Roth's Theorem A Dense Enough Set Has a 3-AP

Exposition by William Gasarch and Kelin Zhu

December 23, 2024

The Erdös-Turan Conjecture

The Erdös-Turan Conjecture

Def Let $N \in \mathbb{N}$. Let $A \subseteq [N]$. The density of A is $|A|/N$.

The Erdös-Turan Conjecture

Def Let $N \in \mathbb{N}$. Let $A \subseteq [N]$. The density of A is $|A|/N$.

Szemerédi's Thm For all $\delta > 0$, for all k, there exists $N = N(\delta, k)$ such that the following holds:

The Erdös-Turan Conjecture

Def Let $N \in \mathbb{N}$. Let $A \subseteq [N]$. The density of A is $|A|/N$.

Szemerédi's Thm For all $\delta > 0$, for all k, there exists $N = N(\delta, k)$ such that the following holds:

If $A \subseteq [N]$ and A has density $\geq \delta$ then A has a k-AP.

The Erdös-Turan Conjecture

Def Let $N \in \mathbb{N}$. Let $A \subseteq [N]$. The density of A is $|A|/N$.

Szemerédi's Thm For all $\delta > 0$, for all k, there exists $N = N(\delta, k)$ such that the following holds:

If $A \subseteq [N]$ and A has density $\geq \delta$ then A has a k-AP.

We won't do the (hard) proof. We will do:

The Erdös-Turan Conjecture

Def Let $N \in \mathbb{N}$. Let $A \subseteq [N]$. The density of A is $|A|/N$.

Szemerédi's Thm For all $\delta > 0$, for all k, there exists $N = N(\delta, k)$ such that the following holds:

If $A \subseteq [N]$ and A has density $\geq \delta$ then A has a k-AP.

We won't do the (hard) proof. We will do:

1) Some easy cases, and

The Erdös-Turan Conjecture

Def Let $N \in \mathbb{N}$. Let $A \subseteq [N]$. The density of A is $|A|/N$.

Szemerédi's Thm For all $\delta > 0$, for all k, there exists $N = N(\delta, k)$ such that the following holds:

If $A \subseteq [N]$ and A has density $\geq \delta$ then A has a k-AP.

We won't do the (hard) proof. We will do:

1) Some easy cases, and

2) The $k = 3$ case which involves the Discrete Fourier Transform.

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.
We can assume $N \equiv 0 \pmod 3$.

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.
We can assume $N \equiv 0 \pmod 3$.
Look at

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.
We can assume $N \equiv 0 \pmod 3$.
Look at

$$\{1,2,3\},\{4,5,6\},\ldots,\{N-2,N-1,N\}.$$

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.
We can assume $N \equiv 0 \pmod 3$.
Look at

$$\{1,2,3\},\{4,5,6\},\ldots,\{N-2,N-1,N\}.$$

Case 1 $\exists x \equiv 1 \pmod 3$, $\{x, x+1, x+2\} \in A$. A has a 3-AP.

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.
We can assume $N \equiv 0 \pmod 3$.
Look at

$$\{1,2,3\},\{4,5,6\},\ldots,\{N-2,N-1,N\}.$$

Case 1 $\exists x \equiv 1 \pmod 3$, $\{x, x+1, x+2\} \in A$. A has a 3-AP.

Case 2 $\forall x \equiv 1 \pmod 3$, $|\{x, x+1, x+2\} \cap A| \leq 2$. Then

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.
We can assume $N \equiv 0 \pmod 3$.
Look at

$$\{1,2,3\},\{4,5,6\},\ldots,\{N-2,N-1,N\}.$$

Case 1 $\exists x \equiv 1 \pmod 3$, $\{x, x+1, x+2\} \in A$. A has a 3-AP.

Case 2 $\forall x \equiv 1 \pmod 3$, $|\{x, x+1, x+2\} \cap A| \leq 2$. Then $|A| \leq 2 \times \frac{N}{3} \leq 0.667N < 0.67N$

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.
We can assume $N \equiv 0 \pmod 3$.
Look at

$$\{1, 2, 3\}, \{4, 5, 6\}, \ldots, \{N-2, N-1, N\}.$$

Case 1 $\exists x \equiv 1 \pmod 3$, $\{x, x+1, x+2\} \in A$. A has a 3-AP.

Case 2 $\forall x \equiv 1 \pmod 3$, $|\{x, x+1, x+2\} \cap A| \leq 2$. Then
$|A| \leq 2 \times \frac{N}{3} \leq 0.667N < 0.67N$
This contradicts A having density ≥ 0.67.

An Easy Case

Thm Let $N \geq 3$. Let $A \subseteq [N]$ of density ≥ 0.67. Then A contains a 3-AP.
We can assume $N \equiv 0 \pmod 3$.
Look at

$$\{1,2,3\}, \{4,5,6\}, \ldots, \{N-2, N-1, N\}.$$

Case 1 $\exists x \equiv 1 \pmod 3$, $\{x, x+1, x+2\} \in A$. A has a 3-AP.

Case 2 $\forall x \equiv 1 \pmod 3$, $|\{x, x+1, x+2\} \cap A| \leq 2$. Then
$|A| \leq 2 \times \frac{N}{3} \leq 0.667N < 0.67N$
This contradicts A having density ≥ 0.67.

There may be a HW where you are asked to prove theorems like the 0.67-Theorem.

Roth's Theorem

Roth's Theorem For all $\delta > 0$ there exists $N = N(\delta)$ such that the following holds

For all $A \subseteq [N]$ of density $\geq \delta$, A has a 3-AP.

Roth's Theorem

Roth's Theorem For all $\delta > 0$ there exists $N = N(\delta)$ such that the following holds
For all $A \subseteq [N]$ of density $\geq \delta$, A has a 3-AP.

The **Intuition** behind the proof will be short and clear.

Roth's Theorem

Roth's Theorem For all $\delta > 0$ there exists $N = N(\delta)$ such that the following holds
For all $A \subseteq [N]$ of density $\geq \delta$, A has a 3-AP.

The **Intuition** behind the proof will be short and clear.

The **formal proof** will be long and use some hard math.

Intuition Behind Roth's Theorem

Given $A \subseteq [N]$ of density δ we show one of the following happens.

Intuition Behind Roth's Theorem

Given $A \subseteq [N]$ of density δ we show one of the following happens.

1) A looks **random**. Then A will have a 3-AP.

Intuition Behind Roth's Theorem

Given $A \subseteq [N]$ of density δ we show one of the following happens.

1) A looks **random**. Then A will have a 3-AP.

2) There is a very large AP $N' \subseteq [N]$

Intuition Behind Roth's Theorem

Given $A \subseteq [N]$ of density δ we show one of the following happens.

1) A looks **random**. Then A will have a 3-AP.

2) There is a very large AP $N' \subseteq [N]$

$$N' = \{a + d, \ldots, a + kd\}$$

Intuition Behind Roth's Theorem

Given $A \subseteq [N]$ of density δ we show one of the following happens.

1) A looks **random**. Then A will have a 3-AP.

2) There is a very large AP $N' \subseteq [N]$

$$N' = \{a + d, \ldots, a + kd\}$$

such that
$A \cap N'$ has density $\delta' > \delta$ in N'.

Intuition Behind Roth's Theorem

Given $A \subseteq [N]$ of density δ we show one of the following happens.

1) A looks **random**. Then A will have a 3-AP.

2) There is a very large AP $N' \subseteq [N]$

$$N' = \{a + d, \ldots, a + kd\}$$

such that
$A \cap N'$ has density $\delta' > \delta$ in N'.

Can view $A \cap N'$ as a denser-than-δ subset of N'.

Intuition Behind Roth's Theorem

Given $A \subseteq [N]$ of density δ we show one of the following happens.

1) A looks **random**. Then A will have a 3-AP.

2) There is a very large AP $N' \subseteq [N]$

$$N' = \{a + d, \ldots, a + kd\}$$

such that
$A \cap N'$ has density $\delta' > \delta$ in N'.

Can view $A \cap N'$ as a denser-than-δ subset of N'.

Repeat this procedure until either you get the **Random** case or the density is ≥ 0.67.

Intuition Behind Roth's Theorem

Given $A \subseteq [N]$ of density δ we show one of the following happens.

1) A looks **random**. Then A will have a 3-AP.

2) There is a very large AP $N' \subseteq [N]$

$$N' = \{a + d, \ldots, a + kd\}$$

such that
$A \cap N'$ has density $\delta' > \delta$ in N'.

Can view $A \cap N'$ as a denser-than-δ subset of N'.

Repeat this procedure until either you get the **Random** case or the density is ≥ 0.67.

Much of what I said here isn't quite right, but thats the intuition.

How Will δ' and δ Relate

What if the δ increase as follows;
δ,

How Will δ' and δ Relate

What if the δ increase as follows;
δ,
$\delta + \frac{\delta^{100}}{2}$,

How Will δ' and δ Relate

What if the δ increase as follows;
δ,
$\delta + \frac{\delta^{100}}{2}$,
$\delta + \frac{\delta^{100}}{2} + \frac{\delta^{100}}{2^2}$.

How Will δ' and δ Relate

What if the δ increase as follows;

δ,

$\delta + \frac{\delta^{100}}{2}$,

$\delta + \frac{\delta^{100}}{2} + \frac{\delta^{100}}{2^2}$.

$\delta + \frac{\delta^2}{2} + \frac{\delta^{100}}{2^2} + \frac{\delta^{100}}{2^3}$.

How Will δ' and δ Relate

What if the δ increase as follows;

δ,

$\delta + \frac{\delta^{100}}{2}$,

$\delta + \frac{\delta^{100}}{2} + \frac{\delta^{100}}{2^2}$.

$\delta + \frac{\delta^2}{2} + \frac{\delta^{100}}{2^2} + \frac{\delta^{100}}{2^3}$.

$\vdots$

How Will δ' and δ Relate

What if the δ increase as follows;

δ,

$\delta + \frac{\delta^{100}}{2}$,

$\delta + \frac{\delta^{100}}{2} + \frac{\delta^{100}}{2^2}$.

$\delta + \frac{\delta^2}{2} + \frac{\delta^{100}}{2^2} + \frac{\delta^{100}}{2^3}$.

$\vdots$

Then density is always

How Will δ' and δ Relate

What if the δ increase as follows;

δ,

$\delta + \frac{\delta^{100}}{2}$,

$\delta + \frac{\delta^{100}}{2} + \frac{\delta^{100}}{2^2}$.

$\delta + \frac{\delta^2}{2} + \frac{\delta^{100}}{2^2} + \frac{\delta^{100}}{2^3}$.

$\vdots$

Then density is always

$< \delta + \delta^{100} \sum_{i=1}^{\infty} \frac{1}{2^i} = \delta + \delta^{100}$.

How Will δ' and δ Relate

What if the δ increase as follows;

δ,

$\delta + \frac{\delta^{100}}{2}$,

$\delta + \frac{\delta^{100}}{2} + \frac{\delta^{100}}{2^2}$.

$\delta + \frac{\delta^{2}}{2} + \frac{\delta^{100}}{2^2} + \frac{\delta^{100}}{2^3}$.

$\vdots$

Then density is always

$< \delta + \delta^{100} \sum_{i=1}^{\infty} \frac{1}{2^i} = \delta + \delta^{100}$.

If $\delta = \frac{1}{10}$ then density is always $< \frac{1}{10} + \frac{1}{10^{100}}$.

How Will δ' and δ Relate

What if the δ increase as follows;
δ,
$\delta + \frac{\delta^{100}}{2}$,
$\delta + \frac{\delta^{100}}{2} + \frac{\delta^{100}}{2^2}$.
$\delta + \frac{\delta^2}{2} + \frac{\delta^{100}}{2^2} + \frac{\delta^{100}}{2^3}$.
$\vdots$
Then density is always
$< \delta + \delta^{100} \sum_{i=1}^{\infty} \frac{1}{2^i} = \delta + \delta^{100}$.
If $\delta = \frac{1}{10}$ then density is always $< \frac{1}{10} + \frac{1}{10^{100}}$.
Much less than 0.67 or any number you could prove Roth's Theorem for.

How Will δ' and δ Relate

What if the δ increase as follows;

δ,

$\delta + \frac{\delta^{100}}{2}$,

$\delta + \frac{\delta^{100}}{2} + \frac{\delta^{100}}{2^2}$.

$\delta + \frac{\delta^2}{2} + \frac{\delta^{100}}{2^2} + \frac{\delta^{100}}{2^3}$.

$\vdots$

Then density is always $< \delta + \delta^{100} \sum_{i=1}^{\infty} \frac{1}{2^i} = \delta + \delta^{100}$.

If $\delta = \frac{1}{10}$ then density is always $< \frac{1}{10} + \frac{1}{10^{100}}$.

Much less than 0.67 or any number you could prove Roth's Theorem for.

It turns out that we increase δ enough so that the density goes to 1.

How Will δ' and δ Relate

We will get $\delta' = \delta + \frac{\delta}{80}$.
Let

How Will δ' and δ Relate

We will get $\delta' = \delta + \frac{\delta}{80}$.
Let
$\delta_0 = \delta$.

How Will δ' and δ Relate

We will get $\delta' = \delta + \frac{\delta}{80}$.
Let
$\delta_0 = \delta$.
$\delta_n = \delta n - 1 + \frac{\delta_{n-1}}{80} = (1 + \frac{1}{80})\delta^{n-1}$.

How Will δ' and δ Relate

We will get $\delta' = \delta + \frac{\delta}{80}$.
Let
$\delta_0 = \delta$.
$\delta_n = \delta n - 1 + \frac{\delta_{n-1}}{80} = (1 + \frac{1}{80})\delta^{n-1}$.
$\delta_n = (1 + \frac{1}{80})^L \delta^{n-L}$

How Will δ' and δ Relate

We will get $\delta' = \delta + \frac{\delta}{80}$.

Let

$\delta_0 = \delta$.

$\delta_n = \delta n - 1 + \frac{\delta_{n-1}}{80} = (1 + \frac{1}{80})\delta^{n-1}$.

$\delta_n = (1 + \frac{1}{80})^L \delta^{n-L}$

Plug in $L = n - 1$

How Will δ' and δ Relate

We will get $\delta' = \delta + \frac{\delta}{80}$.
Let
$\delta_0 = \delta$.
$\delta_n = \delta n - 1 + \frac{\delta_{n-1}}{80} = (1 + \frac{1}{80})\delta^{n-1}$.
$\delta_n = (1 + \frac{1}{80})^L \delta^{n-L}$
Plug in $L = n - 1$
$\delta_n = (1 + \frac{1}{80})^{n-1}\delta$

How Will δ' and δ Relate

We will get $\delta' = \delta + \frac{\delta}{80}$.
Let
$\delta_0 = \delta$.
$\delta_n = \delta n - 1 + \frac{\delta_{n-1}}{80} = (1 + \frac{1}{80})\delta^{n-1}$.
$\delta_n = (1 + \frac{1}{80})^L \delta^{n-L}$
Plug in $L = n - 1$
$\delta_n = (1 + \frac{1}{80})^{n-1}\delta$
Since $1 + \frac{1}{80} > 1$, δ_n goes to infty. In particular, at some point its > 0.67.

Detour:
Discrete Fourier Transform