From Infinite Ramsey To Finite Ramsey

Exposition by William Gasarch

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3. 2^A is the powerset of A.
4. $\binom{A}{a}$ is the set of all a-sized subsets of A.

Let $\mathrm{COL}\colon \binom{A}{2} \rightarrow [2]$. A set $H \subseteq A$ is **homogenous** if COL restricted to $\binom{H}{2}$ is constant. (From now on **homog**.)

Infinite And Finite Ramsey Thm

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We will prove **The Finite Ramsey** from **The Infinite Ramsey.**

Proof of the Finite Ramsey Thm From The Infinite Ramsey Thm

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Lots of Colorings of Finite Graphs

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We use $\mathrm{COL}_0, \mathrm{COL}_1, \ldots$ to form

$\mathrm{COL}\colon \binom{\mathbb{N}}{2} \rightarrow [2]$.

We will use the inf Ramsey Theory to get a contradiction.

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Let $e_1 = (1, 2)$.

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Answer on the next slide

Forming $\mathrm{COL}(\mathbf{1}, \mathbf{2})$

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COL_0 colors $(1, 2)$ **R**

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What about e_2?

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We do the full COL on the next slide.

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Using COL To Get a Contradiction

We have defined $\mathrm{COL}\colon \binom{\mathbb{N}}{2} \to [2]$.

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$$H = \{x_1 < x_2 < x_3 < x_4 < \cdots\}$$

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Look at COL restricted to $\binom{\{x_1,\ldots,x_k\}}{2}$.

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By the construction there is an L (actually infinitely many L) such that COL and COL_L agree on $\binom{\{x_1,\ldots,x_k\}}{2}$.

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Hence there is a homog set of size k for COL_L.

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Hence there is a homog set of size k for COL_L.
This is a contradiction since COL_L has no homog sets of size k.

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STUDENT: Great! what is $R(10)$?

BILL: We showed $R(10)$ exists by showing there is SOME n such that for all $\mathrm{COL}\colon \binom{[n]}{2} \rightarrow [2]$ there is a homog set of size k.

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BILL: We showed $R(10)$ exists by showing there is SOME n such that for all $\mathrm{COL}\colon \binom{[n]}{2} \rightarrow [2]$ there is a homog set of size k.

STUDENT: Surely the proof gives an upper bound on $R(10)$!

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BILL: Then you shall have it! Next lecture!