BILL, RECORD LECTURE!!!!

BILL RECORD LECTURE!!!

Euclidean Ramsey Theory: Triangles

Exposition by William Gasarch

January 23, 2025

Credit Where Credit is Due

The the main thm of these slides is due to
Paul Erdös, Ronald Graham, Peter Montgomery, Bruce L. Rothchild, Joel Spencer, Ernst G. Straus.

Credit Where Credit is Due

The the main thm of these slides is due to
Paul Erdös, Ronald Graham, Peter Montgomery, Bruce L. Rothchild, Joel Spencer, Ernst G. Straus.

Euclidean Ramsey Theorems I

Credit Where Credit is Due

The the main thm of these slides is due to
Paul Erdös, Ronald Graham, Peter Montgomery, Bruce L. Rothchild, Joel Spencer, Ernst G. Straus.

Euclidean Ramsey Theorems I

Journal of Combinaorical Theory (A), Vol. 14, 341-363, 1973

Here is a link.
https://www.cs.umd.edu/~gasarch/TOPICS/eramsey/eramseyOne.pdf

Want an Equilateral Triangle

Def A **mono eq-tri** is an **equilateral triangle** where all the vertices are the same color.

Want an Equilateral Triangle

Def A **mono eq-tri** is an **equilateral triangle** where all the vertices are the same color.

Vote

Want an Equilateral Triangle

Def A **mono eq-tri** is an **equilateral triangle** where all the vertices are the same color.

Vote

1) $\forall\ \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]\ \exists$ a mono eq-tri.

Want an Equilateral Triangle

Def A **mono eq-tri** is an **equilateral triangle** where all the vertices are the same color.

Vote

1) $\forall\ \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]\ \exists$ a mono eq-tri.

2) $\exists\ \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ such that there are no mono eq-tri.

Want an Equilateral Triangle

Def A **mono eq-tri** is an **equilateral triangle** where all the vertices are the same color.

Vote

1) $\forall\ \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ $\exists$ a mono eq-tri.

2) $\exists\ \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ such that there are no mono eq-tri.

3) Unknown to Science!

Want an Equilateral Triangle

Def A **mono eq-tri** is an **equilateral triangle** where all the vertices are the same color.

Vote

1) $\forall\ \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]\ \exists$ a mono eq-tri.

2) $\exists\ \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ such that there are no mono eq-tri.

3) Unknown to Science!

Answer on next slide

$\exists \mathrm{COL}\colon \mathbb{R}^2 \to [2]$ No Mono Eq-Tri

Thm $\exists\ \mathrm{COL}\colon \mathbb{R}^2 \to [2]$ with no mono Eq-Tri.

$\exists \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ No Mono Eq-Tri

Thm $\exists\ \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ with no mono Eq-Tri.

Leave as an exercise.

$\exists \mathrm{COL}\colon \mathbb{R}^2 \to [2]$ No Mono Eq-Tri

Thm $\exists$ $\mathrm{COL}\colon \mathbb{R}^2 \to [2]$ with no mono Eq-Tri.

Leave as an exercise.

So we can't always get a mono Eq-Tri. :-(

$\exists \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ No Mono Eq-Tri

Thm $\exists$ $\mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ with no mono Eq-Tri.

Leave as an exercise.

So we can't always get a mono Eq-Tri. :-(

How about a $2-2-2$ triangle? :-)

$\exists \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ No Mono Eq-Tri

Thm $\exists$ $\mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ with no mono Eq-Tri.

Leave as an exercise.

So we can't always get a mono Eq-Tri. :-(

How about a $2-2-2$ triangle? :-)

Thats stupid! Just scale the coloring. :-(

$\exists$COL: $\mathbb{R}^2 \rightarrow [2]$ No Mono Eq-Tri

Thm $\exists$ COL: $\mathbb{R}^2 \rightarrow [2]$ with no mono Eq-Tri.

Leave as an exercise.

So we can't always get a mono Eq-Tri. :-(

How about a $2-2-2$ triangle? :-)

Thats stupid! Just scale the coloring. :-(

New Question either a mono $1-1-1$ **or** mono $2-2-2$ **or** $\cdots$.

Three Equilateral Triangles Theorem

Three Equilateral Triangles Theorem

Let T_α be the $\alpha - \alpha - \alpha$ Eq Triangle.

Three Equilateral Triangles Theorem

Let T_α be the $\alpha - \alpha - \alpha$ Eq Triangle.

T_α is **mono** if all of the vertices are the same color.

Three Equilateral Triangles Theorem

Let T_α be the $\alpha - \alpha - \alpha$ Eq Triangle.

T_α is **mono** if all of the vertices are the same color.

Thm $\forall \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ either

Three Equilateral Triangles Theorem

Let T_α be the $\alpha - \alpha - \alpha$ Eq Triangle.

T_α is **mono** if all of the vertices are the same color.

Thm $\forall \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ either
$\exists$ a mono T_2, or

Three Equilateral Triangles Theorem

Let T_α be the $\alpha - \alpha - \alpha$ Eq Triangle.

T_α is **mono** if all of the vertices are the same color.

Thm $\forall \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ either
$\exists$ a mono T_2, or

$\exists$ a mono $T_{2\sqrt{3}}$, or

Three Equilateral Triangles Theorem

Let T_α be the $\alpha - \alpha - \alpha$ Eq Triangle.

T_α is **mono** if all of the vertices are the same color.

Thm $\forall \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ either
$\exists$ a mono T_2, or

$\exists$ a mono $T_{2\sqrt{3}}$, or

$\exists$ a mono T_4.

Three Equilateral Triangles Theorem

Let T_α be the $\alpha-\alpha-\alpha$ Eq Triangle.

T_α is **mono** if all of the vertices are the same color.

Thm $\forall \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ either
$\exists$ a mono T_2, or

$\exists$ a mono $T_{2\sqrt{3}}$, or

$\exists$ a mono T_4.

We prove this rather than $T_1 - T_{\sqrt{3}} - T_2$ since this makes the figures easier to draw.

Begin Proof of Three Equilateral Triangles Theorem

Begin Proof of Three Equilateral Triangles Theorem

Thm $\forall \mathrm{COL}\colon \mathbb{R}^2 \to [2]$ either

Begin Proof of Three Equilateral Triangles Theorem

Thm $\forall \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ either
$\exists$ a mono T_2, or

Begin Proof of Three Equilateral Triangles Theorem

Thm $\forall \mathrm{COL}\colon \mathbb{R}^2 \to [2]$ either

$\exists$ a mono T_2, or

$\exists$ a mono $T_{2\sqrt{3}}$, or

Begin Proof of Three Equilateral Triangles Theorem

Thm $\forall \mathrm{COL}\colon \mathbb{R}^2 \rightarrow [2]$ either

$\exists$ a mono T_2, or

$\exists$ a mono $T_{2\sqrt{3}}$, or

$\exists$ a mono T_4.

Begin Proof of Three Equilateral Triangles Theorem

Thm $\forall \mathrm{COL}\colon \mathbb{R}^2 \to [2]$ either
$\exists$ a mono T_2, or
$\exists$ a mono $T_{2\sqrt{3}}$, or
$\exists$ a mono T_4.

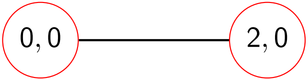
Assume by way of contradiction that there is a $\mathrm{COL}\colon \mathbb{R}^2 \to [2]$ with no mono T_2, $T_{2\sqrt{3}}$ or T_4.

There are Two **R** Points Two Apart

By Thm from last lecture $\exists$ two points, an inch apart, same color.
We can assume that (0,0) and (2,0) are **R**.

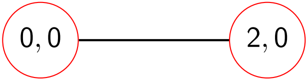
There are Two **R** Points Two Apart

By Thm from last lecture $\exists$ two points, an inch apart, same color.
We can assume that (0,0) and (2,0) are **R**.



There are Two **R** Points Two Apart

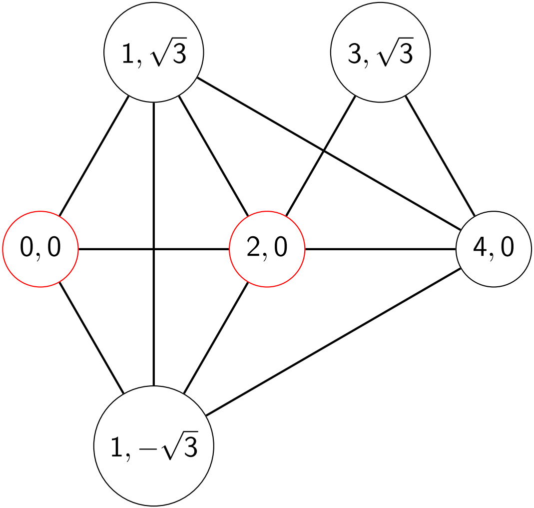
By Thm from last lecture $\exists$ two points, an inch apart, same color.
We can assume that (0,0) and (2,0) are **R**.



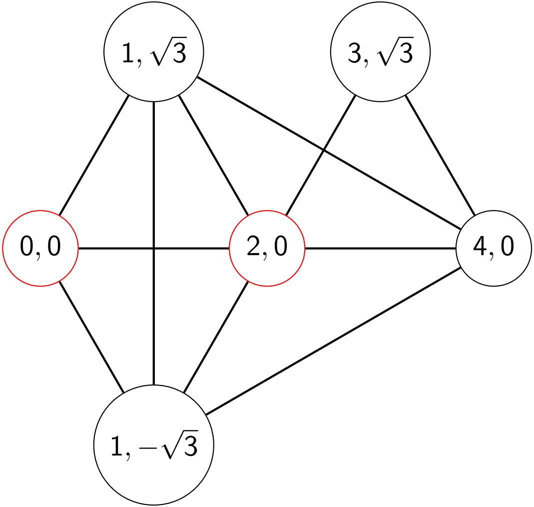
On the next slide we add four more points of interest.

Six Point of Interest

Six Point of Interest

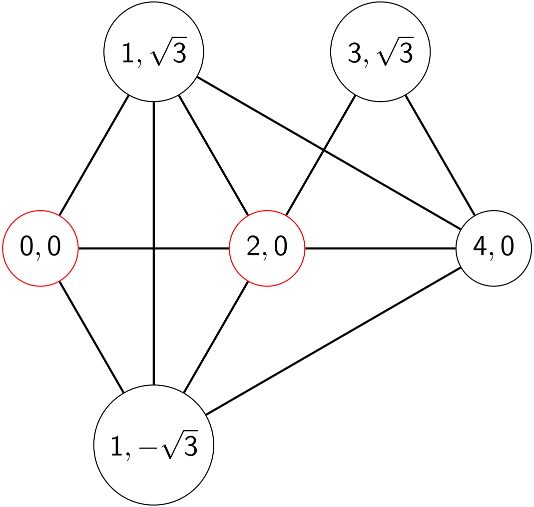


Six Point of Interest



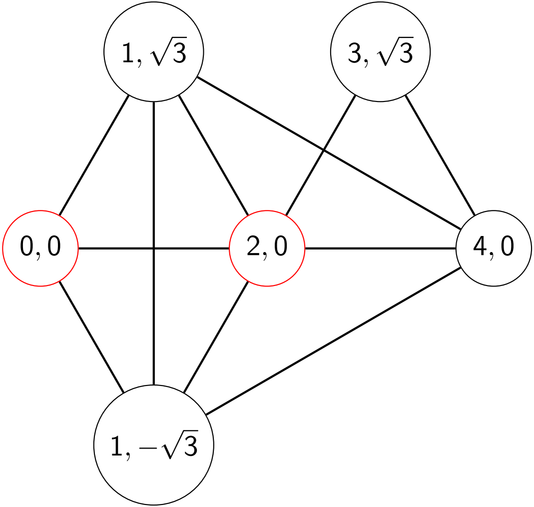
$(0,0) - (1,\sqrt{3}) - (2,0)$ is a T_2 so $\mathrm{COL}(1,\sqrt{3}) = \mathbf{B}$.

Six Point of Interest



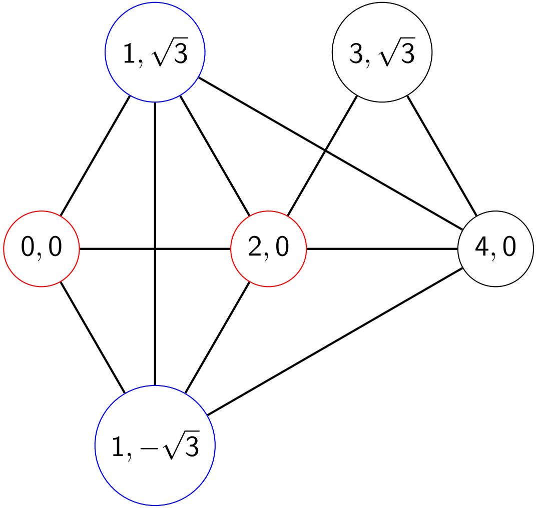
$(0,0) - (1,\sqrt{3}) - (2,0)$ is a T_2 so $\mathrm{COL}(1,\sqrt{3}) =$ **B**.
$(0,0) - (1,-\sqrt{3}) - (2,0)$ is a T_2 so $\mathrm{COL}(1,-\sqrt{3}) =$ **B**.

Six Point of Interest

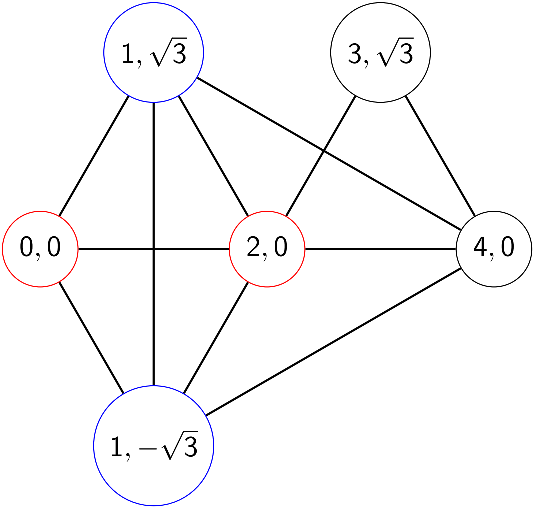


$(0,0) - (1,\sqrt{3}) - (2,0)$ is a T_2 so $\mathrm{COL}(1,\sqrt{3}) =$ **B**.
$(0,0) - (1,-\sqrt{3}) - (2,0)$ is a T_2 so $\mathrm{COL}(1,-\sqrt{3}) =$ **B**.
Next picture has this information.

$(1, \sqrt{3})$ and $(1, -\sqrt{3})$ are B

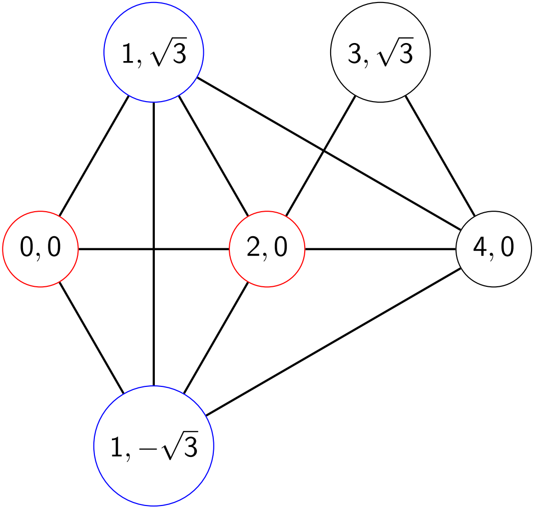


$(1, \sqrt{3})$ and $(1, -\sqrt{3})$ are B



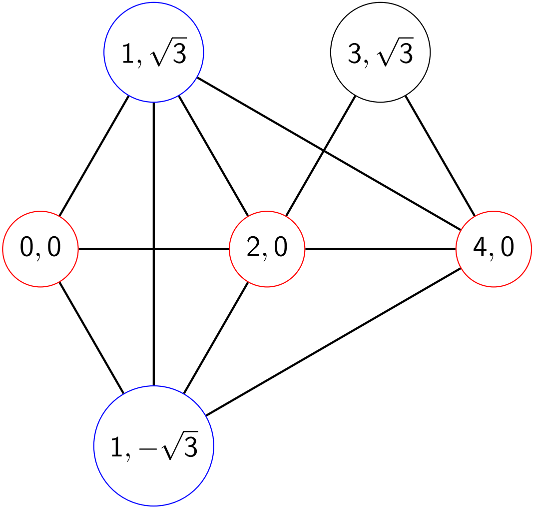
$(1, \sqrt{3}) - (1, -\sqrt{3}) - (4, 0)$ is a $T_{2\sqrt{3}}$ so $\mathrm{COL}(4, 0) =$ **R**.

$(1, \sqrt{3})$ and $(1, -\sqrt{3})$ are B

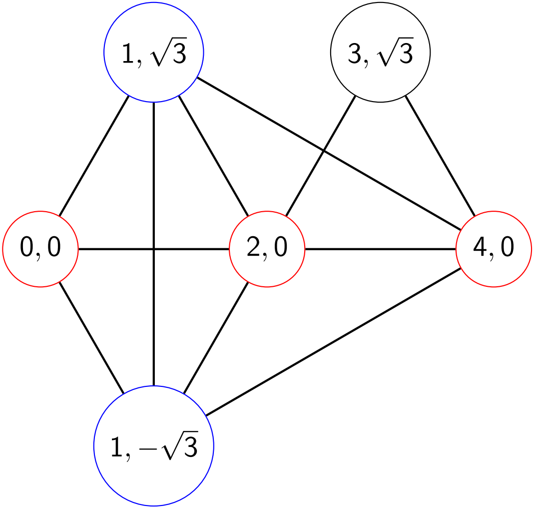


$(1, \sqrt{3}) - (1, -\sqrt{3}) - (4, 0)$ is a $T_{2\sqrt{3}}$ so $\mathrm{COL}(4, 0) =$ **R**.
Next picture has this information.

$(4, 0)$ is R

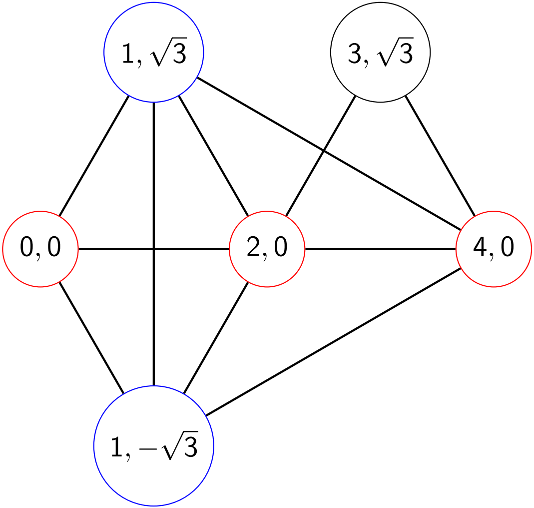


$(4,0)$ is R



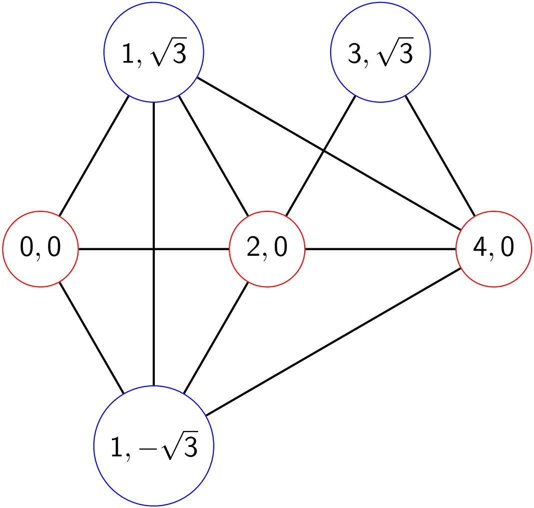
$(2,0) - (4,0) - (3,\sqrt{3})$ is a T_2 so $\mathrm{COL}(3,\sqrt{3}) =$ **B**.

$(4, 0)$ is **R**

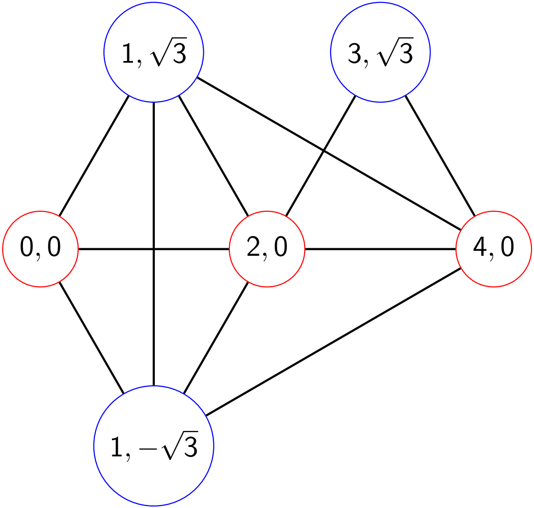


$(2, 0) - (4, 0) - (3, \sqrt{3})$ is a T_2 so $\mathrm{COL}(3, \sqrt{3}) =$ **B**.
Next picture has this info.

$(3, \sqrt{3})$ is B

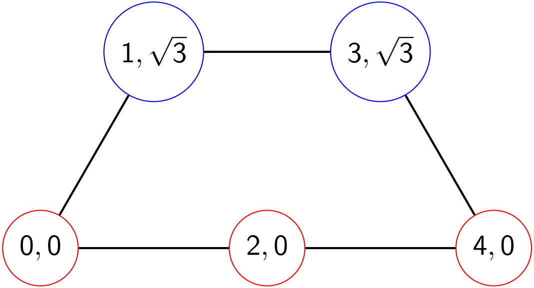


$(3, \sqrt{3})$ is B

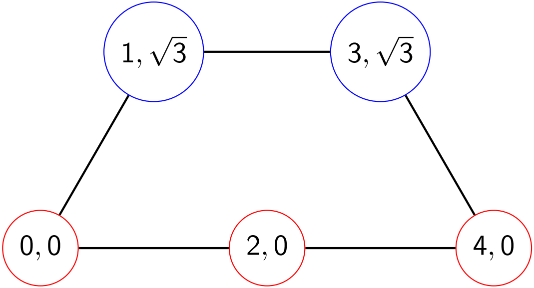


Next picture removes stuff we don't need anymore.

Where We Are Now

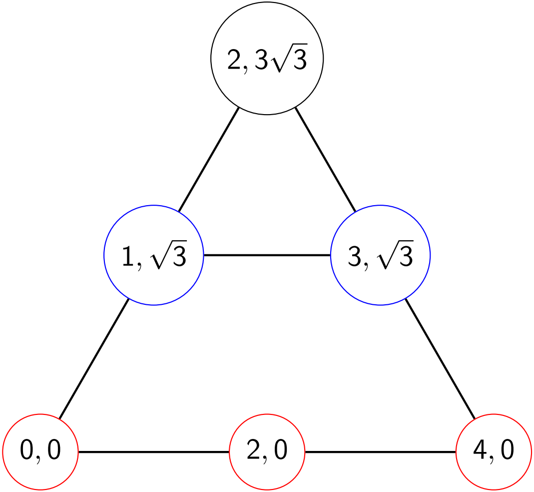


Where We Are Now

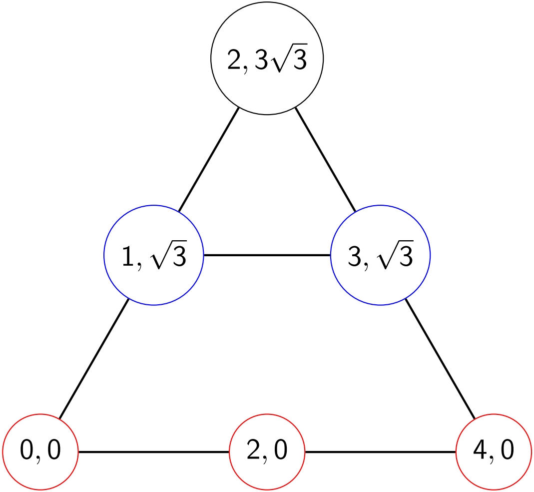


We add a the point $(2, 2\sqrt{3})$ on the next slide.

A Point That Can't be R or B

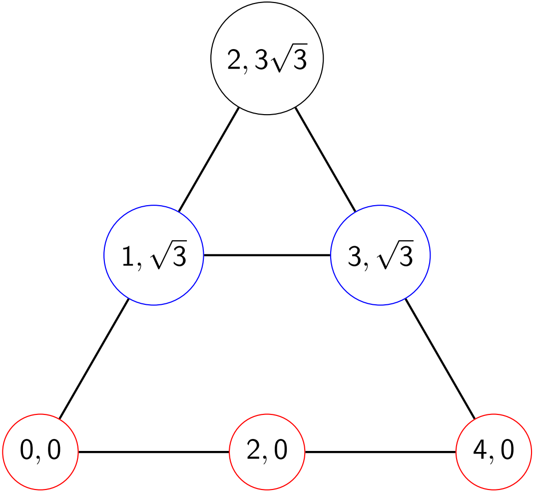


A Point That Can't be R or B



$(2, 3\sqrt{3}) - (1, \sqrt{3}) - (3, \sqrt{3})$ is a T_2 so $\mathrm{COL}(2, 3\sqrt{3}) \neq$ **B**.

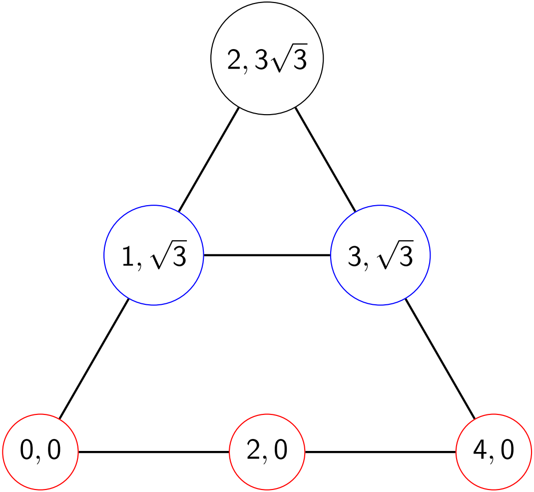
A Point That Can't be R or B



$(2, 3\sqrt{3}) - (1, \sqrt{3}) - (3, \sqrt{3})$ is a T_2 so $\mathrm{COL}(2, 3\sqrt{3}) \neq$ **B**.

$(0, 0) - (4.0) - (2, 3\sqrt{3})$ is a T_4 so $\mathrm{COL}(2, 3\sqrt{3}) \neq$ **R**.

A Point That Can't be R or B



$(2, 3\sqrt{3}) - (1, \sqrt{3}) - (3, \sqrt{3})$ is a T_2 so $\mathrm{COL}(2, 3\sqrt{3}) \neq$ **B**.

$(0, 0) - (4.0) - (2, 3\sqrt{3})$ is a T_4 so $\mathrm{COL}(2, 3\sqrt{3}) \neq$ **R**.

$\mathrm{COL}(2, 3\sqrt{3}) \notin \{$**R**, **B**$\}$. Contradiction!