On Off-diagonal F-Ramsey numbers

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Abstract

A graph is (t_1, t_2)-Ramsey if any red-blue coloring of its edges contains either a red copy of K_{t_1} or a blue copy of K_{t_2}. The size Ramsey number is the minimum number of edges contained in a (t_1, t_2)-Ramsey graph. Generalizing the notion of size Ramsey numbers, the F-Ramsey number $r_F(t_1, t_2)$ is defined to be the minimum number of copies of F in a (t_1, t_2)-Ramsey graph. It is easy to see that $r_{K_s}(t_1, t_2) \le \binom{r(t_1,t_2)}{s}$. Recently, Fox, Tidor, and Zhang showed that equality holds in this bound when $s = 3$ and $t_1 = t_2$, i.e. $r_{K_3}(t, t) = \binom{r(t,t)}{3}$. They further conjectured that $r_{K_s}(t, t) = \binom{r(t,t)}{s}$ for all $s \le t$, in response to a question of Spiro.

In this work, we study the off-diagonal variant of this conjecture: is it true that $r_{K_s}(t_1, t_2) = \binom{r(t_1,t_2)}{s}$ whenever $s \le \max(t_1, t_2)$? Harnessing the constructions used in the recent breakthrough work of Mattheus and Verstraëte on the asymptotics of $r(4, t)$, we show that when t_1 is 3 or 4, the above equality holds up to a lower order term in the exponent.

1 Introduction

We say that a graph G is (t_1, t_2)-*Ramsey* (or t-*Ramsey* if $t_1 = t_2 = t$) if every 2-coloring of its edges in red and blue results in either a red copy of K_{t_1} or a blue copy of K_{t_2}. The smallest n such that K_n is (t_1, t_2)-Ramsey is called the Ramsey number $r(t_1, t_2)$ (or $r(t)$ in the diagonal case). A closely related quantity is the *size Ramsey number* $\hat{r}(t_1, t_2)$, defined to be the minimum number of edges contained in a (t_1, t_2)-Ramsey graph. A classical observation by Chvátal gives the relationship between the usual Ramsey number and size Ramsey number for complete graphs: For any $t_1, t_2 \ge 1$,

$$\hat{r}(t_1, t_2) = \binom{r(t_1, t_2)}{2}.$$

A recent paper of Fox, Tidor, and Zhang [3] initiated the study of a new variant of the Ramsey number: For a given graph F, the F-Ramsey number $r_F(t_1, t_2)$ is the smallest number of copies of F contained in a graph that is (t_1, t_2)-Ramsey. This definition simultaneously generalizes the usual and size Ramsey numbers: taking $F = K_1$ gives the usual Ramsey number, while taking $F = K_2$ yields the size Ramsey number.

In [3], the authors addressed a natural question of Sam Spiro asking whether an analogue of Chvátal's result holds for choices of F other than K_2. They showed that

$$r_{K_3}(t, t) = \binom{r(t)}{3},$$

for all sufficiently large t. They conjectured that for all $s \le t$,

$$r_{K_s}(t, t) = \binom{r(t)}{s}, \quad \forall s \le t. \tag{1}$$

Their conjecture naturally extends to the off-diagonal setting.

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Conjecture 1.1. *For all positive integers s, t_1, t_2 satisfying $s \leq \max(t_1, t_2)$,*

$$r_{K_s}(t_1, t_2) = \binom{r(t_1, t_2)}{s}.$$

It is easy to see that the upper bound in the conjectured equality is true, since $K_{r(t_1,t_2)}$ is (t_1, t_2)-Ramsey. The conjecture is trivially true when $s = 1$, and the case $s = 2$ is Chvátal's observation. The main result of [3] corresponds to the case where $s = 3$ and $t_1 = t_2$ is sufficiently large, but the same argument shows that Conjecture 1.1 is true for $s = 3$ in general. Unfortunately, their proof technique cannot be directly extended to $s \geq 4$, and it appears that significantly different strategies would be required to tackle the conjecture in full generality.

In this paper, we investigate Conjecture 1.1 in the simplest off-diagonal cases $t_1 = 3$ and $t_1 = 4$ (the cases $t_1 \leq 2$ are trivial). In these cases, the asymptotic value of the usual Ramsey number is almost fully determined: for $r(3, t)$, results of Ajtai, Komlós, and Szemerédi [1] and of Kim [4] combine to yield $r(3, t) = \Theta(t^2/\log t)$ as $t \to \infty$; for $r(4, t)$, results of Ajtai, Komlós, and Szemerédi [1] and of Mattheus and Verstraëte [6] combine to yield

$$C \cdot \frac{t^3}{\log^4 t} \leq r(4, t) \leq (1 + o(1))\frac{t^3}{\log^2 t},$$

for some absolute constant $C > 0$ as $t \to \infty$, determining the asymptotics of $r(4, t)$ up to logarithmic factors. Using these known bounds on the usual Ramsey number, we obtain the following results indicating that Conjecture 1.1 holds in these cases up to a lower order term in the exponent.

Theorem 1.2. *For all $s \leq t$,*

$$r_{K_s}(3, t) = \binom{\Omega\left(\frac{t^2}{\log^2(t)}\right)}{s}.$$

In particular,

$$r_{K_s}(3, t) = \binom{r(3, t)}{s}^{1-o(1)}.$$

Theorem 1.3. *For all sufficiently large t and all $s \leq t$,*

$$r_{K_s}(4, t) = \binom{\Omega\left(\frac{t^3}{\log^4(t)}\right)}{s}.$$

In particular,

$$r_{K_s}(4, t) = \binom{r(4, t)}{s}^{1-o(1)}.$$

In both of these results, the implicit constant in the Ω notation is an absolute constant, and the $o(1)$ term in the exponent goes to zero as t grows, with no dependence on s.

Our results provide evidence towards Conjecture 1.1 being at least asymptotically true. In contrast to the arguments in [3], our methods work equally well for all $s \leq t$.

Our arguments utilize the construction given in [6] of a K_4-free graph H with few independent sets of size roughly t (and an analogous construction of a K_3-free graph). The key idea is to produce an edge-coloring contradicting the Ramsey property for a given graph from a random homomorphism into H, which then yields a lower bound on the number of copies of K_t in the relevant Ramsey graph. The Kruskal-Katona theorem then converts this into an appropriate lower bound on the number of copies of K_s for each $s \leq t$.

The paper is organized as follows. In Section 2, we present the proofs of our main results on off-diagonal K_s-Ramsey numbers. We first prove Theorem 1.3 in Section 2.1, then use a similar argument in Section 2.2 to prove Theorem 1.2. Finally, in Section 3, we discuss how our strategy can be applied to the more general setting of F-Ramsey numbers of (H_1, H_2)-Ramsey graphs.

2 Off-diagonal K_s-Ramsey Numbers

This section contains our proofs of the lower bounds on both $r_{K_s}(3,t)$ and $r_{K_s}(4,t)$. While the arguments for the two settings are of similar complexity, we will begin by considering the case of $(4,t)$-Ramsey graphs, because the construction of Mattheus and Verstraëte that we will utilize for this case, while somewhat more intricate than the construction for the case of $(3,t)$-Ramsey graphs, is more explicitly described in [6].

2.1 Asymptotics of $r_{K_s}(4,t)$

We use the following result of Mattheus and Verstraëte, which yields a construction of a family of K_4-free graphs with few independent sets of size t, for all sufficiently large t.

Theorem 2.1 ([6]). *For each prime power $q \geq 2^{40}$, there exists a K_4-free graph H with $q^2(q^2-q+1)$ vertices such that for every set X of at least $m = 2^{24}q^2$ vertices of H,*

$$e(H[X]) \geq \frac{|X|^2}{256q}.$$

The following result is used in [6] to give an upper bound on the number of independent sets of a fixed size $t = 2^{30}q\log^2 q$ in the graph H given by Theorem 2.1. We will use it to bound the number of independent sets of a certain range of sizes.

Proposition 2.2 ([6, Proposition 4]). *Let H be a graph on n vertices, and let $r, R \in \mathbb{N}$, and $\alpha \in [0,1]$ satisfy:*

$$e^{-\alpha r} n \leq R,$$

and, for every subset $X \subseteq V(H)$ of at least R vertices,

$$2e(X) \geq \alpha |X|^2.$$

Then for any $t \geq r$, the number of independent sets of size t in H is at most

$$\binom{n}{r}\binom{R}{t-r}.$$

We also make use of the following form of the Kruskal-Katona theorem due to Lovász.

Lemma 2.3 ([5, Exercise 31(b)]). *Given a set X and positive integers $s \leq t$, let A be a set of t-element subsets of X, and let B be the set of all s-element subsets of the sets in A. If $|A| = \binom{n}{t}$, then $|B| \geq \binom{n}{s}$.*

In the setting of graphs, the Kruskal-Katona theorem implies that if a graph contains $\binom{n}{t}$ copies of K_t, then it must contain at least $\binom{n}{s}$ copies of K_s for $s \leq t$. Now we are ready to prove Theorem 1.3

Proof of Theorem 1.3. We first prove the theorem for the case where $s = t$.

Let $q \geq 2^{40}$ be a power of 2 such that $t \in [2^{33}q\log^2 q, 2^{33}(2q)\log^2(2q)]$. Such a choice of q exists because the union of the relevant intervals contains all sufficiently large integers.

Let H be the graph given by Theorem 2.1 on $n := q^2(q^2-q+1)$ vertices. Then we can apply Proposition 2.2 with $R = 2^{24}q^2$, $r = 2^{10}q\log q$, and $\alpha = 1/(2^8 q)$ to show that, for any $t' \in [t/8, t]$, the number of independent sets of size t' in H is at most

$$\binom{n}{r}\binom{R}{t'-r} \leq n^r \binom{R}{t'} \leq q^{4r}\left(\frac{eR}{t'}\right)^{t'} \leq (q/\log^2 q)^{t'}.$$

Here we have used the fact that by our choices of t, n, R, r, t', we have $n \leq q^4$, $r \leq t' \leq R/2$, $q^{4r} \leq 2^{t'}$, and $2eR/t' \leq q/\log^2 q$.

Let G be a $(4,t)$-Ramsey graph containing exactly $N := r_{K_t}(4,t)$ copies of K_t. Our goal is to prove a lower bound on N. We color the edges of G as follows: Take a uniformly random map $\pi : V(G) \to V(H)$. For an edge $(v,w) \in G$, color it red if $(\pi(v), \pi(w)) \in E(H)$, and blue otherwise (including the case where $\pi(v) = \pi(w)$).

Since H is K_4-free by construction, there are no red K_4's in G. For any copy K of K_t in G, it can be monochromatically blue only if its image $\pi(K)$ is an independent set in H. The probability that this occurs is bounded above by

$$\begin{aligned}
\rho &:= \sum_{i=1}^{t} \mathbb{P}[\pi(K) \text{ is an independent set of size } i] \\
&\leq \sum_{i=1}^{t/8} \binom{n}{i} \left(\frac{i}{n}\right)^t + \sum_{i=t/8+1}^{t} (q/\log^2 q)^i \left(\frac{i}{n}\right)^t \\
&\leq 2n^{-(t-t/8)} t^t + 2(q/\log^2 q)^t \left(\frac{t}{n}\right)^t \\
&= O(t^t q^{-3.5t} + t^t (q^3 \log^2 q)^{-t}) \\
&= O(t^t (q^3 \log^2 q)^{-t}).
\end{aligned}$$

To show the first inequality, we trivially bound the probability that $|\pi(K)| \leq i$ for each $i \leq t/8$, then use the upper bound derived earlier for the number of independent sets of each size $i \in [t/8, t]$. The second-to-last line uses the fact that $n = \Theta(q^4)$.

The expected number of blue copies of K_t in our coloring of G is at most $N\rho$. If $N\rho < 1$, then there is such a coloring with no red K_4 or blue K_t, contradicting the fact that G is $(4,t)$-Ramsey. So, since $t = \Theta(q \log^2 q)$, and thus $q = \Theta(\frac{t}{\log^2 t})$, we have

$$N \geq \frac{1}{\rho} = \Omega\left(\left(t^{-1}\left(q^3 \log^2 q\right)\right)^t\right) = \frac{\left(\Omega(\frac{t^3}{\log^4 t})\right)^t}{t^t} = \binom{\Omega(\frac{t^3}{\log^4(t)})}{t},$$

using the estimate $\binom{n}{k} \leq (\frac{en}{k})^k$. This shows the desired lower bound on $r_{K_t}(4,t)$.

By the Kruskal-Katona theorem, since G contains $\binom{\Omega(\frac{t^3}{\log^4(t)})}{t}$ copies of K_t, it must contain at least $\binom{\Omega(\frac{t^3}{\log^4(t)})}{s}$ copies of K_s for each $s \leq t$.

Finally, since $r(4,t) \leq (1+o(1))\frac{t^3}{\log^2 t} = \left(\frac{t^3}{\log^4 t}\right)^{1+o(1)}$, we have $r_{K_s}(4,t) = \binom{\Omega(\frac{t^3}{\log^4(t)})}{s} \geq \binom{r(4,t)}{s}^{1-o(1)}$. Since $K_{r(4,t)}$ is $(4,t)$-Ramsey by definition, we have a trivial matching upper bound of $r_{K_s}(4,t) \leq \binom{r(4,t)}{s}$. This concludes the proof.

□

2.2 Asymptotics of $r_{K_s}(3,t)$

The asymptotics for $r(3,t)$ have been thoroughly studied. It is known that

$$r(3,t) = \Theta\left(\frac{t^2}{\log t}\right),$$

where the upper bound was shown by Ajtai, Komlós, and Szemerédi [1] and the lower bound by Kim [4]. This lower bound was initially proven by constructing triangle-free graphs on n vertices with chromatic number $\Omega(\sqrt{n/\log n})$ using the Rödl nibble method. However, a different construction, given by Mattheus and Verstraëte [6, Section 3] as an adaptation of their own construction for $r(4,t)$, is more suitable for our approach.

Theorem 2.4 ([6])**.** *There exists a constant $C > 0$ for which the following holds. For each prime power $q \geq 2^{40}$, there exists a triangle-free graph H with q^3+q^2+q+1 vertices such that for every set $X \subseteq V(H)$ of size $|X| \geq C \cdot q^2$,*

$$e(H[X]) \geq \frac{|X|^2}{C \cdot q}.$$

The proof of Theorem 1.2 uses the same strategy as the proof of Theorem 1.3. However, different choices of parameters are used in this proof, so we include the full proof for the sake of completeness.

Proof of Theorem 1.2. As in the proof of Theorem 1.3, we first prove the theorem in the case $s = t$. Let C be the constant from Theorem 2.4, and let $q \geq 2^{40}$ be a power of 2 such that $t \in [2^8 Cq \log^2 q, 2^8 C(2q) \log^2(2q)]$.

Let H be the graph given by Theorem 2.4 on $n := q^3 + q^2 + q + 1$ vertices. Applying Proposition 2.2 with the parameters $R = 2^5 Cq^2, r = 2Cq \log q$ and $\alpha = 1/(C \cdot q)$ gives that for any $t' \in [t/8, t]$, the number of independent sets of size t' in H is at most

$$\binom{n}{r}\binom{R}{t'-r} \leq n^r \binom{R}{t'} \leq (2q^3)^r \left(\frac{eR}{t'}\right)^{t'} \leq (q/\log^2 q)^{t'}.$$

Here we have used the facts that $n \leq 2q^3$, $r \leq t' \leq R/2$, $(2q^3)^r \leq 2^{t'}$, and $2eR/t' \leq q/\log^2 q$.

Let G be a $(3,t)$-Ramsey graph containing exactly $N := r_{K_t}(3,t)$ copies of K_t. Consider the following coloring of G: Take a uniformly random map $\pi : V(G) \to V(H)$. For an edge $(v, w) \in G$, color it red if $(\pi(v), \pi(w)) \in E(H)$, and blue otherwise (including the case where $\pi(v) = \pi(w)$).

Since H is taken to be triangle-free, there are no red triangles in G. For any copy K of K_t in G, it can be monochromatically blue only if its image $\pi(K)$ is an independent set in H. The probability that this occurs is bounded above by

$$\begin{aligned}
\rho &:= \sum_{i=1}^{t} \mathbb{P}[\pi(K) \text{ is an independent set of size } i] \\
&\leq \sum_{i=1}^{t/8} \binom{n}{i}\left(\frac{i}{n}\right)^t + \sum_{i=t/8+1}^{t} (q/\log^2 q)^i \left(\frac{i}{n}\right)^t \\
&\leq 2n^{-(t-t/8)} t^t + 2(q/\log^2 q)^t \left(\frac{t}{n}\right)^t \\
&= O(t^t q^{-21t/8} + t^t (q^2 \log^2 q)^{-t}) \\
&= O(t^t (q^2 \log^2 q)^{-t}),
\end{aligned}$$

by the same reasoning as in the proof of Theorem 1.3. Since the expected number of blue copies of K_t in this coloring is at most $N\rho$, we must have $N\rho \geq 1$ because otherwise there exists is a coloring with no red K_4 or blue K_t, contradicting the fact that G is $(3,t)$-Ramsey. So, by the same reasoning as in the proof of Theorem 1.3, we have

$$N \geq \frac{1}{\rho} = \Omega\left(\left(t^{-1}\left(q^2 \log^2 q\right)\right)^t\right) = \binom{\Omega(\frac{t^2}{\log^2(t)})}{t},$$

showing the lower bound on $r_{K_t}(3,t)$ as desired.

By the Kruskal-Katona theorem, since G contains $\binom{\Omega(\frac{t^2}{\log^2(t)})}{t}$ copies of K_t, it must contain at least $\binom{\Omega(\frac{t^2}{\log^2(t)})}{s}$ copies of K_s for each $s \leq t$. Combining this with the fact that $r(3,t) = \Theta(t^2/\log t)$, as well as the trivial upper bound $K_s(3,t) \leq \binom{r(3,t)}{s}$, we arrive at the second equation in the Theorem statement as before. This concludes the proof. □

3 Cycle-complete K_s-Ramsey numbers

The notion of (t_1, t_2)-Ramsey graphs extends to a more general setting, where monochromatic subgraphs other than cliques are considered: For any graphs H_1 and H_2, we say a graph G is (H_1, H_2)-Ramsey if every 2-coloring of its edges in red and blue results in either a red copy of H_1 or a blue copy of H_2. The definition of the F-Ramsey number can likewise be generalized by letting $r_F(H_1, H_2)$ be the smallest number of copies of F contained in an (H_1, H_2)-Ramsey graph. The diagonal case of this more general definition is briefly discussed in [3]. There, the authors point out that $r_F(H, H) = 0$ in the following two cases:

- $m_2(F) > m_2(H)$ (as shown in [8]), where we define $m_2(G) = \max_{G' \subseteq G} \frac{e(G')-1}{|G'|-2}$;
- The girth of F is smaller than the girth of H (equivalent to a result in [7]).

One can ask whether an analogue of Conjecture 1.1 holds for $r_{K_s}(H_1, H_2)$, for some choices of (H_1, H_2) other than a pair of cliques. Namely, is it true that

$$r_{K_s}(H_1, H_2) = \binom{r(H_1, H_2)}{s},$$

for some appropriate range of values of s? (Here $r(H_1, H_2)$ denotes the usual Ramsey number.) In this section, we study the case where only one of H_1, H_2 is a clique. We obtain lower bounds on $r_{K_s}(H_1, K_t)$ when $t \geq 2$ and $s \leq t$, under certain conditions on H_1.

In recent work of Conlon, Mattheus, Mubayi, and Verstraëte [2], similar techniques as in [6] are used to show lower bounds for the cycle-complete Ramsey numbers $r(C_5, K_t)$ and $r(C_7, K_t)$. They derived these bounds through a more general result conditioned on the existence of a graph with certain properties.

Before stating this result, let us recall some definitions from [2]. Given a graph F, let $\mathcal{E}(F)$ denote the set of sequences of edge-disjoint bipartite subgraphs $F_1, \dots, F_k \subseteq F$ such that the edges of the subgraphs F_i form a partition of the edges of F, each F_i has at least one edge, and each pair $F_i \neq F_j$ shares at most one vertex. Given $H = (F_1, \dots, F_k) \in \mathcal{E}(F)$, define $J(H)$ to be the bipartite graph with parts $[k]$ and $V(F)$, where each $i \in [k]$ is adjacent to the vertices $V(F_i) \subseteq V(F)$. Then let $\mathcal{L}(F) = \{J(H) : H \in \mathcal{E}(F)\} \cup \{C_4\}$. Finally, for $m \leq n$ and $a \leq b$, define an (m, n, a, b)-graph to be a bipartite graph with parts of size m and n such that every vertex in the part with size n has degree a, and every vertex in the part with size m has degree b. Now we are ready to state the main result in [2].

Theorem 3.1 ([2]). *Let F be a graph and let a, b, m, n be positive integers with $a \geq 2^{12}(\log n)^3$ such that there exists an $\mathcal{L}(F)$-free (m, n, a, b)-graph. If $t_0 = 2^8 n(\log n)^2/ab$, then*

$$r(F, K_{t_0}) = \Omega\left(\frac{bt_0}{\log n}\right).$$

Here, we obtain an analogous bound on $r_{K_s}(F, K_t)$ in terms of the lower bound in Theorem 3.1.

Theorem 3.2. *Let F be a graph and let a, b, m, n be positive integers with $a \geq 2^{12}(\log n)^3$ such that there exists an $\mathcal{L}(F)$-free (m, n, a, b)-graph. If $t = 2^{11} n(\log n)^2/ab$, then for all $s \leq t$,*

$$r_{K_s}(F, K_t) = \Omega\left(\binom{g(F, t)}{s}\right),$$

where $g(F, t) = \Omega(\frac{bt_0}{\log n})$ with $t_0 = t/8$ is the lower bound from Theorem 3.1.

Similarly to the proof of Theorem 1.3, we use the following technical result proven in [2].

Lemma 3.3 ([2, Lemma 2]). *Let F be a graph and let a, b, m, n be positive integers with $a \geq 2^{12}(\log n)^3$ such that there exists an $\mathcal{L}(F)$-free (m, n, a, b)-graph. Then there exists an F-free graph H on n vertices such that each $X \subseteq V(H)$ with $|X| \geq 2^{10}(m \log n)/a$ has*

$$e(H[X]) \geq \frac{a^2}{2^8 m}|X|^2.$$

Proof Sketch of Theorem 3.2. As before, we first prove the case where $s = t = 2^{11}n(\log n)^2/ab$. Applying Proposition 2.2 to the graph H given by Lemma 3.3 with $R = 2^{10}m(\log n)/a$, $r = t/(2^3 \log n)$, and $\alpha = \frac{a^2}{2^8 m}$, we conclude that for any $t' \in [t/8, t]$, the number of independent sets of size t' in H is at most

$$\binom{n}{r}\binom{R}{t'-r} \leq n^r \left(\frac{eR}{t'}\right)^{t'} \leq \left(\frac{2^{10}e^2 m \log n}{at'}\right)^{t'},$$

where the second inequality uses the fact that $n^r \leq n^{t'/\log n} = e^{t'}$.

Let G be an (F, K_t)-Ramsey graph and suppose that G contains N copies of K_t as subgraphs. Taking the same uniformly random map $\pi : V(G) \to V(H)$ as before, we color an edge of G red if it gets mapped to an edge in H, and blue otherwise. By the same argument and calculations as before, we can show that

$$N \geq \binom{\Omega\left(g(F, K_t)\right)}{t}.$$

Applying the Kruskal-Katona theorem then yields the desired bound on $r_{K_s}(F, K_t)$ for all $s \leq t$. □

Noting that the conditions of Theorem 3.1 are satisfied for $F = C_5$ and $F = C_7$ for appropriate values of (m, n, a, b), the authors of [2] derive the following bounds on the cycle-complete Ramsey numbers in those two cases.

Theorem 3.4 ([2]). *As $t \to \infty$,*

$$r(C_5, K_t) = \Omega\left(\frac{t^{10/7}}{(\log t)^{13/7}}\right), \quad \text{and} \quad r(C_7, K_t) = \Omega\left(\frac{t^{5/4}}{(\log t)^{3/2}}\right).$$

In the same manner, Theorem 3.2 yields the following bounds.

Theorem 3.5. *As $t \to \infty$, for every $s \leq t$ we have*

$$r_s(C_5, K_t) = \binom{\Omega\left(\frac{t^{10/7}}{(\log t)^{13/7}}\right)}{s}, \quad \text{and} \quad r_s(C_7, K_t) = \binom{\Omega\left(\frac{t^{5/4}}{(\log t)^{3/2}}\right)}{s}.$$

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