BILL, RECORD LECTURE!!!!

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When Does a 2-Coloring Yield a Mono Unit Square?

Exposition by William Gasarch

January 23, 2025

Credit Where Credit is Due!

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Journal of Combinatorial Theory (A), Vol. 14, 341-363, 1973

`https://www.cs.umd.edu/~gasarch/TOPICS/eramsey/eramseyOne.pdf`

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Question Is there a proper 2-coloring of $\mathbb{R}^2$?

Answer Yes. We leave this for an exercise.

What About Higher Dimensions?

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Vote

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The answer is on the next slide.

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We will also have comments on the $\mathbb{R}^4$ proof.

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Let $\mathrm{COL}\colon \mathbb{R}^6 \to [2]$.

We form a coloring $\mathrm{COL}'\colon \binom{[6]}{2} \to [2]$.

We Look at 15 Points in $\mathbb{R}^6$

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Define $\mathrm{COL}'(i,j) = \mathrm{COL}(p_{i,j})$.

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Hence $d(p_{i,i+1}, p_{i+1,i+2}) = \sqrt{(\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2} = 1$.

Improvements On $\mathbb{R}^6$

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We use this in proof on next slide.

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Map those four points with f to get four points in $\mathbb{R}^5$ that form a mono unit square.

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Here is the link to the paper:

`https://www.cs.umd.edu/~gasarch/COURSES/752/S25/slides/R4square.pdf`