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Probabilistic Method For Sum Free Sets

Exposition by William Gasarch

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3) We will use the Prob Method to show there are large sum-free sets.

Sum Free Set Problem

A More Sophisticated Use of Prob Method.

Definition: A set of numbers A is *sum free* if there is NO $x, y, z \in A$ such that $x + y = z$.

Example: Let $y_1, \ldots, y_m \in (1/3, 2/3)$ (so they are all between 1/3 and 2/3). Note that $y_i + y_j > 2/3$, hence $y_i + y_j \notin \{y_1, \ldots, y_m\}$.

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Proof: STUDENTS DO THIS. ITS EASY.
Example: Let $A = \{y_1, \ldots, y_m\}$ all have fractional part in $(1/3, 2/3)$. A is sum free by above Lemma.

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VOTE:

1. There is a sumfree set of size roughly $n/3$.
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STUDENTS - WORK ON THIS IN GROUPS.

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Theorem For all $\epsilon > 0$, for all A that are a set of n real numbers, there is a sum-free subset of A of size $(1/3 - \epsilon)n$.

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For $a \in [L, U]$ let

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$$B_a = \{x \in A : \mathrm{frac}(ax) \in (1/3, 2/3)\}.$$

For all a, B_a is sum-free by Lemma above.
SO we need an a such that B_a is LARGE.

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$$E(|B_a|) = \sum_{x \in A} \Pr_{a \in [L,U]}(\mathrm{frac}(ax) \in (1/3, 2/3))$$

$$= \sum_{x \in A} (1/3 - \epsilon)$$

$$= (1/3 - \epsilon)n.$$

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$$\begin{aligned} E(|B_a|) &= \sum_{x \in A} \Pr_{a \in [L,U]}(\mathrm{frac}(ax) \in (1/3, 2/3)) \\ &= \sum_{x \in A} (1/3 - \epsilon) \\ &= (1/3 - \epsilon)n. \end{aligned}$$

So THERE EXISTS an a such that $|B_a| \geq (1/3 - \epsilon)n$.
What is a? I DON"T KNOW AND I DON"T CARE!
End of Proof