A Variant on $R(3) = 6$

Exposition by William Gasarch

December 20, 2024

Credit Where Credit Was Due

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The questions raised in these slides are due to Paul Erdös.

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The Theorem in these slides is due to Ronald Graham.

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Terminology

Def Let $G = (V, E)$ be a graph. $\mathrm{RAM}(G, c, k)$ means that For all $\mathrm{COL}\colon E \to [c]$ there exists a k-homog set.

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We will mostly be studying $\mathrm{RAM}(G, 2, 3)$.

Vote on $\exists G$ w/o K_6, $\mathrm{RAM}(G)$ Holds

Is there a graph G such that $\mathrm{RAM}(G)$ and K_6 is NOT a subgraph.

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Unknown to Science!

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- Yes
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Answer on the next slide.

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We show the graph and prove it has property (*).

G Such That RAM and No K_6

Let $G = (V, E)$ be the graph

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$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

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Let $G = (V, E)$ be the graph

$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$E = \binom{V}{2} - \{(3,4), (4,5), (5,6), (6,7), (7,8), (8,9), (9,3)\}$

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We mostly do not draw the edges since that would be a mess.

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Exercise Show that G does not have K_6 as a subgraph.

G Such That RAM and No K_6

Let $G = (V, E)$ be the graph

$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$E = \binom{V}{2} - \{(3,4),(4,5),(5,6),(6,7),(7,8),(8,9),(9,3)\}$

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Exercise Show that G does not have K_6 as a subgraph.

We show $\mathrm{RAM}(G)$.

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Exercise Show that G does not have K_6 as a subgraph.

We show $\mathrm{RAM}(G)$.

Assume that $\exists\ \mathrm{COL}\colon \binom{E}{2} \to [2]$ has no mono triangle.

What the Coloring Looks Like

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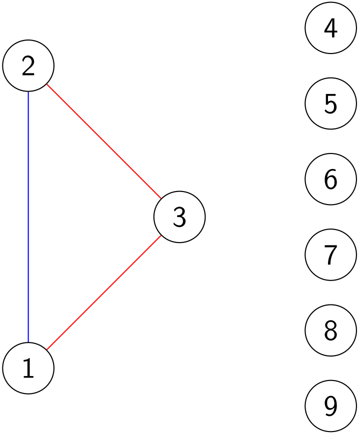

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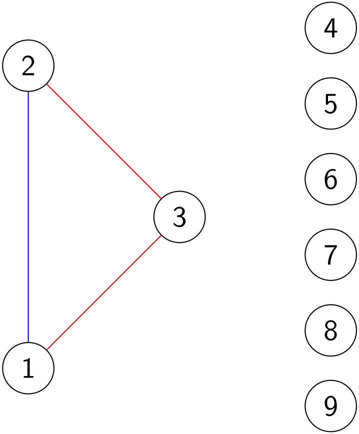

We show that, for all $4 \le i \le 9$, $\mathrm{COL}(3, i) =$ **B**.

$$\mathbf{COL}(3, i) = \mathsf{B}$$

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Assume, BWOC, $\mathrm{COL}(3, 6) = \mathbf{R}$.

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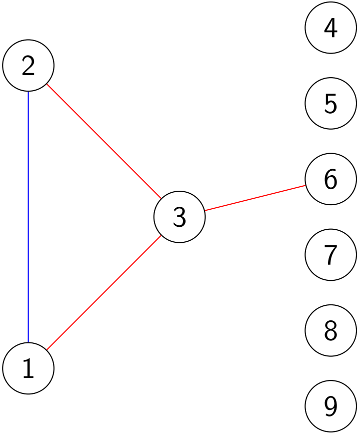

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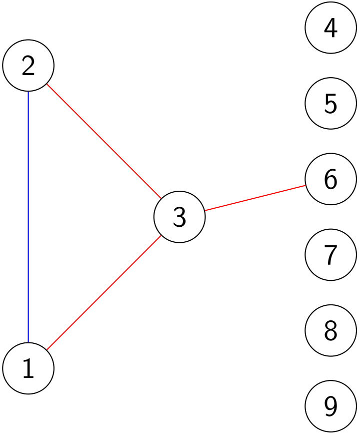

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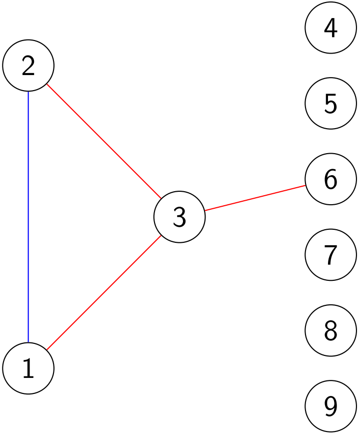

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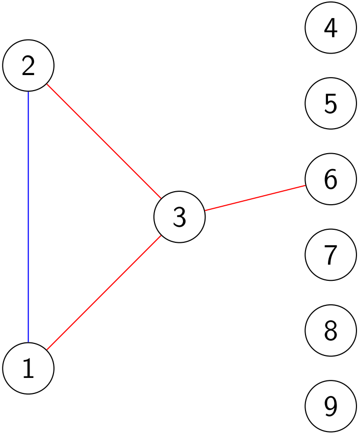

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So $1-2-6$ is a $\mathbf{B}\triangle$.

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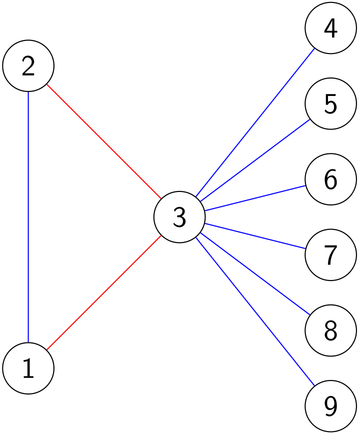

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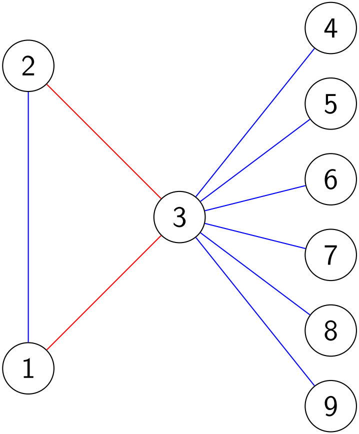

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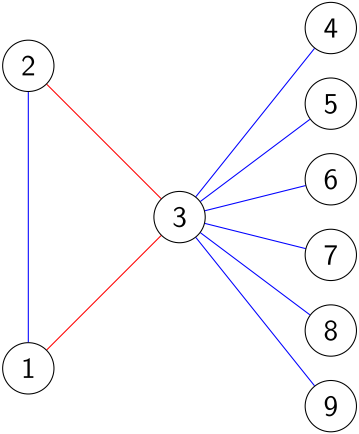

Recall that $(4, 6), (6, 8), (8, 4)$ are edges of G
They must all be **R**.

A R△

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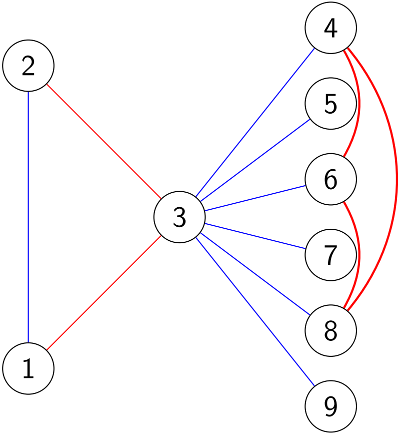

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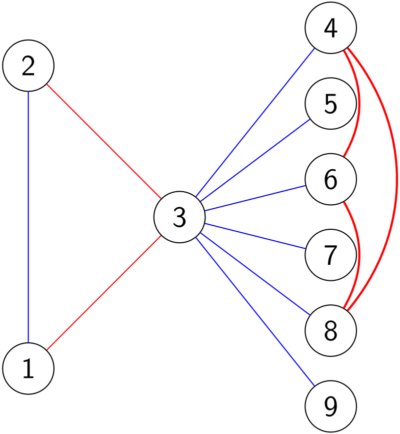

R△: 4 – 6 – 8.