$\mathbf{PH(2) \le 14}$

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Def $\mathrm{PH}(2)$ is the least n such that for all 2-colorings of $\binom{\{2,\ldots,n\}}{2}$ there exists a large homog set.

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Note The graph has 13 vertices so every point has degree 12.

$\deg_{\mathbf{R}}(2) \geq 8$, so $\deg_{\mathbf{B}}(2) \leq 4$

Case 1 $\deg_{\mathbf{R}}(2) \geq 8$. Let the 8 smallest R-neighbors of 2 be $x_1 < \cdots < x_8$.

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- For all $1 \le i < j \le 8$, $\mathrm{COL}(x_i, x_j) = B$ AND $x_1 \le 7$. Large homog set: $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$.

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- For all $1 \le i < j \le 8$, $\mathrm{COL}(x_i, x_j) = B$ AND $x_1 \ge 8$. $x_2 \ge 9$, $x_3 \ge 10$, $\cdots$, $x_8 \ge 15$.

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  $x_2 \ge 9$, $x_3 \ge 10$, $\cdots$, $x_8 \ge 15$.
  Contradiction since we are coloring $\binom{\{2,\ldots,14\}}{2}$.

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For all $1 \le i < j \le 7$, $\mathrm{COL}(x_i, x_j) = B$ AND $x_1 \ge 7$.
Hence $\{x_1, \ldots, x_7\} \subseteq \{7, 8, 9, 10, 11, 12, 13, 14\}$.
Hence B neighbors of 2 are $\supseteq \{3, 4, 5, 6\}$.

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$\deg_{R}(2) \leq 5$

Case 4 $\deg_{B}(2) \leq 5$. Then $\deg_{B}(2) \geq 7$.

If $\deg_{B}(2) = 7$ use the argument used for $\deg_{R}(2) = 7$.

If $\deg_{B}(2) \geq 8$ use the argument used for $\deg_{R}(2) \geq 8$.

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Yes, but don't call me Shirley.