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On the Existence of Regular n-Graphs with Given Girth

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ABSTRACT

In this paper I construct for each g, l, and $m \equiv 0$ modulo n a regular n-graph G of degree g and girth l with $m \geqslant \varphi(g, l, n)$ points, where $\varphi(g, l, n)$ is a certain function.
In [1] Erdös constructed such graphs for $n = 2$.

1. DEFINITIONS. The cardinal number of a set x is denoted by $|x|$. An *n-graph* ($n \geqslant 2$) is an ordered pair of finite sets $G = (V, \Gamma)$, with $\Gamma \subset \{x \mid x \subset V; |x| = n\}$. Elements of V are the *points* of G and elements of Γ are the *edges* of G. I use the notation: $\nu(G) = V$; $e(G) = \Gamma$.

The sequence of edges $e_1, e_2, ..., e_r$ with $e_i \in \Gamma$ and $e_i \cap e_{i+1} \neq 0$ ($1 \leqslant i \leqslant r$) is called a *way of length r* in G if:

(i) $e_i \neq e_j$ $(i \neq j)$,

(ii) $x \in e_i \Rightarrow x \notin e_j$ for $j \notin \{i-1, i, i+1\}$. ($r \geqslant 2$, $r+1$ may be identified with 1).

(iii) $e_i \cap e_j \cap e_k = 0$ $(i \neq j \neq k \neq i)$.

The points $x, y \in V$ are said to be *connected* by the way $e_1, e_2, ..., e_r$, if $x \in e$ and $y \in e$. The *distance* $\rho(x, y)$ between the points x and y is the length of the shortest way connecting x with y. If A, B are non-empty sets of points of G, we define the *distance* $\rho(A, B)$ between A and B to be the length of the shortest way connecting a point of A to a point of B.

A way $e_1, e_2, ..., e_r$ of length r ($r \geqslant 3$) with $e_1 \cap e_r \neq 0$ is called a *circuit* of length r and the length of a shortest circuit in G, the *girth* of G, is denoted by $t(G)$.

If $x \in V$ then $d(x)$, the *degree* of x, is the number of n-edges incident with x.

I denote the set of all regular n-graphs of degree g and girth l by

$$G(g, l, n) = \{G/G \text{ is an } n\text{-graph}; d(x) = g, (x \in \nu(G)), t(G) \geqslant l\}.$$

Suppose $G = (V, \Gamma)$ is an n-graph and $g - 1 \leqslant d(x) \leqslant g$ for all $x \in V$. If $p \in V$ and $d(p) = g - 1$, then

$$|\{x \in V \mid \rho(x, p) \leqslant l - 2\}| \leqslant \sum_{i=1}^{l-2} (g - 1)^i (n - 1)^i = f(g, l, n).$$

Also, if $e \in \Gamma$, then

$$|\{x \in V \mid \rho(x, e) \leqslant l - 2\}| \leqslant nf(g, l, n).$$

I define now the function $\varphi(g, l, n)$:

$$\varphi(g, l, n) = n(n - 1)(f(g, l, n) + (g - 1)^{l-2} (n - 1)^{l-1}) + 1.$$

2. *We will prove the following*:

THEOREM. *If $n \geqslant 2$, $l \geqslant 3$, $g \geqslant 1$, $m \geqslant \varphi(g, l, n)$ and $m \equiv 0 \pmod n$, then there is a graph $G \in G(g, l, n)$ so that $|\nu(G)| = m$.*

The theorem obviously holds if $g = 1$. We assume now $g > 1$ and use induction on g.

Since $m \equiv 0 \pmod n$ and $\varphi(g, l, n) > \varphi(g - 1, l, n)$, there is a graph $G_0 \in G(g - 1, l, n)$ with $\nu(G_0) = m$. Let

$$N = \{H \mid H \text{ is an } n\text{-graph}; g - 1 \leqslant d_H(x) \leqslant g (x \in \nu(H)); \; t(H) \geqslant l; |\nu(H)| = m\}.$$

Then, $N \neq 0$, since $G_0 \in N$. Therefore, there is a graph $G \in N$ so that

$$|e(G)| \geqslant |e(H)| \quad \text{for all} \quad H \in N. \tag{1}$$

To prove our theorem it is sufficient to prove that $d_G(x) = g$ for all $x \in \nu(G)$. We will assume that

$$V' = \{x \mid x \in G; d_G(x) = g - 1\} \neq \emptyset \tag{2}$$

and obtain a contradiction.

The number of distinct pairs (x, y) with $x \in \nu(G)$ and $x \in y \in e(G)$ is

$$n \, |e(G)| = mg - |V'|.$$

Therefore $|V'|$ is a multiple of n and by (2), $|V'| \geqslant n$. Let $A \subset V'$, $|A| = n$. We will show that there are $n - 1$ distinct edges

$$y_1, y_2, \ldots, y_{n-1} \in e(G)$$

such that

$$\rho(y_i, A) \geqslant l - 1, \tag{3}$$

and

$$\rho(y_i, y_j) \geqslant l - 1 \qquad (i \neq j). \tag{4}$$

Suppose there are at most r edges $y_1, y_2, ..., y_r$ which satisfy (3) and (4) and that $0 \leqslant r < n - 1$. Put

$$B = A \cup \bigcup_{1 \leqslant j \leqslant r} y_j$$

and let $C = \{x \in \nu(G)/\rho(x, B) \leqslant l - 2\}$. Then

$$| C | \leqslant (r + 1)\, nf(g, l, n).$$

Therefore, if $D = \nu(G) - C$, then

$$| D | > n(n-1)(g-1)^{l-2}\,(n-1)^{l-1}.$$

The set D contains no edge of G by the maximality condition on r.

Let $E = \{x \in \nu(G)/\rho(x, B) = l - 2\}$. Then

$$| E | \leqslant (r + 1)\, n(g-1)^{l-2}\,(n-1)^{l-2}.$$

Let $D' = \{y \in e(G)/y \cap D \neq 0\}$. Since D contains no edge of G, and the points of D are at distance at least $l - 1$ from B, it follows that if $y \in D'$, then $y \cap E \neq 0$ and $y \subset E \cap D$. Since each point of E is incident with at most $g - 1$ edges in D' it follows that

$$| D' | \leqslant (g - 1)\, | E |.$$

Also, since each point of D is incident with at least $(g - 1)$ edges of D' and every edge of D' has at most $n - 1$ points in D, we have

$$(g - 1)\, | D | \leqslant (n - 1)\, | D' |.$$

We now have the contradiction

$$n(n-1)(g-1)^{l-2}\,(n-1)^{l-1}$$
$$< | D | \leqslant (n-1)\, | E | \leqslant (n-1)\, n(g-1)^{l-2}\,(n-1)^{l-1}.$$

This proves our assertion that there are $y_1, ..., y_{n-1} \in e(G)$ so that (3) and (4) hold.

Since $A, y_1, ..., y_{n-1}$ are n disjoint sets each with n elements, there are disjoint sets $z_1, ..., z_n$ such that $| z_i | = n$ and

$$| z_i \cap A | = 1, \qquad | z_i \cap y_j | = 1 \qquad (1 \leqslant i \leqslant n;\;\; 1 \leqslant j \leqslant n - 1).$$

Consider now the graphs $G_1 = (\nu(G), \Gamma_1)$, $G_2 = (\nu(G), \Gamma_2)$, where

$$\Gamma_1 = e(G) - \{y_1, y_2, ..., y_{n-1}\}, \qquad \Gamma_2 = \Gamma_1 \cup \{z_1, ..., z_n\}.$$

Clearly G_2 is an n-graph, $|\nu(G_2)| = m$ and $d_{G_2}(x) = g - 1$ or g for $x \in \nu(G)$.

Suppose G_2 contains a circuit $e_1, e_2, ..., e_r$ of length $r < l$. Since G contains no such circuit one of the edges z_j must be included and we can assume $e_1 = z_1$. If $p \in e_1 \cap e_2$ and $q \in e_1 \cap e_r$, then $p \neq q$ and by the definition of z_1 we may assume $p \notin A$.

Since the z_j are mutually disjoint $e_2 \notin \{z_1, ..., z_n\}$, and, hence, there is some $s \leqslant r$ so that $e_2, ..., e_s \in \Gamma_1$ and the way $e_2, ..., e_s$ joins p to some other point of $A \cup y_1 \cup \cdots \cup y_n$. This is impossible by (3) and (4).

This proves that $G_2 \in N$ and, since $|e(G_2)| = |e(G)| + 1$, we have a contradiction against (1). This proves that (2) is false and hence that G is a regular graph of degree g.

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Reference

1. P. Erdös and H. Sachs, Regulare Graphen gegebener Taillenweite mit minimaler Knotenzahl, *Wiss. Z. Univ. Halle, Math.-Nat.* **12** (March 1963), 251–258.