Finite Ramsey Theorem For 3-Hypergraph: Better Bounds

Exposition by William Gasarch

December 10, 2024

Credit Where Credit is Due

The main theorem in these slides, in fact not just the 3-ary case but also the *a*-ary case, appeared in
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Here is a link

`https://www.cs.umd.edu/users/gasarch/TOPICS/canramsey/ErdosRado2.pdf`

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We will do much better.

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We will later see how big we need n to be.

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Hence H is homog for COL.

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We take $n = 2^{2^{4k}}$.

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