BILL, RECORD LECTURE!!!!

BILL RECORD LECTURE!!!

Lower Bounds on $R(k)$

Exposition by William Gasarch

Lower Bounds on $R(k)$

Def $R(k)$ is the least n such that

Lower Bounds on $R(k)$

Def $R(k)$ is the least n such that
for all $\mathrm{COL}\colon \binom{[n]}{2} \rightarrow [2]$ there exists a homog set of size k.

Lower Bounds on $R(k)$

Def $R(k)$ is the least n such that
for all $\mathrm{COL}\colon \binom{[n]}{2} \rightarrow [2]$ there exists a homog set of size k.

We showed $R(k) \le 2^{2k-1}$.

Lower Bounds on $R(k)$

Def $R(k)$ is the least n such that
for all $\mathrm{COL}\colon \binom{[n]}{2} \rightarrow [2]$ there exists a homog set of size k.

We showed $R(k) \le 2^{2k-1}$.
We stated $R(k) \le (4-\epsilon)^k$.

Lower Bounds on $R(k)$

Def $R(k)$ is the least n such that
for all $\mathrm{COL}\colon \binom{[n]}{2} \rightarrow [2]$ there exists a homog set of size k.

We showed $R(k) \le 2^{2k-1}$.
We stated $R(k) \le (4-\epsilon)^k$.

How good are there upper bounds (UB)?

Lower Bounds on $R(k)$

Def $R(k)$ is the least n such that
for all $\mathrm{COL}\colon \binom{[n]}{2} \to [2]$ there exists a homog set of size k.

We showed $R(k) \le 2^{2k-1}$.
We stated $R(k) \le (4-\epsilon)^k$.

How good are there upper bounds (UB)?
We show lower bounds (LB).

Lower Bounds on $R(k)$

Def $R(k)$ is the least n such that
for all $\mathrm{COL}\colon \binom{[n]}{2} \rightarrow [2]$ there exists a homog set of size k.

We showed $R(k) \le 2^{2k-1}$.
We stated $R(k) \le (4-\epsilon)^k$.

How good are there upper bounds (UB)?
We show lower bounds (LB).

We compare our LBs to the UB 2^{2k-1} for convenience.

How to Show A Lower Bounds

To show that $R(k) \geq f(k)$ we need to construct a coloring

How to Show A Lower Bounds

To show that $R(k) \geq f(k)$ we need to construct a coloring

$\mathrm{COL}\colon \binom{f(k)}{2} \rightarrow [2]$ such that there is no homog set of size k.

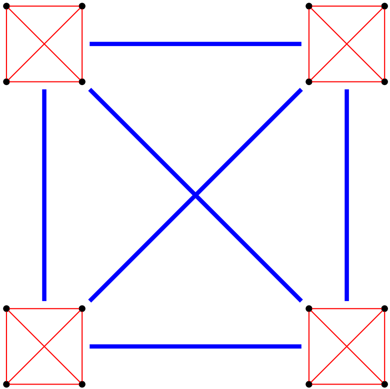
$R(k)) \geq (k-1)^2$

Thm $R(k) \geq (k-1)^2$.

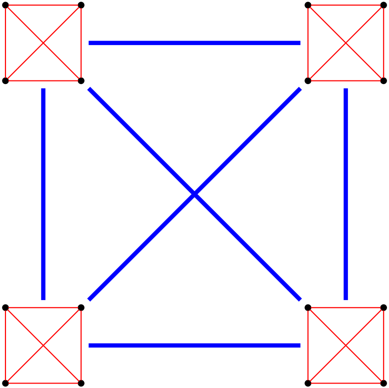
$R(k)) \geq (k-1)^2$

Thm $R(k) \geq (k-1)^2$.
We first give an example, on the next slide.

Example: The $k = 5$ Case

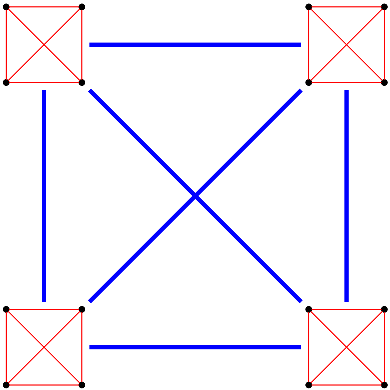


Example: The $k = 5$ Case



The thick **blue** lines between two K_4's, X and Y, means that there is a blue edge between **every** pair $\{x, y\}$ with $x \in X$ and $y \in Y$.

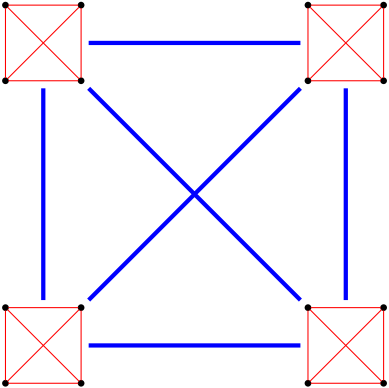
Example: The $k = 5$ Case



The thick **blue** lines between two K_4's, X and Y, means that there is a blue edge between **every** pair $\{x, y\}$ with $x \in X$ and $y \in Y$.

$4 \times 4 = 16$ vertices.

Example: The $k = 5$ Case



The thick **blue** lines between two K_4's, X and Y, means that there is a blue edge between **every** pair $\{x, y\}$ with $x \in X$ and $y \in Y$.

$4 \times 4 = 16$ vertices. No mono K_5.

General Case

General Case

Here is a coloring of the edges of $K_{(k-1)^2}$ with no mono K_k:

General Case

Here is a coloring of the edges of $K_{(k-1)^2}$ with no mono K_k:

First partition $[(k-1)^2]$ into $k-1$ groups of $k-1$ each.

General Case

Here is a coloring of the edges of $K_{(k-1)^2}$ with no mono K_k:
First partition $[(k-1)^2]$ into $k-1$ groups of $k-1$ each.

$$COL(x,y) = \begin{cases} \textbf{RED} & \text{if } x,y \text{ are in same } V_i \\ \textbf{BLUE} & \text{if } x,y \text{ are in different } V_i \end{cases} \quad (1)$$

General Case

Here is a coloring of the edges of $K_{(k-1)^2}$ with no mono K_k:

First partition $[(k-1)^2]$ into $k-1$ groups of $k-1$ each.

$$COL(x,y) = \begin{cases} \textbf{RED} & \text{if } x, y \text{ are in same } V_i \\ \textbf{BLUE} & \text{if } x, y \text{ are in different } V_i \end{cases} \quad (1)$$

Look at any k vertices.

General Case

Here is a coloring of the edges of $K_{(k-1)^2}$ with no mono K_k:
First partition $[(k-1)^2]$ into $k-1$ groups of $k-1$ each.

$$COL(x,y) = \begin{cases} \textbf{RED} & \text{if } x,y \text{ are in same } V_i \\ \textbf{BLUE} & \text{if } x,y \text{ are in different } V_i \end{cases} \tag{1}$$

Look at any k vertices.

- They can't all be in one V_i, so it can't have RED K_k.

General Case

Here is a coloring of the edges of $K_{(k-1)^2}$ with no mono K_k:
First partition $[(k-1)^2]$ into $k-1$ groups of $k-1$ each.

$$COL(x,y) = \begin{cases} \textbf{RED} & \text{if } x,y \text{ are in same } V_i \\ \textbf{BLUE} & \text{if } x,y \text{ are in different } V_i \end{cases} \tag{1}$$

Look at any k vertices.

- They can't all be in one V_i, so it can't have RED K_k.
- They can't all be in different V_i, so it can't have BLUE K_k.

Compare Upper and Lower Bound

Compare Upper and Lower Bound

So we have

$$k^2 - 2k + 1 \leq R(k) \leq 2^{2k-1}$$

Compare Upper and Lower Bound

So we have

$$k^2 - 2k + 1 \leq R(k) \leq 2^{2k-1}$$

The upper and lower bounds are far apart.

Compare Upper and Lower Bound

So we have

$$k^2 - 2k + 1 \leq R(k) \leq 2^{2k-1}$$

The upper and lower bounds are far apart.

We will do better!

We Show

$$R(k) \geq \Omega(k^3)$$

$R(k) \geq \Omega(k^3)$

$R(k) \geq \Omega(k^3)$

1) In 1975 Nagy showed

$R(k) \geq \Omega(k^3)$

1) In 1975 Nagy showed

Thm Let $k \geq 16$. There exists a graph (V, E) such that:

$R(k) \geq \Omega(k^3)$

1) In 1975 Nagy showed

Thm Let $k \geq 16$. There exists a graph (V, E) such that:

$|V| = \binom{k}{3}$,

$R(k) \ge \Omega(k^3)$

1) In 1975 Nagy showed
Thm Let $k \ge 16$. There exists a graph (V, E) such that:
$|V| = \binom{k}{3}$,
G has no clique of size $\ge \frac{k-1}{2}$,

$R(k) \geq \Omega(k^3)$

1) In 1975 Nagy showed
Thm Let $k \geq 16$. There exists a graph (V, E) such that:
$|V| = \binom{k}{3}$,
G has no clique of size $\geq \frac{k-1}{2}$,
G has no ind. set of size $\geq \frac{k}{2}$.

$R(k) \geq \Omega(k^3)$

1) In 1975 Nagy showed

Thm Let $k \geq 16$. There exists a graph (V, E) such that:
$|V| = \binom{k}{3}$,
G has no clique of size $\geq \frac{k-1}{2}$,
G has no ind. set of size $\geq \frac{k}{2}$.

Hence $R(k) \geq \Omega(k^3)$.

$R(k) \geq \Omega(k^3)$

1) In 1975 Nagy showed
Thm Let $k \geq 16$. There exists a graph (V, E) such that:
$|V| = \binom{k}{3}$,
G has no clique of size $\geq \frac{k-1}{2}$,
G has no ind. set of size $\geq \frac{k}{2}$.

Hence $R(k) \geq \Omega(k^3)$.
Note

$R(k) \geq \Omega(k^3)$

1) In 1975 Nagy showed

Thm Let $k \geq 16$. There exists a graph (V, E) such that:

$|V| = \binom{k}{3}$,

G has no clique of size $\geq \frac{k-1}{2}$,

G has no ind. set of size $\geq \frac{k}{2}$.

Hence $R(k) \geq \Omega(k^3)$.

Note

The proof is elementary. **Yeah!**

$R(k) \geq \Omega(k^3)$

1) In 1975 Nagy showed

Thm Let $k \geq 16$. There exists a graph (V, E) such that:
$|V| = \binom{k}{3}$,
G has no clique of size $\geq \frac{k-1}{2}$,
G has no ind. set of size $\geq \frac{k}{2}$.

Hence $R(k) \geq \Omega(k^3)$.

Note

The proof is elementary. **Yeah!**

The paper is in Hungarian.

$R(k) \geq \Omega(k^3)$

1) In 1975 Nagy showed

Thm Let $k \geq 16$. There exists a graph (V, E) such that:
$|V| = \binom{k}{3}$,
G has no clique of size $\geq \frac{k-1}{2}$,
G has no ind. set of size $\geq \frac{k}{2}$.

Hence $R(k) \geq \Omega(k^3)$.

Note

The proof is elementary. **Yeah!**
The paper is in Hungarian. And not online.

$R(k) \geq \Omega(k^3)$

1) In 1975 Nagy showed

Thm Let $k \geq 16$. There exists a graph (V, E) such that:

$|V| = \binom{k}{3}$,

G has no clique of size $\geq \frac{k-1}{2}$,

G has no ind. set of size $\geq \frac{k}{2}$.

Hence $R(k) \geq \Omega(k^3)$.

Note

The proof is elementary. **Yeah!**

The paper is in Hungarian. And not online. **Boo!**

$R(k) \geq \Omega(k^3)$

1) In 1975 Nagy showed
Thm Let $k \geq 16$. There exists a graph (V, E) such that:
$|V| = \binom{k}{3}$,
G has no clique of size $\geq \frac{k-1}{2}$,
G has no ind. set of size $\geq \frac{k}{2}$.

Hence $R(k) \geq \Omega(k^3)$.
Note
The proof is elementary. **Yeah!**
The paper is in Hungarian. And not online. **Boo!**
I found a brief sketch of it on the web.

$R(k) \geq \Omega(k^3)$

1) In 1975 Nagy showed
Thm Let $k \geq 16$. There exists a graph (V, E) such that:
$|V| = \binom{k}{3}$,
G has no clique of size $\geq \frac{k-1}{2}$,
G has no ind. set of size $\geq \frac{k}{2}$.

Hence $R(k) \geq \Omega(k^3)$.
Note
The proof is elementary. **Yeah!**
The paper is in Hungarian. And not online. **Boo!**
I found a brief sketch of it on the web. **Yeah!**

$R(k) \geq \Omega(k^3)$

1) In 1975 Nagy showed
Thm Let $k \geq 16$. There exists a graph (V, E) such that:
$|V| = \binom{k}{3}$,
G has no clique of size $\geq \frac{k-1}{2}$,
G has no ind. set of size $\geq \frac{k}{2}$.

Hence $R(k) \geq \Omega(k^3)$.
Note
The proof is elementary. **Yeah!**
The paper is in Hungarian. And not online. **Boo!**
I found a brief sketch of it on the web. **Yeah!**
The only English written version is

$R(k) \ge \Omega(k^3)$

1) In 1975 Nagy showed

Thm Let $k \ge 16$. There exists a graph (V, E) such that:

$|V| = \binom{k}{3}$,

G has no clique of size $\ge \frac{k-1}{2}$,

G has no ind. set of size $\ge \frac{k}{2}$.

Hence $R(k) \ge \Omega(k^3)$.

Note

The proof is elementary. **Yeah!**

The paper is in Hungarian. And not online. **Boo!**

I found a brief sketch of it on the web. **Yeah!**

The only English written version is on these slides!

$R(k) \ge \Omega(k^3)$

1) In 1975 Nagy showed

Thm Let $k \ge 16$. There exists a graph (V, E) such that:

$|V| = \binom{k}{3}$,

G has no clique of size $\ge \frac{k-1}{2}$,

G has no ind. set of size $\ge \frac{k}{2}$.

Hence $R(k) \ge \Omega(k^3)$.

Note

The proof is elementary. **Yeah!**

The paper is in Hungarian. And not online. **Boo!**

I found a brief sketch of it on the web. **Yeah!**

The only English written version is on these slides! **Really!**

$R(k) \ge \Omega(k^3)$

1) In 1975 Nagy showed

Thm Let $k \ge 16$. There exists a graph (V, E) such that:
$|V| = \binom{k}{3}$,
G has no clique of size $\ge \frac{k-1}{2}$,
G has no ind. set of size $\ge \frac{k}{2}$.

Hence $R(k) \ge \Omega(k^3)$.

Note

The proof is elementary. **Yeah!**
The paper is in Hungarian. And not online. **Boo!**
I found a brief sketch of it on the web. **Yeah!**
The only English written version is on these slides! **Really!**
I am working on a survey of lower bounds with Rishi.

$R(k) \ge \Omega(k^3)$

1) In 1975 Nagy showed

Thm Let $k \ge 16$. There exists a graph (V, E) such that:

$|V| = \binom{k}{3}$,

G has no clique of size $\ge \frac{k-1}{2}$,

G has no ind. set of size $\ge \frac{k}{2}$.

Hence $R(k) \ge \Omega(k^3)$.

Note

The proof is elementary. **Yeah!**

The paper is in Hungarian. And not online. **Boo!**

I found a brief sketch of it on the web. **Yeah!**

The only English written version is on these slides! **Really!**

I am working on a survey of lower bounds with Rishi. **Yeah!**

The Graph

Let $G = (V, E)$ where

The Graph

Let $G = (V, E)$ where
$V = \binom{[k]}{3}$.

The Graph

Let $G = (V, E)$ where
$V = \binom{[k]}{3}$.
$E = \{(A, B) \colon |A \cap B| = 1\}$.

The Graph

Let $G = (V, E)$ where
$V = \binom{[k]}{3}$.
$E = \{(A, B) \colon |A \cap B| = 1\}$.
We show that G has no $\geq \frac{k-1}{2}$-clique and no $\geq \frac{k}{2}$-ind. set.

The Largest Clique. Case 1

Let C be a clique in G. We show $|C| \leq \frac{k-1}{2}$.

The Largest Clique. Case 1

Let C be a clique in G. We show $|C| \leq \frac{k-1}{2}$.

Case 1 $\exists w \in \{1, \ldots, k\}$ that appears in 4 vertices of C.

The Largest Clique. Case 1

Let C be a clique in G. We show $|C| \leq \frac{k-1}{2}$.
Case 1 $\exists w \in \{1,\ldots,k\}$ that appears in 4 vertices of C.
Renumber: $\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\}$. (so $w = 1$).

The Largest Clique. Case 1

Let C be a clique in G. We show $|C| \leq \frac{k-1}{2}$.
Case 1 $\exists w \in \{1,\ldots,k\}$ that appears in 4 vertices of C.
Renumber: $\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\}$. (so $w=1$).
Claim 1 Let $\{x,y,z\}$ be another vertex of C. Then $1 \in \{x,y,z\}$.

The Largest Clique. Case 1

Let C be a clique in G. We show $|C| \leq \frac{k-1}{2}$.
Case 1 $\exists w \in \{1, \ldots, k\}$ that appears in 4 vertices of C.
Renumber: $\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\}$. (so $w = 1$).
Claim 1 Let $\{x,y,z\}$ be another vertex of C. Then $1 \in \{x,y,z\}$.
Assume, by way of contraction, that $1 \notin \{x,y,z\}$.

The Largest Clique. Case 1

Let C be a clique in G. We show $|C| \le \frac{k-1}{2}$.
Case 1 $\exists w \in \{1, \ldots, k\}$ that appears in 4 vertices of C.
Renumber: $\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{1,8,9\}$. (so $w = 1$).
Claim 1 Let $\{x, y, z\}$ be another vertex of C. Then $1 \in \{x, y, z\}$.
Assume, by way of contraction, that $1 \notin \{x, y, z\}$.
Since $|\{x, y, z\} \cap \{1, 2, 3\}| = 1$, either 2 or 3 is in $\{x, y, z\}$.

The Largest Clique. Case 1

Let C be a clique in G. We show $|C| \leq \frac{k-1}{2}$.
Case 1 $\exists w \in \{1, \ldots, k\}$ that appears in 4 vertices of C.
Renumber: $\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{1,8,9\}$. (so $w = 1$).
Claim 1 Let $\{x, y, z\}$ be another vertex of C. Then $1 \in \{x, y, z\}$.
Assume, by way of contraction, that $1 \notin \{x, y, z\}$.
Since $|\{x, y, z\} \cap \{1, 2, 3\}| = 1$, either 2 or 3 is in $\{x, y, z\}$.
Since $|\{x, y, z\} \cap \{1, 4, 5\}| = 1$, either 4 or 5 is in $\{x, y, z\}$.

The Largest Clique. Case 1

Let C be a clique in G. We show $|C| \leq \frac{k-1}{2}$.
Case 1 $\exists w \in \{1, \ldots, k\}$ that appears in 4 vertices of C.
Renumber: $\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{1,8,9\}$. (so $w = 1$).
Claim 1 Let $\{x, y, z\}$ be another vertex of C. Then $1 \in \{x, y, z\}$.
Assume, by way of contraction, that $1 \notin \{x, y, z\}$.
Since $|\{x, y, z\} \cap \{1,2,3\}| = 1$, either 2 or 3 is in $\{x, y, z\}$.
Since $|\{x, y, z\} \cap \{1,4,5\}| = 1$, either 4 or 5 is in $\{x, y, z\}$.
Since $|\{x, y, z\} \cap \{1,6,7\}| = 1$, either 6 or 7 is in $\{x, y, z\}$.

The Largest Clique. Case 1

Let C be a clique in G. We show $|C| \leq \frac{k-1}{2}$.
Case 1 $\exists w \in \{1, \ldots, k\}$ that appears in 4 vertices of C.
Renumber: $\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\}$. (so $w = 1$).
Claim 1 Let $\{x,y,z\}$ be another vertex of C. Then $1 \in \{x,y,z\}$.
Assume, by way of contraction, that $1 \notin \{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,2,3\}| = 1$, either 2 or 3 is in $\{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,4,5\}| = 1$, either 4 or 5 is in $\{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,6,7\}| = 1$, either 6 or 7 is in $\{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,8,9\}| = 1$, either 8 or 9 is in $\{x,y,z\}$.

The Largest Clique. Case 1

Let C be a clique in G. We show $|C| \leq \frac{k-1}{2}$.
Case 1 $\exists w \in \{1, \ldots, k\}$ that appears in 4 vertices of C.
Renumber: $\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\}$. (so $w = 1$).
Claim 1 Let $\{x,y,z\}$ be another vertex of C. Then $1 \in \{x,y,z\}$.
Assume, by way of contraction, that $1 \notin \{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,2,3\}| = 1$, either 2 or 3 is in $\{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,4,5\}| = 1$, either 4 or 5 is in $\{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,6,7\}| = 1$, either 6 or 7 is in $\{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,8,9\}| = 1$, either 8 or 9 is in $\{x,y,z\}$.
Hence $|\{x,y,z\}| \geq 4$ which is impossible.

The Largest Clique. Case 1

Let C be a clique in G. We show $|C| \leq \frac{k-1}{2}$.
Case 1 $\exists w \in \{1, \ldots, k\}$ that appears in 4 vertices of C.
Renumber: $\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\}$. (so $w=1$).
Claim 1 Let $\{x,y,z\}$ be another vertex of C. Then $1 \in \{x,y,z\}$.
Assume, by way of contraction, that $1 \notin \{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,2,3\}| = 1$, either 2 or 3 is in $\{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,4,5\}| = 1$, either 4 or 5 is in $\{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,6,7\}| = 1$, either 6 or 7 is in $\{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,8,9\}| = 1$, either 8 or 9 is in $\{x,y,z\}$.
Hence $|\{x,y,z\}| \geq 4$ which is impossible.

We assume k is odd (k even, proof similar). Renumber.

The Largest Clique. Case 1

Let C be a clique in G. We show $|C| \le \frac{k-1}{2}$.
Case 1 $\exists w \in \{1,\ldots,k\}$ that appears in 4 vertices of C.
Renumber: $\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\}$. (so $w=1$).
Claim 1 Let $\{x,y,z\}$ be another vertex of C. Then $1 \in \{x,y,z\}$.
Assume, by way of contraction, that $1 \notin \{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,2,3\}| = 1$, either 2 or 3 is in $\{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,4,5\}| = 1$, either 4 or 5 is in $\{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,6,7\}| = 1$, either 6 or 7 is in $\{x,y,z\}$.
Since $|\{x,y,z\} \cap \{1,8,9\}| = 1$, either 8 or 9 is in $\{x,y,z\}$.
Hence $|\{x,y,z\}| \ge 4$ which is impossible.

We assume k is odd (k even, proof similar). Renumber.
$C = \{\{1,2,3\},\{1,4,5\},\ldots,\{1,k-1,k\}\}$.

The Largest Clique. Case 1

Let C be a clique in G. We show $|C| \le \frac{k-1}{2}$.
Case 1 $\exists w \in \{1, \ldots, k\}$ that appears in 4 vertices of C.
Renumber: $\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{1,8,9\}$. (so $w = 1$).
Claim 1 Let $\{x, y, z\}$ be another vertex of C. Then $1 \in \{x, y, z\}$.
Assume, by way of contraction, that $1 \notin \{x, y, z\}$.
Since $|\{x, y, z\} \cap \{1,2,3\}| = 1$, either 2 or 3 is in $\{x, y, z\}$.
Since $|\{x, y, z\} \cap \{1,4,5\}| = 1$, either 4 or 5 is in $\{x, y, z\}$.
Since $|\{x, y, z\} \cap \{1,6,7\}| = 1$, either 6 or 7 is in $\{x, y, z\}$.
Since $|\{x, y, z\} \cap \{1,8,9\}| = 1$, either 8 or 9 is in $\{x, y, z\}$.
Hence $|\{x, y, z\}| \ge 4$ which is impossible.

We assume k is odd (k even, proof similar). Renumber.
$C = \{\{1,2,3\}, \{1,4,5\}, \ldots, \{1, k-1, k\}\}$. $|C| = \frac{k-1}{2}$.

The Largest Clique. Case 2

Case 2: $\forall w$, w appears in at most 3 vertices of C.

The Largest Clique. Case 2

Case 2: $\forall w$, w appears in at most 3 vertices of C.
Let $\{1, 2, 3\}$ be a vertex of C.
Every neighbor of $\{1, 2, 3\}$ in C must have either 1 or 2 or 3 in it.

The Largest Clique. Case 2

Case 2: $\forall w$, w appears in at most 3 vertices of C.
Let $\{1, 2, 3\}$ be a vertex of C.
Every neighbor of $\{1, 2, 3\}$ in C must have either 1 or 2 or 3 in it.
At most 2 neighbors have 1 in it.

The Largest Clique. Case 2

Case 2: $\forall w$, w appears in at most 3 vertices of C.
Let $\{1, 2, 3\}$ be a vertex of C.
Every neighbor of $\{1, 2, 3\}$ in C must have either 1 or 2 or 3 in it.
At most 2 neighbors have 1 in it.
At most 2 neighbors have 2 in it.

The Largest Clique. Case 2

Case 2: $\forall w$, w appears in at most 3 vertices of C.
Let $\{1, 2, 3\}$ be a vertex of C.
Every neighbor of $\{1, 2, 3\}$ in C must have either 1 or 2 or 3 in it.
At most 2 neighbors have 1 in it.
At most 2 neighbors have 2 in it.
At most 2 neighbors have 3 in it.

The Largest Clique. Case 2

Case 2: $\forall w$, w appears in at most 3 vertices of C.
Let $\{1, 2, 3\}$ be a vertex of C.
Every neighbor of $\{1, 2, 3\}$ in C must have either 1 or 2 or 3 in it.
At most 2 neighbors have 1 in it.
At most 2 neighbors have 2 in it.
At most 2 neighbors have 3 in it.
Hence $|N(\{1, 2, 3\})| \leq 6$.

The Largest Clique. Case 2

Case 2: $\forall w$, w appears in at most 3 vertices of C.
Let $\{1, 2, 3\}$ be a vertex of C.
Every neighbor of $\{1, 2, 3\}$ in C must have either 1 or 2 or 3 in it.
At most 2 neighbors have 1 in it.
At most 2 neighbors have 2 in it.
At most 2 neighbors have 3 in it.
Hence $|N(\{1, 2, 3\})| \leq 6$.

Since C is a clique and every vertex Taking $\{1, 2, 3\}$ and its neighbors there are only 7 vertices in the $|C|$.

The Largest Clique. Case 2

Case 2: $\forall w$, w appears in at most 3 vertices of C.
Let $\{1, 2, 3\}$ be a vertex of C.
Every neighbor of $\{1, 2, 3\}$ in C must have either 1 or 2 or 3 in it.
At most 2 neighbors have 1 in it.
At most 2 neighbors have 2 in it.
At most 2 neighbors have 3 in it.
Hence $|N(\{1, 2, 3\})| \leq 6$.

Since C is a clique and every vertex Taking $\{1, 2, 3\}$ and its neighbors there are only 7 vertices in the $|C|$.

One can show that $k \leq 15$ contrary to hypothesis.

The Largest Ind. Set

The Largest Ind. Set

Let I be a maximum sized ind. set in G.

The Largest Ind. Set

Let I be a maximum sized ind. set in G. We show $|I| \leq \frac{k}{2}$.

The Largest Ind. Set

Let I be a maximum sized ind. set in G. We show $|I| \leq \frac{k}{2}$.

We give an example of an ind. set for $k = 17$. We write it as three clusters of vertices to illustrate how we classify the vertices.

The Largest Ind. Set

Let I be a maximum sized ind. set in G. We show $|I| \leq \frac{k}{2}$.

We give an example of an ind. set for $k = 17$. We write it as three clusters of vertices to illustrate how we classify the vertices.

I) $\{1, 2, 3\}$, $\{1, 2, 4\}$, ..., $\{1, 2, 17\}$. (Do not confuse this **I** with the name of the ind. set which we call I.)

The Largest Ind. Set

Let I be a maximum sized ind. set in G. We show $|I| \leq \frac{k}{2}$.

We give an example of an ind. set for $k = 17$. We write it as three clusters of vertices to illustrate how we classify the vertices.

I) $\{1, 2, 3\}$, $\{1, 2, 4\}$, ..., $\{1, 2, 17\}$. (Do not confuse this **I** with the name of the ind. set which we call I.)

II) $\{8, 9, 10\}$, $\{8, 9, 11\}$, $\{8, 10, 11\}$, $\{9, 10, 11\}$.

The Largest Ind. Set

Let I be a maximum sized ind. set in G. We show $|I| \leq \frac{k}{2}$.

We give an example of an ind. set for $k = 17$. We write it as three clusters of vertices to illustrate how we classify the vertices.

I) $\{1, 2, 3\}$, $\{1, 2, 4\}$, ..., $\{1, 2, 17\}$. (Do not confuse this **I** with the name of the ind. set which we call I.)

II) $\{8, 9, 10\}$, $\{8, 9, 11\}$, $\{8, 10, 11\}$, $\{9, 10, 11\}$.

III) $\{12, 13, 14\}$, $\{15, 16, 17\}$.

Types I Clusters

Types I Clusters

Type I Clusters

A set of vertices such that

Types I Clusters

Type I Clusters

A set of vertices such that

(1) every pair has intersection of size 2, and

Types I Clusters

Type I Clusters

A set of vertices such that
(1) every pair has intersection of size 2, and
(2) any three of them have an intersection of size 2.

Types I Clusters

Type I Clusters

A set of vertices such that
(1) every pair has intersection of size 2, and
(2) any three of them have an intersection of size 2.
(Equiv: $\exists x, y$, the cluster is all vertices of the form $\{x, y, z\}$.)

Types I Clusters

Type I Clusters
A set of vertices such that
(1) every pair has intersection of size 2, and
(2) any three of them have an intersection of size 2.
(Equiv: $\exists x, y$, the cluster is all vertices of the form $\{x, y, z\}$.)
Example $k = 17$ (which matters here)

Types I Clusters

Type I Clusters
A set of vertices such that
(1) every pair has intersection of size 2, and
(2) any three of them have an intersection of size 2.
(Equiv: $\exists x, y$, the cluster is all vertices of the form $\{x, y, z\}$.)
Example $k = 17$ (which matters here)
$\{1, 2, 3\}$, $\{1, 2, 4\}$, ..., $\{1, 2, 17\}$.

Types I Clusters

Type I Clusters
A set of vertices such that
(1) every pair has intersection of size 2, and
(2) any three of them have an intersection of size 2.
(Equiv: $\exists x, y$, the cluster is all vertices of the form $\{x, y, z\}$.)
Example $k = 17$ (which matters here)
$\{1, 2, 3\}$, $\{1, 2, 4\}$, ..., $\{1, 2, 17\}$.
Facts left to the reader to prove.

Types I Clusters

Type I Clusters
A set of vertices such that
(1) every pair has intersection of size 2, and
(2) any three of them have an intersection of size 2.
(Equiv: $\exists x, y$, the cluster is all vertices of the form $\{x, y, z\}$.)
Example $k = 17$ (which matters here)
$\{1, 2, 3\}$, $\{1, 2, 4\}$, ..., $\{1, 2, 17\}$.
Facts left to the reader to prove.
Assume C is a type I cluster.

Types I Clusters

Type I Clusters
A set of vertices such that
(1) every pair has intersection of size 2, and
(2) any three of them have an intersection of size 2.
(Equiv: $\exists x, y$, the cluster is all vertices of the form $\{x, y, z\}$.)
Example $k = 17$ (which matters here)
$\{1, 2, 3\}$, $\{1, 2, 4\}$, . . ., $\{1, 2, 17\}$.
Facts left to the reader to prove.
Assume C is a type I cluster.
1) If $|C| = s$ then the union of the vertices has $s + 2$ numbers.

Types I Clusters

Type I Clusters
A set of vertices such that
(1) every pair has intersection of size 2, and
(2) any three of them have an intersection of size 2.
(Equiv: $\exists x, y$, the cluster is all vertices of the form $\{x, y, z\}$.)
Example $k = 17$ (which matters here)
$\{1, 2, 3\}$, $\{1, 2, 4\}$, ..., $\{1, 2, 17\}$.
Facts left to the reader to prove.
Assume C is a type I cluster.
1) If $|C| = s$ then the union of the vertices has $s + 2$ numbers.
2) If $v \notin C$ then for all $v' \in C$, $v \cap v' = \emptyset$.

Type II Clusters

Type II Clusters

Type II Clusters A set of vertices such that

Type II Clusters

Type II Clusters A set of vertices such that
(1) every pair has intersection of size 2, and

Type II Clusters

Type II Clusters A set of vertices such that
(1) every pair has intersection of size 2, and
(2) no three have an intersection of size 2.

Type II Clusters

Type II Clusters A set of vertices such that
(1) every pair has intersection of size 2, and
(2) no three have an intersection of size 2.
Example $k = 17$ (which does not matter here)

Type II Clusters

Type II Clusters A set of vertices such that
(1) every pair has intersection of size 2, and
(2) no three have an intersection of size 2.
Example $k = 17$ (which does not matter here)
$\{8, 9, 10\}$, $\{8, 9, 11\}$, $\{8, 10, 11\}$, $\{9, 10, 11\}$.

Type II Clusters

Type II Clusters A set of vertices such that
(1) every pair has intersection of size 2, and
(2) no three have an intersection of size 2.
Example $k = 17$ (which does not matter here)
$\{8, 9, 10\}$, $\{8, 9, 11\}$, $\{8, 10, 11\}$, $\{9, 10, 11\}$.
Facts left to the reader to prove.

Type II Clusters

Type II Clusters A set of vertices such that
(1) every pair has intersection of size 2, and
(2) no three have an intersection of size 2.
Example $k = 17$ (which does not matter here)
$\{8, 9, 10\}$, $\{8, 9, 11\}$, $\{8, 10, 11\}$, $\{9, 10, 11\}$.
Facts left to the reader to prove.
Assume C is a type II cluster.

Type II Clusters

Type II Clusters A set of vertices such that
(1) every pair has intersection of size 2, and
(2) no three have an intersection of size 2.
Example $k = 17$ (which does not matter here)
$\{8, 9, 10\}$, $\{8, 9, 11\}$, $\{8, 10, 11\}$, $\{9, 10, 11\}$.
Facts left to the reader to prove.
Assume C is a type II cluster.
1) The number of vertices in C is ≤ 4.

Type II Clusters

Type II Clusters A set of vertices such that
(1) every pair has intersection of size 2, and
(2) no three have an intersection of size 2.
Example $k = 17$ (which does not matter here)
$\{8, 9, 10\}$, $\{8, 9, 11\}$, $\{8, 10, 11\}$, $\{9, 10, 11\}$.
Facts left to the reader to prove.
Assume C is a type II cluster.
1) The number of vertices in C is ≤ 4.
2) The union of the vertices in C has ≤ 4 numbers.

Type II Clusters

Type II Clusters A set of vertices such that
(1) every pair has intersection of size 2, and
(2) no three have an intersection of size 2.
Example $k = 17$ (which does not matter here)
$\{8, 9, 10\}$, $\{8, 9, 11\}$, $\{8, 10, 11\}$, $\{9, 10, 11\}$.
Facts left to the reader to prove.
Assume C is a type II cluster.
1) The number of vertices in C is ≤ 4.
2) The union of the vertices in C has ≤ 4 numbers.
3) If $v \notin C$ then for all $v' \in C$, $v \cap v' = \emptyset$.

Type III Clusters

Type III Clusters

Type III Clusters A set of vertices such that every vertex is disjoint from every vertex in the ind. set (including other vertices in the cluster).

Type III Clusters

Type III Clusters A set of vertices such that every vertex is disjoint from every vertex in the ind. set (including other vertices in the cluster).
Example $k = 17$ which does not matter.

Type III Clusters

Type III Clusters A set of vertices such that every vertex is disjoint from every vertex in the ind. set (including other vertices in the cluster).
Example $k = 17$ which does not matter.
$\{12, 13, 14\}$, $\{15, 16, 17\}$.

Type III Clusters

Type III Clusters A set of vertices such that every vertex is disjoint from every vertex in the ind. set (including other vertices in the cluster).
Example $k = 17$ which does not matter.
$\{12, 13, 14\}$, $\{15, 16, 17\}$.

Facts Left To The Reader
Let C be a cluster of type **III**.

Type III Clusters

Type III Clusters A set of vertices such that every vertex is disjoint from every vertex in the ind. set (including other vertices in the cluster).
Example $k = 17$ which does not matter.
$\{12, 13, 14\}$, $\{15, 16, 17\}$.

Facts Left To The Reader
Let C be a cluster of type **III**.

1) If $|C| = s$ then the union of the vertices in C has $3s$ numbers.

Type III Clusters

Type III Clusters A set of vertices such that every vertex is disjoint from every vertex in the ind. set (including other vertices in the cluster).
Example $k = 17$ which does not matter.
$\{12, 13, 14\}$, $\{15, 16, 17\}$.

Facts Left To The Reader
Let C be a cluster of type **III**.

1) If $|C| = s$ then the union of the vertices in C has $3s$ numbers.
2) If $v \notin C$ then for all $v' \in E$, $v \cap v' = \emptyset$. (this is by definition of a type III cluster).

Type III Clusters

Type III Clusters A set of vertices such that every vertex is disjoint from every vertex in the ind. set (including other vertices in the cluster).
Example $k = 17$ which does not matter.
$\{12, 13, 14\}$, $\{15, 16, 17\}$.

Facts Left To The Reader
Let C be a cluster of type **III**.

1) If $|C| = s$ then the union of the vertices in C has $3s$ numbers.
2) If $v \notin C$ then for all $v' \in E$, $v \cap v' = \emptyset$. (this is by definition of a type III cluster).
3) There is at most one cluster of type **III** (if not you can union them).

No Need for Type I Clusters

Lemma Assume I is a maximum sized ind. set. Then $\exists I'$, an ind. set, $|I| = |I'|$ and I has no type I clusters.

No Need for Type I Clusters

Lemma Assume I is a maximum sized ind. set. Then $\exists I'$, an ind. set, $|I| = |I'|$ and I has no type I clusters.

We do four examples of how to take a cluster of type I and rearrange the numbers in it to form clusters of type **II** or **III**, while not decreasing the number of vertices.

No Need for Type I Clusters

Lemma Assume I is a maximum sized ind. set. Then $\exists I'$, an ind. set, $|I| = |I'|$ and I has no type I clusters.

We do four examples of how to take a cluster of type I and rearrange the numbers in it to form clusters of type **II** or **III**, while not decreasing the number of vertices.

Since the numbers in the cluster are not in any other vertex, this rearranging will not affect any other vertex in I.

No Need for Type I Clusters

Lemma Assume I is a maximum sized ind. set. Then $\exists I'$, an ind. set, $|I| = |I'|$ and I has no type I clusters.

We do four examples of how to take a cluster of type I and rearrange the numbers in it to form clusters of type **II** or **III**, while not decreasing the number of vertices.

Since the numbers in the cluster are not in any other vertex, this rearranging will not affect any other vertex in I.

We leave it to the reader to take our examples and make general proofs out of them (which is easy).

Getting Rid of Type I Clusters of size $\equiv 0 \pmod 4$

$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\},$
$\{1,2,7\}, \{1,2,8\}, \{1,2,9\}, \{1,2,10\},$
$\{1,2,11\}, \{1,2,12\}$

Getting Rid of Type I Clusters of size $\equiv 0 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\}$

has 10 vertices.

Getting Rid of Type I Clusters of size $\equiv 0 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\}$

has 10 vertices.

Rearrange into 3 clusters of type **II**, which is 12 vertices:

Getting Rid of Type I Clusters of size $\equiv 0 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\}$

has 10 vertices.

Rearrange into 3 clusters of type **II**, which is 12 vertices:

$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$
$\{5,6,7\},\{5,6,8\},\{5,7,8\},\{6,7,8\}$
$\{9,10,11\},\{9,10,12\},\{9,11,12\},\{10,11,12\}.$

Getting Rid of Type I Clusters of size $\equiv 1 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\},\{1,2,13\}$

Getting Rid of Type I Clusters of size $\equiv 1 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\},\{1,2,13\}$

has 11 vertices.

Getting Rid of Type I Clusters of size $\equiv 1 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\},\{1,2,13\}$

has 11 vertices.

We rearrange this into 3 clusters of type **II**, which is 12 vertices:

Getting Rid of Type I Clusters of size $\equiv 1 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\},\{1,2,13\}$

has 11 vertices.

We rearrange this into 3 clusters of type **II**, which is 12 vertices:

$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$
$\{5,6,7\},\{5,6,8\},\{5,7,8\},\{6,7,8\}$
$\{9,10,11\},\{9,10,12\},\{9,11,12\},\{10,11,12\}.$

Getting Rid of Type I Clusters of size $\equiv 1 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\},\{1,2,13\}$

has 11 vertices.

We rearrange this into 3 clusters of type **II**, which is 12 vertices:

$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$
$\{5,6,7\},\{5,6,8\},\{5,7,8\},\{6,7,8\}$
$\{9,10,11\},\{9,10,12\},\{9,11,12\},\{10,11,12\}.$

We cannot use the number 13. Oh well.

Getting Rid of Type I Clusters of size $\equiv 1 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\},\{1,2,13\}$

has 11 vertices.

We rearrange this into 3 clusters of type **II**, which is 12 vertices:

$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$
$\{5,6,7\},\{5,6,8\},\{5,7,8\},\{6,7,8\}$
$\{9,10,11\},\{9,10,12\},\{9,11,12\},\{10,11,12\}.$

We cannot use the number 13. Oh well.
Even so, rearranging lead to $12 > 11$ vertices.

Getting Rid of Type I Clusters of size $\equiv 2 \pmod 4$

$$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$$
$$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$$
$$\{1,2,11\},\{1,2,12\},\{1,2,13\},\{1,2,14\}$$

Getting Rid of Type I Clusters of size $\equiv 2 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\},\{1,2,13\},\{1,2,14\}$

has 12 vertices.

Getting Rid of Type I Clusters of size $\equiv 2 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\},\{1,2,13\},\{1,2,14\}$

has 12 vertices.

We rearrange this into 3 clusters of type **II**, which is 12 vertices:
$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$
$\{5,6,7\},\{5,6,8\},\{5,7,8\},\{6,7,8\}$
$\{9,10,11\},\{9,10,12\},\{9,11,12\},\{10,11,12\}$.

Getting Rid of Type I Clusters of size $\equiv 2 \pmod{4}$

$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\},$
$\{1,2,7\}, \{1,2,8\}, \{1,2,9\}, \{1,2,10\},$
$\{1,2,11\}, \{1,2,12\}, \{1,2,13\}, \{1,2,14\}$

has 12 vertices.

We rearrange this into 3 clusters of type **II**, which is 12 vertices:
$\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}$
$\{5,6,7\}, \{5,6,8\}, \{5,7,8\}, \{6,7,8\}$
$\{9,10,11\}, \{9,10,12\}, \{9,11,12\}, \{10,11,12\}.$

We cannot use the numbers 13 or 14. Oh well.

Getting Rid of Type I Clusters of size $\equiv 2 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\},\{1,2,13\},\{1,2,14\}$

has 12 vertices.

We rearrange this into 3 clusters of type **II**, which is 12 vertices:
$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$
$\{5,6,7\},\{5,6,8\},\{5,7,8\},\{6,7,8\}$
$\{9,10,11\},\{9,10,12\},\{9,11,12\},\{10,11,12\}.$

We cannot use the numbers 13 or 14. Oh well.

Even so, rearranging let to $12 = 12$ vertices.

Getting Rid of Type I Clusters of size $\equiv 3 \pmod 4$

$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\},$
$\{1,2,7\}, \{1,2,8\}, \{1,2,9\}, \{1,2,10\},$
$\{1,2,11\}, \{1,2,12\}, \{1,2,13\}, \{1,2,14\},$
$\{1,2,15\}$

Getting Rid of Type I Clusters of size $\equiv 3 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\},\{1,2,13\},\{1,2,14\},$
$\{1,2,15\}$

has 13 vertices.

Getting Rid of Type I Clusters of size $\equiv 3 \pmod 4$

$\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\},$
$\{1,2,7\}, \{1,2,8\}, \{1,2,9\}, \{1,2,10\},$
$\{1,2,11\}, \{1,2,12\}, \{1,2,13\}, \{1,2,14\},$
$\{1,2,15\}$

has 13 vertices.

We rearrange this into 3 clusters of type **II** and one size-1 cluster of type **III**, which is 13 vertices.

Getting Rid of Type I Clusters of size $\equiv 3 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\},\{1,2,13\},\{1,2,14\},$
$\{1,2,15\}$

has 13 vertices.

We rearrange this into 3 clusters of type **II** and one size-1 cluster of type **III**, which is 13 vertices.

$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$
$\{5,6,7\},\{5,6,8\},\{5,7,8\},\{6,7,8\}$
$\{9,10,11\},\{9,10,12\},\{9,11,12\},\{10,11,12\}.$

Getting Rid of Type I Clusters of size $\equiv 3 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\},\{1,2,13\},\{1,2,14\},$
$\{1,2,15\}$

has 13 vertices.

We rearrange this into 3 clusters of type **II** and one size-1 cluster of type **III**, which is 13 vertices.

$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$
$\{5,6,7\},\{5,6,8\},\{5,7,8\},\{6,7,8\}$
$\{9,10,11\},\{9,10,12\},\{9,11,12\},\{10,11,12\}.$
$\{13,14,15\}$

Getting Rid of Type I Clusters of size $\equiv 3 \pmod 4$

$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},$
$\{1,2,7\},\{1,2,8\},\{1,2,9\},\{1,2,10\},$
$\{1,2,11\},\{1,2,12\},\{1,2,13\},\{1,2,14\},$
$\{1,2,15\}$

has 13 vertices.

We rearrange this into 3 clusters of type **II** and one size-1 cluster of type **III**, which is 13 vertices.

$\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}$
$\{5,6,7\},\{5,6,8\},\{5,7,8\},\{6,7,8\}$
$\{9,10,11\},\{9,10,12\},\{9,11,12\},\{10,11,12\}.$
$\{13,14,15\}$

The number of vertices stayed the same.

Ind. Sets That have Type II,III Clusters Only

Ind. Sets That have Type II,III Clusters Only

We bound the $|I|$ if I has clusters of type **II** and **III** only.

Ind. Sets That have Type II,III Clusters Only

We bound the $|I|$ if I has clusters of type **II** and **III** only.
Let I be an independent set that has the following.

Ind. Sets That have Type II,III Clusters Only

We bound the $|I|$ if I has clusters of type **II** and **III** only.
Let I be an independent set that has the following.
1) x clusters of type **II**. This is $4x$ vertices and uses up $4x$ numbers that cannot be used in any other vertex.

Ind. Sets That have Type II,III Clusters Only

We bound the $|I|$ if I has clusters of type **II** and **III** only.
Let I be an independent set that has the following.
1) x clusters of type **II**. This is $4x$ vertices and uses up $4x$ numbers that cannot be used in any other vertex.
2) One cluster of type **III** with y vertices. This uses up $3y$ numbers that cannot be used in any other vertex.

Ind. Sets That have Type II,III Clusters Only

We bound the $|I|$ if I has clusters of type **II** and **III** only.
Let I be an independent set that has the following.
1) x clusters of type **II**. This is $4x$ vertices and uses up $4x$ numbers that cannot be used in any other vertex.
2) One cluster of type **III** with y vertices. This uses up $3y$ numbers that cannot be used in any other vertex.

Number of vertices: $4x + y$.

Ind. Sets That have Type II,III Clusters Only

We bound the $|I|$ if I has clusters of type **II** and **III** only.
Let I be an independent set that has the following.
1) x clusters of type **II**. This is $4x$ vertices and uses up $4x$ numbers that cannot be used in any other vertex.
2) One cluster of type **III** with y vertices. This uses up $3y$ numbers that cannot be used in any other vertex.

Number of vertices: $4x + y$. Number of numbers: $4x + 3y$.

Ind. Sets That have Type II,III Clusters Only

We bound the $|I|$ if I has clusters of type **II** and **III** only.
Let I be an independent set that has the following.
1) x clusters of type **II**. This is $4x$ vertices and uses up $4x$ numbers that cannot be used in any other vertex.
2) One cluster of type **III** with y vertices. This uses up $3y$ numbers that cannot be used in any other vertex.

Number of vertices: $4x + y$. Number of numbers: $4x + 3y$.
Maximize $4x + y$ relative to $4x + 3y \leq k$.

Ind. Sets That have Type II,III Clusters Only

We bound the $|I|$ if I has clusters of type **II** and **III** only.
Let I be an independent set that has the following.
1) x clusters of type **II**. This is $4x$ vertices and uses up $4x$ numbers that cannot be used in any other vertex.
2) One cluster of type **III** with y vertices. This uses up $3y$ numbers that cannot be used in any other vertex.

Number of vertices: $4x + y$. Number of numbers: $4x + 3y$.
Maximize $4x + y$ relative to $4x + 3y \leq k$.
Left to reader to show max is $\leq \frac{k}{2}$.

Ind. Sets That have Type II,III Clusters Only

We bound the $|I|$ if I has clusters of type **II** and **III** only.
Let I be an independent set that has the following.
1) x clusters of type **II**. This is $4x$ vertices and uses up $4x$ numbers that cannot be used in any other vertex.
2) One cluster of type **III** with y vertices. This uses up $3y$ numbers that cannot be used in any other vertex.

Number of vertices: $4x + y$. Number of numbers: $4x + 3y$.
Maximize $4x + y$ relative to $4x + 3y \le k$.
Left to reader to show max is $\le \frac{k}{2}$.
Hint Do four cases based on $x \pmod 4$.

Ind. Sets That have Type II,III Clusters Only

We bound the $|I|$ if I has clusters of type **II** and **III** only.
Let I be an independent set that has the following.
1) x clusters of type **II**. This is $4x$ vertices and uses up $4x$ numbers that cannot be used in any other vertex.
2) One cluster of type **III** with y vertices. This uses up $3y$ numbers that cannot be used in any other vertex.

Number of vertices: $4x + y$. Number of numbers: $4x + 3y$.
Maximize $4x + y$ relative to $4x + 3y \le k$.
Left to reader to show max is $\le \frac{k}{2}$.
Hint Do four cases based on $x \pmod 4$.
PROOF IS DONE

Where Are We Now?

We had $k^2 - 2k + 1 \leq R(k) \leq 2^{2k-1}$
and asked **can we do better?**

Where Are We Now?

We had $k^2 - 2k + 1 \leq R(k) \leq 2^{2k-1}$
and asked **can we do better?**

We can!

Where Are We Now?

We had $k^2 - 2k + 1 \leq R(k) \leq 2^{2k-1}$
and asked **can we do better?**

We can! We now have $\Omega(k^3) \leq R(k) \leq 2^{2k-1}$

Where Are We Now?

We had $k^2 - 2k + 1 \leq R(k) \leq 2^{2k-1}$
and asked **can we do better?**

We can! We now have $\Omega(k^3) \leq R(k) \leq 2^{2k-1}$
The bounds are still far apart.

Where Are We Now?

We had $k^2 - 2k + 1 \leq R(k) \leq 2^{2k-1}$
and asked **can we do better?**

We can! We now have $\Omega(k^3) \leq R(k) \leq 2^{2k-1}$
The bounds are still far apart.

We ask again **can we do better?**

Where Are We Now?

We had $k^2 - 2k + 1 \leq R(k) \leq 2^{2k-1}$
and asked **can we do better?**

We can! We now have $\Omega(k^3) \leq R(k) \leq 2^{2k-1}$
The bounds are still far apart.

We ask again **can we do better?**

The next slide gives a summary of improvements.

Where Are We Now?

We had $k^2 - 2k + 1 \le R(k) \le 2^{2k-1}$
and asked **can we do better?**

We can! We now have $\Omega(k^3) \le R(k) \le 2^{2k-1}$
The bounds are still far apart.

We ask again **can we do better?**

The next slide gives a summary of improvements.
PRO We have an exponential lower bound.

Where Are We Now?

We had $k^2 - 2k + 1 \le R(k) \le 2^{2k-1}$
and asked **can we do better?**

We can! We now have $\Omega(k^3) \le R(k) \le 2^{2k-1}$
The bounds are still far apart.

We ask again **can we do better?**

The next slide gives a summary of improvements.
PRO We have an exponential lower bound.
CON The proof is beyond the scope of the students in this class

Where Are We Now?

We had $k^2 - 2k + 1 \le R(k) \le 2^{2k-1}$
and asked **can we do better?**

We can! We now have $\Omega(k^3) \le R(k) \le 2^{2k-1}$
The bounds are still far apart.

We ask again **can we do better?**

The next slide gives a summary of improvements.
PRO We have an exponential lower bound.
CON The proof is beyond the scope of the students in this class
And the professor of this class until Rishi and I finish our survey.

Where Are We Now?

We had $k^2 - 2k + 1 \le R(k) \le 2^{2k-1}$
and asked **can we do better?**

We can! We now have $\Omega(k^3) \le R(k) \le 2^{2k-1}$
The bounds are still far apart.

We ask again **can we do better?**

The next slide gives a summary of improvements.
PRO We have an exponential lower bound.
CON The proof is beyond the scope of the students in this class
And the professor of this class until Rishi and I finish our survey.
Next Slide Packet: **Are Better Lower Bounds Possible Given the Current State Of Our Knowledge?**.

Where Are We Now?

We had $k^2 - 2k + 1 \le R(k) \le 2^{2k-1}$
and asked **can we do better?**

We can! We now have $\Omega(k^3) \le R(k) \le 2^{2k-1}$
The bounds are still far apart.

We ask again **can we do better?**

The next slide gives a summary of improvements.
PRO We have an exponential lower bound.
CON The proof is beyond the scope of the students in this class
And the professor of this class until Rishi and I finish our survey.
Next Slide Packet: **Are Better Lower Bounds Possible Given the Current State Of Our Knowledge?**.
And, as always, I ask **Will the proofs be suitable for HS students? For Ilya?**

Summary of What is Known

The following results were obtained by constructing the appropriate colorings.

Result	Comments	Paper
$R(k) \geq (k-1)^2$	Elt	Folklore, 1950s
$R(k) \geq \Omega(k^{2.3\cdots})$	Elt	[1], 1971
$R(k) \geq \Omega(k^3)$	Elt	[9], 1981, 1975
$R(k) \geq 2^{\Omega(\log^2(k)/\log\log k)}$	Set Systems	[3], 1981
$R(k) \geq 2^{\Omega(\log^2(k)/\log\log k)}$	Info. Theory & Lin. Alg.	[2], 1998
$R(k) \geq 2^{\Omega(\log^2(k)/\log\log k)}$	Representing OR	[5], 2000
$R(k) \geq 2^{\Omega(\log^2(k)/\log\log k)}$	Representing OR	[4], 2014
$R(k) \geq$ Better than [3]	Extractors	[6], 2017,
$R(k) \geq$ Better than [3]	Extractors	[7], 2019,
$R(k) \geq 2^{\Omega(k^{\delta})} (\delta < 1)$	Extractors	[8], 2023

Summary of What is Known

The following results were obtained by constructing the appropriate colorings.

Result	Comments	Paper
$R(k) \geq (k-1)^2$	Elt	Folklore, 1950s
$R(k) \geq \Omega(k^{2.3\ldots})$	Elt	[1], 1971
$R(k) \geq \Omega(k^3)$	Elt	[9], 1981, 1975
$R(k) \geq 2^{\Omega(\log^2(k)/\log\log k)}$	Set Systems	[3], 1981
$R(k) \geq 2^{\Omega(\log^2(k)/\log\log k)}$	Info. Theory & Lin. Alg.	[2], 1998
$R(k) \geq 2^{\Omega(\log^2(k)/\log\log k)}$	Representing OR	[5], 2000
$R(k) \geq 2^{\Omega(\log^2(k)/\log\log k)}$	Representing OR	[4], 2014
$R(k) \geq$ Better than [3]	Extractors	[6], 2017,
$R(k) \geq$ Better than [3]	Extractors	[7], 2019,
$R(k) \geq 2^{\Omega(k^\delta)} (\delta < 1)$	Extractors	[8], 2023

Refs at end of slide packet. Some include pointers to the papers.

We Want Easier Proofs

Proof of $R(k) \geq \Omega(k^3)$ is HS.

We Want Easier Proofs

Proof of $R(k) \ge \Omega(k^3)$ is HS.

Proof of $R(k) \ge 2^{\Omega(\log^2(k)/\log\log k)}$ is not HS (debatable).

We Want Easier Proofs

Proof of $R(k) \geq \Omega(k^3)$ is HS.

Proof of $R(k) \geq 2^{\Omega(\log^2(k)/\log\log k)}$ is not HS (debatable).

All of the later results are not HS (not debatable).

We Want Easier Proofs

Proof of $R(k) \ge \Omega(k^3)$ is HS.

Proof of $R(k) \ge 2^{\Omega(\log^2(k)/\log\log k)}$ is not HS (debatable).

All of the later results are not HS (not debatable).

Challenge

We Want Easier Proofs

Proof of $R(k) \ge \Omega(k^3)$ is HS.

Proof of $R(k) \ge 2^{\Omega(\log^2(k)/\log\log k)}$ is not HS (debatable).

All of the later results are not HS (not debatable).

Challenge

1) Want HS proof that $R(k) \ge \Omega(k^4)$. Or higher degree.

We Want Easier Proofs

Proof of $R(k) \ge \Omega(k^3)$ is HS.

Proof of $R(k) \ge 2^{\Omega(\log^2(k)/\log\log k)}$ is not HS (debatable).

All of the later results are not HS (not debatable).

Challenge

1) Want HS proof that $R(k) \ge \Omega(k^4)$. Or higher degree.
2) Want easier proofs of the results in the table beyond $\Omega(k^3)$.

First Slide of References

1) H. Abbott. Lower bounds on some Ramsey numbers. *Discrete Mathematics*, 2:289–293, 1971.

2) N. Alon. The Shannon capacity of a union. *Combinatorica*, 18(3):301–310, 1998.
https://doi.org/10.1007/PL00009824.

3) P. Frankl and R. Wilson. Intersection theorems with geometric consequences. *Combinatorica*, 1:357–368, 1981.
https://www.cs.umd.edu/~gasarch/TOPICS/CRT/FW.pdf

Second Slide of References

4) P. Gopalan. Constructing Ramsey graphs from Boolean function representations. *Combinatorica*, 34(2):173–206, 2014. https://doi.org/10.1007/s00493-014-2367-1.

5) V. Grolmusz. Superpolynomial size set-systems with restricted intersections mod 6 and explicit Ramsey graphs. *Combinatorica*, 20(1):71–86, 2000. https://doi.org/10.1007/s004930070032.

6) X. Li. Improved non-malleable extractors, non-malleable codes and independent source extractors. In H. Hatami, P. McKenzie, and V. King, editors, *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, Montreal, QC, Canada, June 19-23, 2017*, pages 1144–1156. ACM, 2017. https://doi.org/10.1145/3055399.3055486.

Third and Last Slide of References

7) X. Li. Non-malleable extractors and non-malleable codes: Partially optimal constructions. In A. Shpilka, editor, *34th Computational Complexity Conference, CCC 2019, July 18-20, 2019, New Brunswick, NJ, USA*, volume 137 of *LIPIcs*, pages 28:1–28:49. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.
https://doi.org/10.4230/LIPIcs.CCC.2019.28.

8) X. Li. Two source extractors for asymptotically optimal entropy, and (many) more. In *64th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2023, Santa Cruz, CA, USA, November 6-9, 2023*, pages 1271–1281. IEEE, 2023.
https://doi.org/10.1109/FOCS57990.2023.00075.

9) Z. Nagy. A constructive estimate of the Ramsey numbers (In Hungarian). *Mat. Lapok*, pages 301–302, 1975.