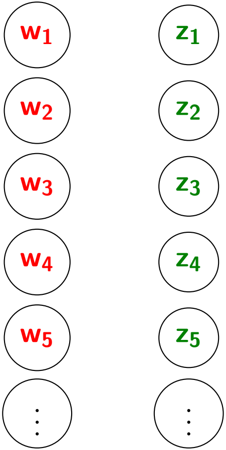
Finite number of colors (though could be large)

$(\exists c)(\exists^\infty x)[\mathrm{COL}'(x) = c]$. Assume R.

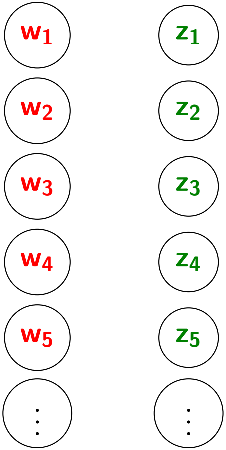
$(\exists d)(\exists^\infty x)[\mathrm{COL}'(y) = d]$. Assume G.

Kill all those who disagree

We Are Done!

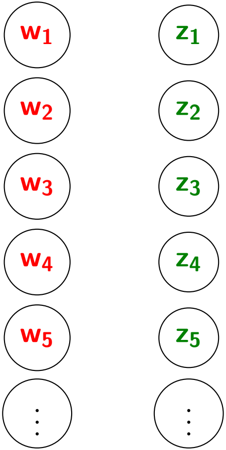


We Are Done!



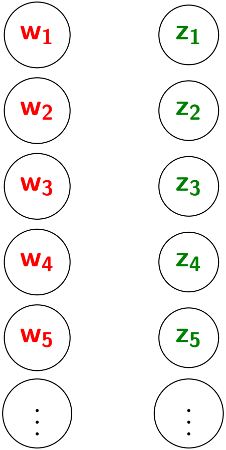
$H_1 = \{w_1, w_2, \ldots\}$

We Are Done!



$H_1 = \{w_1, w_2, \dots\}$ $\qquad$ $H_2 = \{z_1, z_2, \dots\}$.

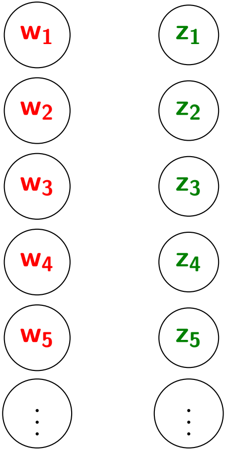
We Are Done!



$H_1 = \{w_1, w_2, \ldots\}$ $\qquad$ $H_2 = \{z_1, z_2, \ldots\}$.

Began with finite $\mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \rightarrow [1,000,000]$.

We Are Done!

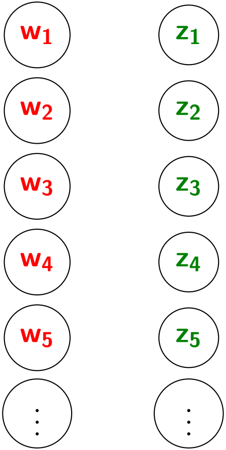


$H_1 = \{w_1, w_2, \ldots\}$ $\quad$ $H_2 = \{z_1, z_2, \ldots\}$.

Began with finite $\mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [1,000,000]$.

$\mathrm{COL}\colon H_1 \times H_2$ uses only two colors.

We Are Done!



$H_1 = \{w_1, w_2, \ldots\}$ $\qquad$ $H_2 = \{z_1, z_2, \ldots\}$.

Began with finite $\mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [1,000,000]$.

$\mathrm{COL}\colon H_1 \times H_2$ uses only two colors.

Restate theorem we just proved on next slide.

Recap

Recap

We have shown the following

Recap

We have shown the following
Thm $\exists\ \mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [2]$ such that there is no 1-homog (H_1, H_2).

Recap

We have shown the following
Thm $\exists$ $\mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [2]$ such that there is no 1-homog (H_1, H_2).
Thm $\forall$ $\mathrm{COL}\colon \mathbb{N} \times \mathbb{N} \to [1,000,000]$ $\exists$ 2-homog (H_1, H_2).

Back to $\mathbb{Z}$

Theorem for $\mathbb{Z}$

Theorem for $\mathbb{Z}$

Thm 1 $\exists\ \mathrm{COL}\colon \binom{\mathbb{Z}}{2} \rightarrow [2]$ such that there is no 3-homog $H \equiv \mathbb{Z}$.

Theorem for $\mathbb{Z}$

Thm 1 $\exists\ \mathrm{COL}\colon \binom{\mathbb{Z}}{2} \to [2]$ such that there is no 3-homog $H \equiv \mathbb{Z}$.
Thm 2 $\forall\ \mathrm{COL}\colon \binom{\mathbb{Z}}{2} \to [1,000,000]$ $\exists$ 4-homog $H \equiv \mathbb{Z}$.

Theorem for $\mathbb{Z}$

Thm 1 $\exists$ COL: $\binom{\mathbb{Z}}{2} \rightarrow [2]$ such that there is no 3-homog $H \equiv \mathbb{Z}$.
Thm 2 $\forall$ COL: $\binom{\mathbb{Z}}{2} \rightarrow [1,000,000]$ $\exists$ 4-homog $H \equiv \mathbb{Z}$.

Thm 1 we proved earlier.

Theorem for $\mathbb{Z}$

Thm 1 $\exists\ \mathrm{COL}\colon \binom{\mathbb{Z}}{2} \rightarrow [2]$ such that there is no 3-homog $H \equiv \mathbb{Z}$.
Thm 2 $\forall\ \mathrm{COL}\colon \binom{\mathbb{Z}}{2} \rightarrow [1,000,000]$ $\exists$ 4-homog $H \equiv \mathbb{Z}$.

Thm 1 we proved earlier.

Thm 2 we prove on the next slide.

Theorem for $\mathbb{Z}$

Let $\mathrm{COL}\colon \binom{\mathbb{Z}}{2} \to [1,000,000]$.

Theorem for $\mathbb{Z}$

Let $\mathrm{COL}\colon \binom{\mathbb{Z}}{2} \rightarrow [1,000,000]$.
1) Use Inf Ramsey on the COL restricted to first $\mathbb{N}$. Homog set H_1. Color c_1.

Theorem for $\mathbb{Z}$

Let $\mathrm{COL}\colon \binom{\mathbb{Z}}{2} \to [1,000,000]$.
1) Use Inf Ramsey on the COL restricted to first $\mathbb{N}$. Homog set H_1. Color c_1.
2) Use Inf Ramsey on the COL restricted to $\binom{-\mathbb{N}}{2}$. Homog set H_2. Color c_2.

Theorem for $\mathbb{Z}$

Let $\mathrm{COL}\colon \binom{\mathbb{Z}}{2} \rightarrow [1,000,000]$.
1) Use Inf Ramsey on the COL restricted to first $\mathbb{N}$. Homog set H_1. Color c_1.
2) Use Inf Ramsey on the COL restricted to $\binom{-\mathbb{N}}{2}$. Homog set H_2. Color c_2.
3) View the edges from H_1 to H_2 as a bipartite graph. Use Bipartite Thm. Colors c_3, c_4.

Theorem for $\mathbb{Z}$

Let $\mathrm{COL}\colon \binom{\mathbb{Z}}{2} \to [1,000,000]$.
1) Use Inf Ramsey on the COL restricted to first $\mathbb{N}$. Homog set H_1. Color c_1.
2) Use Inf Ramsey on the COL restricted to $\binom{-\mathbb{N}}{2}$. Homog set H_2. Color c_2.
3) View the edges from H_1 to H_2 as a bipartite graph. Use Bipartite Thm. Colors c_3, c_4.
At most 4 colors. DONE!

Theorem for $\omega + \omega$

Let $\mathrm{COL}\colon \binom{\omega+\omega}{2} \to [1,000,000]$.

Theorem for $\omega + \omega$

Let $\mathrm{COL}\colon \binom{\omega+\omega}{2} \to [1,000,000]$.
1) Use Inf Ramsey on the COL restricted to first ω. Homog set H_1. Color c_1.

Theorem for $\omega + \omega$

Let $\mathrm{COL}\colon \binom{\omega+\omega}{2} \rightarrow [1,000,000]$.
1) Use Inf Ramsey on the COL restricted to first ω. Homog set H_1. Color c_1.
2) Use Inf Ramsey on the COL restricted to second ω. Homog set H_2. Color c_2.

Theorem for $\omega + \omega$

Let $\mathrm{COL}\colon \binom{\omega+\omega}{2} \to [1,000,000]$.
1) Use Inf Ramsey on the COL restricted to first ω. Homog set H_1. Color c_1.
2) Use Inf Ramsey on the COL restricted to second ω. Homog set H_2. Color c_2.
3) View the edges from H_1 to H_2 as a bipartite graph. Use Bipartite Thm. Colors c_3, c_4.

Theorem for $\omega + \omega$

Let $\mathrm{COL}\colon \binom{\omega+\omega}{2} \to [1,000,000]$.
1) Use Inf Ramsey on the COL restricted to first ω. Homog set H_1. Color c_1.
2) Use Inf Ramsey on the COL restricted to second ω. Homog set H_2. Color c_2.
3) View the edges from H_1 to H_2 as a bipartite graph. Use Bipartite Thm. Colors c_3, c_4.
At most 4 colors. DONE!

Theorem for $\omega + \omega$

Let $\mathrm{COL}\colon \binom{\omega+\omega}{2} \to [1,000,000]$.
1) Use Inf Ramsey on the COL restricted to first ω. Homog set H_1. Color c_1.
2) Use Inf Ramsey on the COL restricted to second ω. Homog set H_2. Color c_2.
3) View the edges from H_1 to H_2 as a bipartite graph. Use Bipartite Thm. Colors c_3, c_4.
At most 4 colors. DONE!
Proof that you need 4 colors similar to that for $\binom{\mathbb{Z}}{2}$.

What Else is Known?

What Else is Known?

Lots of linear orders have been looked at.

What Else is Known?

Lots of linear orders have been looked at.

Hypergraph versions have been looked at.

What Else is Known?

Lots of linear orders have been looked at.

Hypergraph versions have been looked at.

Other structures, more complicated than linear orders, have been looked at.

What Else is Known?

Lots of linear orders have been looked at.

Hypergraph versions have been looked at.

Other structures, more complicated than linear orders, have been looked at.

If I wasn't making up this slide at 10:30AM for a class at 11:00AM I would go into more detail.

What Else is Known?

Lots of linear orders have been looked at.

Hypergraph versions have been looked at.

Other structures, more complicated than linear orders, have been looked at.

If I wasn't making up this slide at 10:30AM for a class at 11:00AM I would go into more detail.

Instead some of the other results might be on a HW.