Application! Restricting Domains To Stop Being Onto

Exposition by William Gasarch

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Credit Where Credit Was Due

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He is interested in the logical strength of **The Thin Set Theorem**. This will not be our concern.

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$f : \mathbb{Z} \to \mathbb{Z}$ via $f(x) = x + 1$ is onto
$f : \mathbb{N} \to \mathbb{Z}$ via $f(x) = x + 1$ is NOT onto.

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Good Question For which $f : \mathbb{Z} \to \mathbb{Z}$ is there an INFINITE set $\mathbb{D} \subseteq \mathbb{Z}$ such that $f : \mathbb{D} \to \mathbb{Z}$ is not onto.

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That wasn't stupid, but it was easy.

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Answer on next page.

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Why 3? We will discuss that later.

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Now look at f restricted to $(x,y) \in H_1 \times H_1$ with $x > y$.

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Define $\mathrm{COL}_3\colon \binom{H_1}{2} \rightarrow [4]$
Recall that the coloring is on **unordered pairs**
COL_3 takes input $\{x, y\}$ and we can assume $x > y$.

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If $\mathrm{COL}_3(H_3) = 0$ then $(\forall x, y \in H_3, x > y)[f(x, y) = 0]$.

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If $\mathrm{COL}_3(H_3) = \mathbf{R}$ then $(\forall x, y \in H_1, x > y)[f(x,y) \notin \{0,1,2\}]$.

Second Use of Ramsey

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If $\mathrm{COL}_3(H_3) = \mathbf{R}$ then $(\forall x, y \in H_1, x > y)[f(x,y) \notin \{0,1,2\}]$.
We show that f on $H_3 \times H_3$ is not onto.

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$f(x, y)$ with $x < y$ is always 0, always 1, always 2, or always $\notin \{0, 1, 2\}$.

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One of the four colors is not here. Which one?

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if color 0 then 0 not in the image, so NOT onto.

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if color 2 then 2 not in the image, so NOT onto.

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if color 0 then 0 not in the image, so NOT onto.

if color 1 then 1 not in the image, so NOT onto.

if color 2 then 2 not in the image, so NOT onto.

if color **R** then image is subset of $\{0,1,2\}$, so NOT onto.

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We pick 4 since it is one more than the number possible types of inputs.

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We will discuss this more after we do the 3-hypergraph Ramsey Theorem and can examine $f(x, y, z)$.