BILL, RECORD LECTURE!!!!

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Euclidean Ramsey Theory // Chromatic Number of the Plane

Exposition by William Gasarch

January 23, 2025

Ramsey Theory VS Euclidean Ramsey Theory

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A) The objects we are coloring are **discrete**.
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B) We do not care about the geometric size. For example, the Square can be any size.
In Euclidean Ramsey Theory we will be seek an object of a certain size, for example the unit square.

For All 2-Colorings of the Plane. . .

Thm $\forall \text{ COL}: \mathbb{R}^2 \to [2]$ $\exists$ 2 points, same color, 1 inch apart.

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Proof on the next page.

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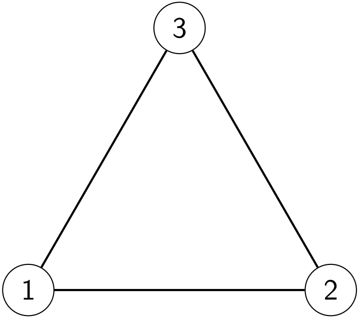

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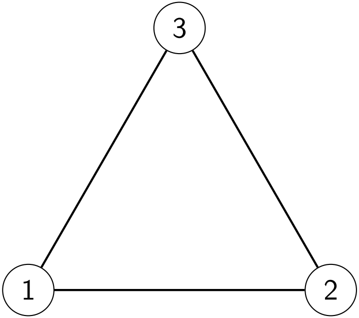

3 vertices and 2 colors. So 2 of the vertices are the same color.

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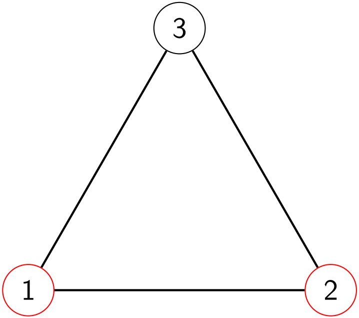

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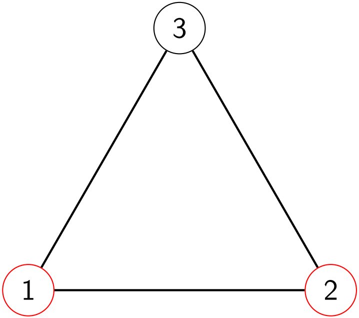

Vertices 1 and 2 are an inch apart.

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We investigate what χ can be.

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Answer on next slide

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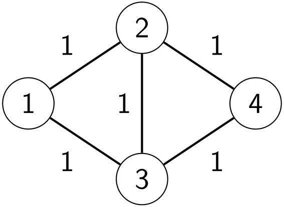

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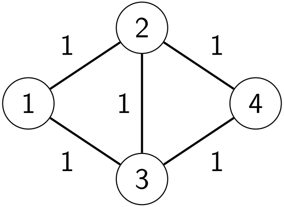

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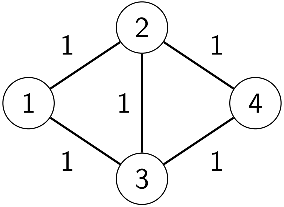

$\mathrm{COL}(1) =$ **R**, so $\mathrm{COL}(2) \neq$ **R** and $\mathrm{COL}(3) \neq$ **R**.

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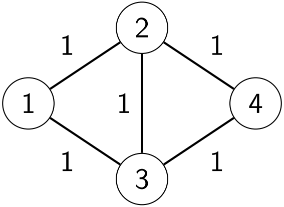

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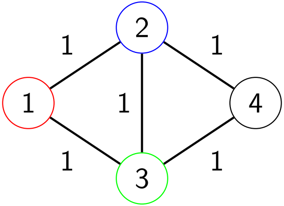

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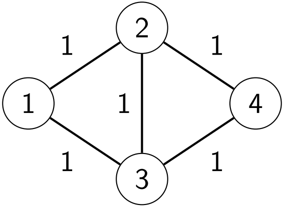

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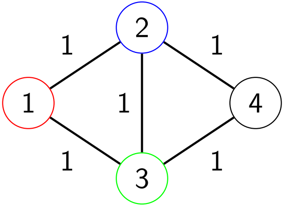

Hence $\mathrm{COL}(4) = \mathbf{R}$.

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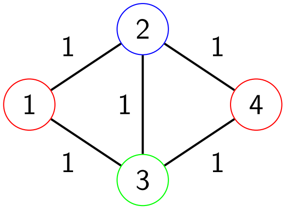

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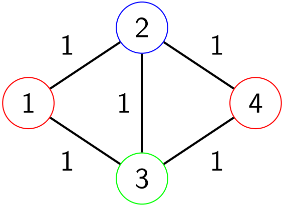

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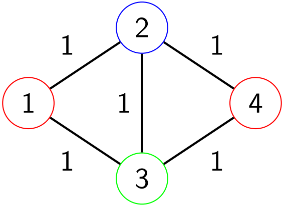

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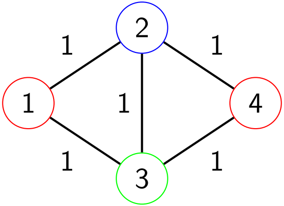

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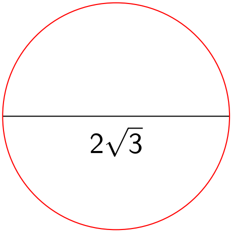

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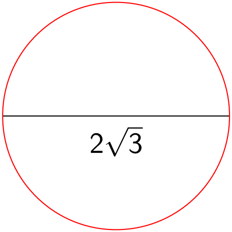

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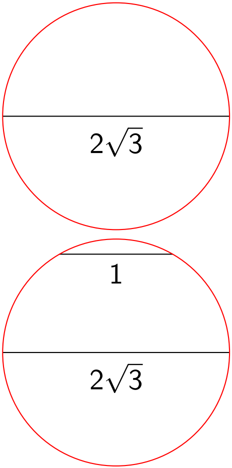

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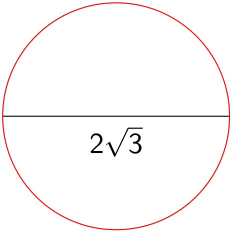

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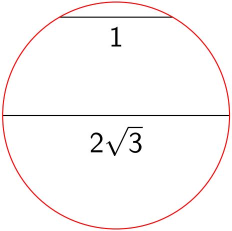

Endpoints of chord are **R** and an inch apart.

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It is a 7-vertex graph drawn in the plane with all sides of length 1.
It is an easy exercise to show that this graph is not 3-colorable.

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So we know that $\chi \geq 5$.

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Here is the 7-coloring:
`https://thatsmaths.com/2022/03/24/the-chromatic-number-of-the-plane/`