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Generalized and geometric Ramsey numbers for cycles

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Abstract

Let C_n denote the cycle of length n. The generalized Ramsey number of the pair (C_n, C_k), denoted by $R(C_n, C_k)$, is the smallest positive integer R such that any complete graph with R vertices whose edges are coloured with two different colours contains either a monochromatic cycle of length n in the first colour or a monochromatic cycle of length k in the second colour. Generalized Ramsey numbers for cycles were completely determined by Faudree–Schelp and Rosta, based on earlier works of Bondy, Erdős and Gallai. Unfortunately, both proofs are quite involved and difficult to follow. In the present paper we treat this problem in a unified, self-contained and simplified way. We also extend this study to a related geometric problem, where we colour the straight-line segments determined by a finite number of points in the plane. In this case, the monochromatic subgraphs are required to satisfy an additional (non-crossing) geometric condition.

1. Introduction

Let n be a positive integer. The *Ramsey number* $R(n)$ is the smallest integer R with the property that any complete graph of at least R vertices whose edges are partitioned into two colour classes contains a monochromatic complete subgraph with n vertices. This purely graph theoretic concept has its roots in different branches of mathematics, and the theory developed from it influenced such diverse areas as number theory, ergodic theory, or theoretical computer science.

The existence of $R(n)$ was proved in a more general setting and applied to formal logic by Ramsey [19]. Even earlier Schur [21] obtained a result of similar flavour in number theory, in connection with Fermat's Last Theorem. Dilworth's classical theorem [6] is another typical example in the same spirit. The notion and existence of $R(n)$,

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together with an effective upper bound, were rediscovered and applied to geometry by Erdős and Szekeres [9]. The probabilistic proof technique, introduced by Erdős [7] to establish a lower bound on $R(n)$, is often the starting point in the analysis of randomized algorithms. For more on Ramsey theory in general, we refer to the monograph of Graham et al. [13].

For a pair of (simple, undirected) graphs (G, H), the *generalized* Ramsey number $R(G, H)$ is the smallest integer R with the property that any complete graph of at least R vertices whose edges are coloured with two colours (red and blue, say) contains either a subgraph isomorphic to G all of whose edges are red or a subgraph isomorphic to H all of whose edges are blue.

It is very difficult to determine the Ramsey numbers for complete graphs. Even their order of magnitude is unknown in general. It is easier to deal with paths, trees and cycles. See the survey paper on generalized Ramsey numbers by Burr [3] and the regularly updated unnotated bibliography by Radziszowski [18]. In this paper we study $R(G, H)$ in the case when both graphs G and H are cycles.

The results concerning (generalized) Ramsey numbers for cycles were established by Chartrand and Schuster [4] (for $k < 7$), by Bondy and Erdős [2] (for $n = k$ odd, and for the case when k is much smaller than n), and for all the remaining values by the second author [20] and by Faudree and Schelp [10] independently. These results are summarized in the following theorem.

Theorem 1.1. *Let $3 \leqslant k \leqslant n$ be integers. Then*

$$R(C_n, C_k) = \begin{cases} 6 & \text{if } k = n = 3 \text{ or } 4, \\ n + k/2 - 1 & \text{if } n, k \text{ are even}, \\ \max\{n + k/2 - 1, 2k - 1\} & \text{if } n \text{ is odd},\ k \text{ is even}, \\ 2n - 1 & \text{otherwise (i.e., if } k \text{ is odd).} \end{cases} \tag{1}$$

(For convenience, we use the notation K_n, C_n and P_n for complete graphs, cycles and paths with n vertices, respectively. P_n and C_n are often referred to as the path of length $n - 1$ (resp. the cycle of length n). The vertex set of a graph G is denoted by $V(G)$.)

All existing proofs of this theorem depend on Turán-type results due to Bondy [1] (resp. Erdős and Gallai [8]). Turán-type theorems, often referred to as density results, assert that any graph whose edge set is dense enough contains certain subgraphs: a large complete subgraph or a large cycle, for example. They can be used in a natural way to obtain upper bounds on Ramsey numbers. This phenomenon is studied in detail in a recent work by Faudree and Simonovits [11].

The above-mentioned results, however, have a certain weakness in the sense that they only can be applied to determine the generalized Ramsey numbers for cycles if the length of the smaller cycle is an odd number. Thus, instead of using these theorems we present a proof scheme which works without parity restrictions. We briefly sketch the proof in the case when k is even; the other case can be treated with some slight modifications.

Let G be a complete graph with at least as many vertices as indicated in the theorem. Assume that its edges are coloured with red and blue. If G contains a large monochromatic cycle (that is, one whose length is at least n) then it is easy to prove, based on Lemma 2.1, that there is either a blue C_k or a monochromatic C_n in G. If there is a blue C_n but no blue C_k then the vertex set of the blue C_n can be partitioned into red cycles that are connected with appropriate red edges such that a red C_n can be formed from the vertex set of the blue C_n switching from one red cycle to an other along a connecting red edge. This is worked out in Lemma 3.3. The same idea works if the largest monochromatic cycle is a blue C_L with $k \leqslant L < n$: either there is a blue C_k or we can construct a red C_L. On the other hand, if the largest monochromatic cycle is a red C_L, $L < n$, then it is shown in the proof of Lemma 3.1, how to construct a blue path of length $k-4$ which alternates between the vertex set of the red cycle and the rest of G, and how to close it to form a blue C_k. The very same argument yields a contradiction if $k > L$, making the proof complete.

In the remaining part of this section we mention a geometric analogue of the problem we studied so far. A *geometric graph* is a graph drawn in the plane so that every vertex corresponds to a point, and every edge is a closed straight-line segment connecting two vertices but not passing through a third. The $\binom{N}{2}$ segments determined by N points in the plane, no three of which are collinear, form a *complete* geometric graph with N vertices. A subgraph of a geometric graph is said to be *non-crossing*, if no two of its edges have an interior point in common.

For a pair of (planar) graphs (G,H), the *geometric* Ramsey number $R_g(G,H)$ is the smallest integer R with the property that any complete geometric graph with at least R vertices whose edges are coloured with red and blue contains either a non-crossing subgraph isomorphic to G all of whose edges are red or a non-crossing subgraph isomorphic to H all of whose edges are blue. This concept was initiated, in a more general framework, by Károlyi et al. [16].

Partial results on geometric Ramsey numbers for cycles and paths were found recently by Károlyi et al. [17]. Note that they only focused on the symmetrical case (that is, when $G = H$). They also used a Turán-type result which asserts the existence of a large non-crossing path in any dense geometric graph whose vertex set is a convex polygon. We feel that there is a close relationship between the abstract and the geometric problems which still has to be revealed. See Conjecture 5.1, for example.

With a minor modification in the proof, a result of [17] can be extended to obtain some bounds on asymmetrical geometric Ramsey numbers, as follows.

Theorem 1.2. *Let k and l be integers greater than* 2. *Then*

$$\begin{aligned}(k-1)(l-1)+1 &= R_g(C_k,P_l) \leqslant R_g(C_k,C_l)\\ &\leqslant (k-1)(l-2)+(k-2)(l-1)+2.\end{aligned}$$

This paper is organized as follows. Section 2 contains a collection of standard observations which we refer to frequently throughout the paper. Section 3 contains the proof of Theorem 1.1 based on two main lemmas which we prove separately in Section 4. Finally, we prove Theorem 1.2 and raise an open problem in Section 5.

2. Preliminary results

In this section we gather a few simple observations that go back to Bondy and Erdős [2]. For the purpose of our proof of Theorem 1.1, some of the original results of [2] are presented in a slightly extended form in the following lemma.

Lemma 2.1. *Let G be any complete graph whose edges are coloured with red and blue.*

(1) *Suppose that G contains a monochromatic C_{2l+1} for some $l \geqslant 3$. Then G also contains a monochromatic C_{2l}.*

(2) *Suppose that G contains a monochromatic C_{2l} for some $l \geqslant 3$. Then G also contains a monochromatic C_{2l-2}.*

(3) *Suppose that G contains a (monochromatic) blue cycle $C = x_1x_2 \dots x_{2l}x_1$, but does not contain any monochromatic C_{2l-1}. Then each of the complete subgraphs G_1 and G_2, induced on the vertex sets $\{x_1, x_3, \dots, x_{2l-1}\}$ (resp. $\{x_2, x_4, \dots, x_{2l}\}$), is a red K_l.*

(4) *Suppose that G contains a blue cycle $C = x_1x_2 \dots x_{2l}x_1$ such that $G_1 = \{x_1, x_3, \dots, x_{2l-1}\}$ and $G_2 = \{x_2, x_4, \dots, x_{2l}\}$ are red complete subgraphs with l vertices. Then one of the following 3 possibilities occur.*
   (i) *G contains a red C_m for each $3 \leqslant m \leqslant 2l$.*
   (ii) *G contains a blue C_k for each $3 < k \leqslant 2l+1$ resp. for each $3 \leqslant k \leqslant 2l+1$, according to the cases $l \geqslant 3$ and $l = 2$, respectively.*
   (iii) *G contains a blue C_k for each even number $4 \leqslant k \leqslant 2l$ and a red C_m for each $3 \leqslant m \leqslant \min\{\lceil |V(G)|/2 \rceil, 2l\}$.*

We include the full proof of this lemma for convenience. Before turning to the actual proof, let us make a useful note first. Let $C = x_1x_2 \dots x_tx_1$ denote any cycle. An edge of the form x_ix_{i+j} will be referred to as a chord of length j, or simply as a j-chord, of C. (Here, and throughout the paper, indices are always meant modulo the length of the cycle we consider.)

Proposition 2.2. *Let $C = x_1x_2 \dots x_tx_1$ be a monochromatic blue (red) cycle in G. Then either G contains a blue (red) C_{t-1} or every 2-chord of C is red (blue).*

Proof of Lemma 2.1.

(1) Suppose that $C = x_1x_2 \dots x_{2l+1}x_1$ is a monochromatic cycle, say blue, and there is no monochromatic C_{2l}. A repeated application of Proposition 2.2 yields that every 2-chord of C is red, and every 4-chord of C is blue. Were some 3-chord, say x_1x_4, blue, the cycle $x_1x_4x_3x_2x_6x_7 \dots x_{2l+1}x_1$ would be a blue C_{2l}. Thus, we may conclude

that every 3-chord of C is red, hence the cycle $x_1x_4x_7x_9\dots x_{2l+1}x_2x_{2l}x_{2l-2}\dots x_6x_3x_1$ is a red C_{2l}, a contradiction.

(2) If $l=3$, then the statement follows from the fact that $R(C_4,C_4)=6$, whose proof is left to the reader. Thus, assume that $l\geqslant 4$, and let $C=x_1x_2\dots x_{2l}x_1$ be a monochromatic cycle, say blue. If there is a monochromatic C_{2l-1} then we are done in view of (1). Otherwise, it follows from Proposition 2.2 that every 2-chord of C is red. Were there no blue C_{2l-2}, every 3-chord of C would be red, in which case $x_1x_{2l-1}\dots x_3x_{2l}x_{2l-2}\dots x_4x_1$ would be a red cycle of length $2l-1$, a contradiction.

(3) We have to prove that all $2j$-chords of C are red for $1\leqslant j\leqslant l/2$. This is immediate from Proposition 2.2, if $j=1$. Consider any $2j$-chord, say x_1x_{2j+1}. In each of the cycles $x_1x_{2j+1}x_{2j}x_{2j-1}\dots x_2x_{2j+3}x_{2j+4}\dots x_{2l}x_1$ and $x_1x_2\dots x_{2j-1}x_{2l}x_{2l-1}\dots x_{2j+1}x_1$, all edges, except possibly two, are blue. Since these are cycles of length $2l-1$, we can conclude that either x_1x_{2j+1} is a red edge or both edges x_2x_{2j+3} and $x_{2j-1}x_{2l}$ are red. The second possibility can be excluded by considering the $(2l-1)$-cycle $x_2x_4\dots x_{2l}x_{2j-1}x_{2j-3}\dots x_{2j+3}x_2$.

(4) If there are two independent red edges connecting G_1 and G_2, then it is easy to form a red C_m for each $3\leqslant m\leqslant 2l$. Otherwise, all the edges connecting G_1 and G_2 are blue with the possible exception of some edges that have a common vertex x. We may assume that $x\in G_1$. Note that $x\in C$, hence there are at least two blue edges connecting x to G_2. It is obvious then, how to find a blue C_k for any even number $2<k\leqslant 2l$. Let $D=V(G)\setminus V(C)$. If a vertex of D is connected to both G_1 and G_2 by some blue edges, then it is easy to find a blue C_k for any $3<k\leqslant 2l+1$, and even a blue C_3 in the special case when $l=2$. Otherwise, there is an $i\in\{1,2\}$ and a set F of at least $\lceil |D|/2\rceil$ vertices in D such that every edge connecting F with G_i is red. In this case $|F\cup G_i|\geqslant\lceil |V(G)|/2\rceil$. It is not difficult then to find a red C_m for any $3\leqslant m\leqslant\min\{\lceil |V(G)|/2\rceil, 2l\}$, making the proof complete. □

Finally, note that in parts (3) and (4) of Lemma 2.1, the roles played by the two colours can be interchanged, just like in Proposition 2.2.

3. Proof of Theorem 1.1

The proof of $R(C_3,C_3)=R(C_4,C_4,)=6$ is an easy and well-known exercise, which is left to the reader. In the sequel we only consider the remaining cases.

Let us indicate the lower bounds first. In K_{2n-2}, colour the edges of a complete bipartite graph $K_{n-1,n-1}$ with blue and all the remaining edges with red. Then it does not contain a red C_n. Moreover, it does not contain a blue C_k either, if k is odd. Similar constructions, based on the complete bipartite graphs $K_{k-1,k-1}$ and $K_{n-1,k/2-1}$, respectively, provide examples for lower bounds when n is odd (resp. when k is even).

Turning to the upper bounds, first we will assume that $k\neq 3$ if $n>4$ is an even number. *This assumption will also be made tacitly throughout Lemmas* 3.2 and 3.3.

The main technical details of the proof are contained in Lemmas 3.1 and 3.3, whose proofs we postpone until the next section.

Lemma 3.1. *Let G be a complete graph with $n+k/2-1$ vertices, where k is even and $n>4$. If we colour the edges of G with red and blue, then either there exists a monochromatic cycle of length at least n or there is a blue C_k in G. In particular, if $m>4$ is an even number, then any complete graph with $(3/2)m-1$ vertices, whose edges are coloured with two colours, contains a monochromatic cycle of length at least m.*

This lemma can be combined with Lemma 2.1 to obtain the following result.

Lemma 3.2. *Let G be any complete graph with (at least) as many vertices as it is indicated in Theorem 1.1. If we colour the edges of G with red and blue, then G contains either a monochromatic C_n or a blue C_k.*

Proof. Assume that there is no blue C_k in G. According to Lemma 3.1, there exists a monochromatic cycle of length at least n. This follows immediately from the first part of the lemma, if k is even. On the other hand, if k is odd, then $|V(G)|\geqslant 2n-1\geqslant(3/2)(n+1)-1$. In this case we can apply the second part of the lemma with either $m=n$ or $m=n+1$, according to the parity of n.

Assume now, by way of contradiction, that G does not contain any monochromatic C_n. It follows from Lemma 2.1(1) and (2) that n is odd and G contains a monochromatic cycle C of length $n+1$. Suppose first that C is blue, and apply Lemma 2.1(3) and (4) with $l=\lceil n/2\rceil$. It follows, in each possible case, that G contains either a red C_n or a blue C_k, a contradiction. We can arrive at a contradiction in a similar way if C is red: we only need to reverse the roles of the two colours when applying the lemma. □

Now the upper bounds of the theorem follow immediately from the following lemma.

Lemma 3.3. *Let G be a complete graph such that $|V(G)|\geqslant 2n-1$ if n is even and k is odd. Consider any two-colouring of its edges with red and blue. If G contains a blue C_n, then it also contains either a red C_n or a blue C_k.*

Consider finally the case when $k=3$ and $n>4$ is an even number. Let G be any complete graph with at least $2n-1$ vertices whose edges are coloured with red and blue. We have already proved that $R(C_n,C_{n-1})=2n-1$. Assume that there is no red C_n and let $C=x_1x_2\ldots x_{n-1}x_1$ be a blue C_{n-1}. If the edge x_ix_{i+2} is blue for some $1\leqslant i\leqslant n-1$, then $x_ix_{i+1}x_{i+2}x_i$ is a blue triangle and we are done. Thus, we may assume that all edges x_ix_{i+2} are red. Choose a vertex $v\in V(G)\backslash V(C)$. If both edges vx_{i-1} and vx_{i+1} were red for some i, then $x_{i-1}vx_{i+1}x_{i+3}\ldots x_{i-3}x_{i-1}$ would be a red C_n. Thus, we can conclude that there is an i such that vx_i and vx_{i+1} are blue edges, and then $vx_ix_{i+1}v$ is a blue triangle. This completes the proof of the theorem. □

4. Proof of the main lemmas

The proof of Lemma 3.1 depends on Lemma 3.3, so we prove this second lemma first.

4.1. Proof of Lemma 3.3

Let $C = x_1x_2 \dots x_nx_1$ be a blue cycle. If C has a blue chord of length $k-1$, then G contains a blue C_k. Thus, we will assume that all $(k-1)$-chords of C are red.

First, we prove the lemma under the assumption that all 2-chords of C are red, too. If n is odd, then they form a red C_n, and we are done. If both n and k are even, then the cycle $x_3x_5 \dots x_1x_kx_{k-2}x_{k-4} \dots x_{k+2}x_3$ is a red C_n. Otherwise, n is even and k is odd.

If $G_1 = \{x_1, x_3, \dots, x_{n-1}\}$ and $G_2 = \{x_2, x_4, \dots, x_n\}$ are complete red subgraphs (this is the case if, for example, $n \leqslant 6$, in particular, if $k = 3$ and $n = 4$), then we are done on account of Lemma 2.1(4). If not, then there exists a blue $2j$-chord, say x_1x_{1+2j}, for some $2 < 2j \leqslant n/2$, $2j \neq k-1$. We may assume that $n \geqslant 8$ and $k > 3$.

If there are indices γ, δ of different parity such that both edges $x_\gamma x_\delta$ and $x_{\gamma-2}x_{\delta+2}$ are red, then the cycle $x_\gamma x_{\gamma+2} \dots x_{\gamma-2}x_{\delta+2}x_{\delta+4} \dots x_\delta x_\gamma$ is a red C_n. In the sequel we assume that this is not the case.

Suppose first that $k < 2j+1$ and choose $\gamma = k-2$, $\delta = 2j+2$. Either the edge $x_\gamma x_\delta$ is blue, in which case $x_1x_2 \dots x_\gamma x_\delta x_{2j+1}x_1$ is a blue C_k, or the edge $x_{\gamma-2}x_{\delta+2}$ is blue, in which case $x_1x_2 \dots x_{\gamma-2}x_{\delta+2}x_{\delta+1}x_\delta x_{2j+1}x_1$ is a blue cycle of length k.

Assume next that $2j+1 < k < n-1$. Choose $\gamma = 2j$, $\delta = k$. Again, either $x_\gamma x_\delta$ or $x_{\gamma-2}x_{\delta+2}$ is a blue edge, and it is easy to find a blue cycle of length k.

Suppose finally that $k = n-1$. Either there is a blue C_k or, as in the proof of Lemma 2.1(3), the edges x_nx_{2j-1}, x_2x_{2j+3}, $x_{n-1}x_{2j}$ and x_3x_{2j+2} are red. Consider now any vertex $v \in D = V(G) \setminus V(C)$. If both edges vx_1 and vx_2 are blue, then either x_ix_{i+3} is a blue edge for some $2 \leqslant i \leqslant n-2$, in which case $x_1vx_2x_3 \dots x_ix_{i+3}x_{i+4} \dots x_1$ is a blue C_k, or the cycle $x_2x_n \dots x_4x_7x_9 \dots x_5x_2$ is a red C_n. Thus, we may assume that one of the edges vx_i and vx_{i+1} is red for any $v \in D$ and $1 \leqslant i \leqslant n$. Note that $|D| \geqslant n/2$. Thus, if every edge that connects C and D is red, then it is straightforward to find a red cycle of length n, alternating between C and D. Otherwise, there is a vertex $v \in D$ and an index $1 \leqslant i \leqslant n$ such that vx_i is a blue edge. In this case, vx_{i-1} and vx_{i+1} are red edges. If $i \neq 1, 2j, 2j+2$, then v can be inserted, between x_{i-1} and x_{i+1}, in the red $(n-1)$-cycle $x_2x_4 \dots x_nx_{2j-1}x_{2j-3} \dots x_{2j+3}x_2$, and a red C_n is obtained this way. Otherwise, we can do the same with the $(n-1)$-cycle $x_3x_5 \dots x_{n-1}x_{2j}x_{2j-2} \dots x_{2j+2}x_3$. This completes the proof of the lemma in the case when all 2-chords of C are red.

For the remaining part of the proof we assume that there is a blue 2-chord, say x_1x_3.

Proposition 4.1. *If x_ix_{i+2} is a blue edge, then either G contains a blue C_k or the edges $x_{i+1}x_{i+k-1}$, $x_{i+1}x_{i-k+3}$ and x_jx_{j+k} are red for $i-k+2 \leqslant j \leqslant i$.*

Introduce $d = \gcd(n, k-1)$, $n' = n/d$ and the red paths $P_j^+ = x_jx_{j+(k-1)} \dots x_{j+(n'-1)(k-1)}$ and $P_j^- = x_jx_{j-(k-1)} \dots x_{j-(n'-1)(k-1)}$. Each of the cycles $P_j^+x_j$ and $P_j^-x_j$ is a red $C_{n'}$.

These cycles are formed by $(k-1)$-chords of C, and they define a partition of the vertex set of C into d parts. In particular, if $d=1$ then $P_1^+x_1$ is a red C_n. Thus, assume that $d>1$ and there is no blue C_k in G. We have to make a slight distinction.

Suppose first that $d=k-1$. Let P^* denote the path obtained from P_{k+1}^+, omitting its last vertex x_2 and the edge incident to it. It follows from Proposition 4.1, that either $P_4^+P_5^+\ldots P_k^+P^*P_3^+x_2x_4$ is a red C_n or the edge x_2x_4 is blue. If we repeat this argument, then we eventually find a red C_n, unless all 2-chords of C are blue. In the second case, every edge of the cycle $x_3x_4x_6\ldots x_{2k}x_3$ is blue, except x_3x_{2k}, which must then be red. We can apply Proposition 4.1 again to show that $P_3^+P_4^+\ldots P_k^+P_{k+1}^-x_3$ is a red C_n in this case.

Assume finally that $1<d<k-1$. Write $\alpha=(k-1)-(d-1)$. It follows from Proposition 4.1, that either one of the cycles $P_j^+P_{j+1}^+\ldots P_{j+d-1}^+x_j$ is a red C_n or the α-chords $x_{j-\alpha}x_j$ are blue for every $2\leqslant j\leqslant k+1-(d-1)$. In the second case, define the blue paths $Q_i=x_ix_{i-\alpha}x_{i-\alpha+1}x_{i+1}$ and $R_i=x_ix_{i+1}\ldots x_nx_1$. If d is odd, then the cycle $Q_3Q_5\ldots Q_dx_{d+2}R_{d+2-\alpha}x_2x_3$ is a blue C_k, which is impossible according to the above made assumption. We arrive at a similar contradiction in the case when d is even if we consider the cycle $Q_3Q_5\ldots Q_{d+1}x_{d+3}R_{d+3-\alpha}x_3$. This means that we can always prove the existence of a red C_n if G does not contain any blue C_k, and the proof is complete. □

4.2. *Proof of Lemma 3.1*

First, we prove the first part of the lemma. Assume that the largest monochromatic cycle in G has $L\leqslant n-1$ vertices. It follows from $R(C_4,C_4)=6$ that $L\geqslant 4$ if $n>4$. Note that it is enough to establish the existence of a blue C_k under the assumption that G contains a red C_L. Indeed, assume that G contains a blue C_L. If we exchange the roles of the two colours, we obtain that G contains a red C_k, which is only possible if $k\leqslant L$. We can then apply Lemma 3.3 to prove that G contains either a red C_L or a blue C_k. Thus, from now on we will assume that G contains a red cycle $C=x_1x_2\ldots x_Lx_1$.

Let $D=V(G)\backslash V(C)$, then $|D|\geqslant k/2$. The following simple observation is an easy consequence of the fact that G contains no red cycle longer than L. Note that if the edge ux_i is red for some $u\in D$ then ux_{i-1} and ux_{i+1} are blue edges.

Proposition 4.2. *Let $u,v\in D$ and consider 4 consecutive vertices $x_i,x_{i+1},x_{i+2},x_{i+3}$ of C. There is an $i\leqslant j\leqslant i+3$ such that both edges ux_j and vx_j are blue.*

If $k=4$ then the existence of a blue C_4 follows from Proposition 4.2 with an easy case analysis. Thus, in the sequel we assume that $n\geqslant k\geqslant 6$. The central notion of the proof is the following. A *blue* path $P=u_1z_1u_2z_2\ldots u_{s-1}z_{s-1}u_s$ is said to be *alternating* if (i) $u_1,\ldots,u_s\in D$, (ii) $z_1,\ldots,z_{s-1}\in V(C)$ and if $z_i=x_j$ then $z_{i+1}\in\{x_{j+1},x_{j+2}\}$, (iii) $s\leqslant k/2-1$, and (iv) if $z_i=x_{j_i}$ then $\sum_{i=1}^{s-2}\{(j_{i+1}-j_i)\bmod L\}\leqslant L-4$. (For an integer N, $N \bmod L$ denotes the smallest non-negative integer congruent modulo L to N.)

Let $P = u_1z_1u_2z_2 \dots u_{s-1}z_{s-1}u_s$ be any maximal alternating path, that is, an alternating (blue) path of maximum length. It follows from Proposition 4.2 that $s \geqslant 2$. For notational simplicity, we will assume that $z_1 = x_1$, then P can be written as $P = u_1x_1u_2 \dots u_{s-1}x_tu_s$, where $t \leqslant L-3$. For $1 \leqslant i \leqslant 2s-2$, denote by P_{-i} the path obtained from P by deleting its last i vertices and the corresponding edges.

Lemma 4.3. *Let $P = u_1x_1u_2 \dots u_{s-1}x_tu_s$ be a maximal alternating path. Then either $s = k/2-1$ or $L-5 \leqslant t \leqslant L-3$. Moreover, if $s < k/2-1$ and $t = L-5$, then $x_{t-1} \in P$.*

Proof. Assume that $t \leqslant L-5$ and $s < k/2-1$, then there exist $y_1, y_2 \in D \backslash P$. It follows from the maximality of P that $u_sx_{t+m}y_j$ is not a blue path for $1 \leqslant j, m \leqslant 2$. Were the edge u_sx_{t+1} blue, y_jx_{t+1} would be red, y_jx_t and y_jx_{t+2} would be blue and the path $P_{-1}y_1x_{t+2}y_2$ would be an alternating (blue) path longer than P. Thus, u_sx_{t+1} is red, u_sx_{t+2} is blue, and the edges y_jx_{t+2} are red. Moreover, y_jx_t are also red edges, otherwise $P_{-1}y_1x_{t+1}y_2$ would be blue. Now it follows from the maximality of L that the edges u_sx_{t+m} are blue for $m = -1, -2, 3, 4$. In particular, $P_{-3}u_sx_tu_{s-1}$ is a (blue) alternating path as long as P, for which the whole argument can be repeated. This indicates that $u_{s-1}x_{t+m}$ are also blue edges for $m = -1, -2, 3, 4$.

It follows that $x_{t-1} \in P$, otherwise the blue path $P_{-2}x_{t-1}y_1x_{t+1}y_2$ would be an alternating path longer than P. Moreover, were $t \leqslant L-6$, the blue path $P_{-3}y_1x_{t+1}y_2x_{t+3}u_{s-1}$ would be an alternating path longer than P. This completes the proof of the lemma. □

Now the proof can be completed easily based on the following lemma.

Lemma 4.4. *Let $P = u_1x_1u_2 \dots u_{s-1}x_tu_s$ be a maximal alternating path. If $t \leqslant L-4$ then P can be altered to a blue C_{2s+2}. Similarly, if $t = L-3$, then P can be altered to either a blue C_{2s+2} or to a blue C_{2s+1}.*

Indeed, let $P = u_1x_1u_2 \dots u_{s-1}x_tu_s$ be a maximal alternating path. Note that it follows from property (ii) that $t \leqslant 2s-3$. Thus, were $t = L-3$ or $t = L-4$, we would have $L \leqslant 2s$ resp. $L \leqslant 2s+1$. Given that G does not contain any cycle longer than L, this would then contradict to Lemma 4.4. Similarly, were $t = L-5$ and $x_{t-1} \in P$, property (ii) would imply that $t-1 \leqslant 2(s-1)-3$, that is, $L \leqslant 2s+1$, and again we would arrive at a contradiction. Thus, it follows from Lemma 4.3 that $t \leqslant L-5$ and $s = k/2-1$. In view of Lemma 4.4, P can be altered to a blue cycle of length $2s+2 = k$ and we are done. To complete the proof of the first part of Lemma 3.1 it only remains to prove Lemma 4.4.

Proof of Lemma 4.4. Let $u \in D \backslash P$ and $j \in \{1, s\}$. If the edge u_ju is red then it follows from the maximality of L that there is a blue path u_jx_iu for some $t+1 \leqslant i \leqslant L$. Moreover, if $t \leqslant L-4$ then there is such a blue path, even if u_ju is blue, according to Proposition 4.2. Thus, let Q_j be one of those blue path if such a blue path exists, and let Q_j be the blue 'path' $u_j u$ otherwise. We have to distinguish 4 cases, according to the sizes of Q_1 and Q_s. In the particular case when $|Q_1|=|Q_s|=3$, the union of these two paths may have either 4 or 5 vertices. In this case it is important that Q_1 and Q_s are chosen so that $|Q_1 \cup Q_s|$ is as large as possible. For convenience, we denote by Q_1' the (blue) path obtained from Q_1 by reversing the order of its vertices.

If $|Q_1|=|Q_s|=3$ and $|Q_1 \cup Q_s|=5$ then PQ_sQ_1' is a blue C_{2s+2}. Similarly, if either $|Q_1|=3$, $|Q_s|=2$ or $|Q_1|=2$, $|Q_s|=3$ then $t=L-3$ and PQ_sQ_1' is a blue C_{2s+1}.

Suppose next that $|Q_1|=|Q_s|=2$, in this case $t=L-3$ and the path $u_s u u_1$ is blue. From the maximality of $|L|$ and $|Q_j|$ it follows that either ux_{t+1}, ux_{t+3}, $u_s x_{t+2}$ are red edges or $u_s x_{t+1}$, $u_s x_{t+3}$, ux_{t+2} are red edges. If $x_t x_{t+2}$ were red then it would be easy to construct, in either case, a red cycle which is longer than L. Thus, we can conclude that either $P_{-1}x_{t+2}u_s u u_1$ is a blue C_{2s+1} or $P_{-1}x_{t+2}uu_s x_{t+3}u_1$ is a blue C_{2s+2}, completing the proof in this case.

The only remaining case, when $|Q_1|=|Q_s|=3$ and $|Q_1 \cup Q_s|=4$, can be treated with similar arguments. However, it requires a short case analysis whose details we leave to the reader. □

The second part of Lemma 3.1 follows immediately from the first part if we choose $n=k=m$. Note that it implies a result of Gerencsér and Gyárfás [12], namely that for any 2-colouring of the edges of a complete graph with $3n-1$ vertices, there exists a monochromatic path of length $2n-1$. In fact, our proof of Lemma 3.1 can be viewed as an extension of the original idea of [12] adopted to this more difficult problem, and it is not likely that the length of the proof can essentially be reduced. □

5. Geometric Ramsey numbers

Proof of Theorem 1.2. First we prove that $R_g(C_k,P_l) \geqslant (k-1)(l-1)+1$. Indeed, take $(k-1)(l-1)$ points on a circle and partition them into $k-1$ groups, each containing $l-1$ consecutive points. Colour with red all edges between points in different groups, and colour with blue all edges between points belonging to the same group. Any red non-crossing cycle contains at most one point from each group, hence it can not have more than $k-1$ points. On the other hand, all vertices of a blue connected subgraph are from the same group, so there is no blue path with more than $l-1$ points.

Next, we show that $R_g(C_k,P_l) \leqslant (k-1)(l-1)+1$. Let $p_1, p_2, \ldots, p_{(k-1)(l-1)+1}$ be vertices of a complete geometric graph G of at least $(k-1)(l-1)+1$ vertices whose edges are coloured with red and blue. Suppose that they are listed in increasing order of their x-coordinates, which are all distinct. A path $p_{i_1}p_{i_2}\ldots p_{i_j}$ is said to be *monotone* if $i_1<i_2<\cdots<i_j$. Define a partial ordering of the vertices, as follows. Let $p_i<p_j$ if $i<j$ and there is a monotone blue path connecting p_i to p_j. By Dilworth's theorem [6], one can find either l elements that form a totally ordered subset $Q\subseteq\{p_1, p_2, \ldots, p_{(k-1)(l-1)+1}\}$ or k elements that are pairwise incomparable. In the first case, there is a monotone blue path visiting every vertex of Q. In the second case,

there is a complete red subgraph of k vertices, because any two incomparable elements are connected by a red edge. By a result of Gritzmann et al. [14], this contains a non-crossing cycle of length k.

To establish the last inequality, let P denote the vertex set of a complete geometric graph G of at least $(k-1)(l-2)+(l-1)(k-2)+2$ vertices, whose edges are coloured with red and blue. Let p be a vertex of the convex hull of P. Consider the edges incident to p, either at least $(k-1)(l-2)+1$ of them are blue, or at least $(l-1)(k-2)+1$ of them are red. Suppose, without loss of generality, that the first possibility is the case. Let $p_1, p_2, \ldots, p_{(k-1)(l-2)+1}$ be vertices of G, listed in clockwise order of visibility from p, such that each edge pp_i is blue. As in the previous proof, we say that a path $p_{i_1}p_{i_2}\ldots p_{i_j}$ is monotone if $i_1 < i_2 < \cdots < i_j$. Define a partial ordering of the vertices $p_1, p_2, \ldots, p_{(k-1)(l-2)+1}$, as follows. Let $p_i < p_j$ if $i < j$ and there is a monotone blue path connecting p_i to p_j. Applying Dilworth's theorem again, there are either $l-1$ elements that form a totally ordered subset, or k elements that are pairwise incomparable. In the first case, there is a monotone blue path $q_1q_2\ldots q_{l-1}$, and we can complete it to a non-crossing blue cycle $pq_1q_2\ldots q_{l-1}p$ of length ℓ. In the second case, there is a complete red subgraph of k vertices, and we can argue as in the previous proof. □

In fact, it is inherent in the first part of the proof, that $R_g(C_k, G_l) \geqslant (k-1)(l-1)+1$ for any connected graph G_l with l vertices. On the other hand, a result of Chvátal [5] states that $R(K_k, T_l) = (k-1)(l-1)+1$ for every fixed tree T_l on l vertices. This means, in the view of [14], that any complete geometric graph with at least $(k-1)(l-1)+1$ vertices whose edges are coloured with red and blue contains either a red non-crossing C_k or a blue T_l (whose edges may possibly cross). In particular, we have $R_g(C_k, S_l) = (k-1)(l-1)+1$, where S_l denotes the star with l vertices. Since P_l and S_l represents the two extremes among trees with l vertices, we have a good reason to believe that the following conjecture is true.

Conjecture 5.1. *Let T_l denote any (fixed) tree with l vertices. Is it true that $R_g(C_k, T_l) = (k-1)(l-1)+1$?*

Since completing this paper, it has been communicated to us that H. Harborth and H. Lefmann [15], has obtained similar results about geometric Ramsey numbers, though they restrict their study for points in convex position.

References

[1] J.A. Bondy, Large cycles in graphs, Discrete Math. 1 (1971) 121–132.
[2] J.A. Bondy, P. Erdős, Ramsey numbers for cycles in graphs, J. Combin. Theory B 14 (1973) 46–54.
[3] S.A. Burr, Generalized Ramsey theory for graphs – a survey, in: R. Bari, F. Harary (Eds.), Graphs and Combinatorics Lecture Notes in Mathematics, vol. 406, Springer, Berlin, 1974,, pp. 52–75.
[4] G. Chartrand, S. Schuster, On a variation of the Ramsey number, Trans. Amer. Math. Soc. 173 (1972) 353–362.

[5] V. Chvátal, Tree – complete graph Ramsey numbers, J. Graph Theory 1 (1977) 93.
[6] R.P. Dilworth, A decomposition theorem for partially ordered sets, Ann. Math. 51 (1950) 161–166.
[7] P. Erdős, Some remarks on the theory of graphs, Bull. Amer. Math. Soc. 53 (1947) 292–294.
[8] P. Erdős, T. Gallai, On maximal paths and circuits of graphs, Acta. Math. Acad. Sci. Hungar. 10 (1959) 337–356.
[9] P. Erdős, G. Szekeres, A combinatorial problem in geometry, Comput. Math. 2 (1935) 53–62.
[10] R.J. Faudree, R.H. Schelp, All Ramsey numbers for cycles in graphs, Discrete Math. 8 (1974) 313–329.
[11] R.J. Faudree, M. Simonovits, Ramsey problems and their connections to Turán-type extremal problems, J. Graph Theory 16 (1992) 25–50.
[12] L. Gerencsér, A. Gyárfás, On Ramsey-type problems, Ann. Univ. Sci. R. Eötvös Sect. Math. X (1967) 167–170.
[13] R.L. Graham, B.L. Rothschild, J.H. Spencer, Ramsey Theory, 2nd ed., Wiley, New York, 1990.
[14] P. Gritzmann, B. Mohar, J. Pach, R. Pollack, Embedding a planar triangulation with vertices at specified points (solution to problem E3341), Amer. Math. Monthly 98 (1991) 165–166.
[15] H. Harborth, H. Lefmann, Coloring arca of convex sets, Discrete Math. 220 (2000) 107–117.
[16] Gy. Károlyi, J. Pach, G. Tóth, Ramsey-type results for geometric graphs. I, Proc. 12th Annual ACM Symp. on Computational Geometry, Philadelphia, 1996, pp. 359–365; Discrete Comput. Geom. 18 (1997) 247–255.
[17] Gy. Károlyi, J. Pach, G. Tóth, P. Valtr, Ramsey-type results for geometric graphs. II, Proc. 13th Annu. ACM Symp. on Computational Geometry, Nice, 1997, pp. 94–103; Discrete Comput. Geom. 20 (1998) 375–388.
[18] S.P. Radziszowski, Small Ramsey numbers, Electron. J. Combin. 1 (DS1) (1994) 27.
[19] F.P. Ramsey, On a problem of formal logic, Proc. London Math. Soc. 30 (1930) 264–286.
[20] V. Rosta, On a Ramsey type problem of J.A. Bondy and P. Erdős. I–II, J. Combin. Theory B 15 (1973) 94–120.
[21] I. Schur, Über die Kongruenz $x^m + y^m \equiv z^m (\bmod p)$, Jahresber. Deutschen Math.-Verein. 25 (1916) 114–116.