

BILL, RECORD LECTURE!!!!

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Monochromatic C_4

Exposition by William Gasarch

When Do You Get A Mono C_4 ?

Thm For all COL: $\binom{[18]}{2} \rightarrow [2]$ there exists a mono C_4 .

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We have shown $R(C_4) \leq 18$.

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Vote

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1) $R(C_4) = 18$.

2) $10 \leq R(C_4) \leq 17$.

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- 2) $10 \leq R(C_4) \leq 17$.
- 3) $5 \leq R(C_4) \leq 9$.

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Answer on the next page.

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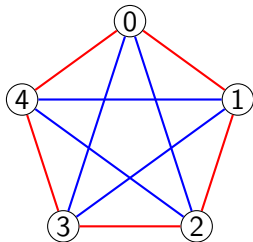
We present a COL: $\binom{[5]}{2} \rightarrow [2]$ with no mono C_4 .

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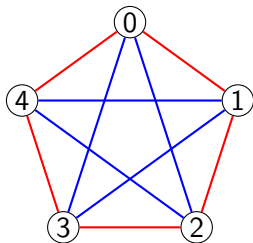


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Note: There is a mono C_5 but not a mono C_4 .

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Let $\text{COL}(T_i)$ be the color of all the edges of T_i .

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- 1) $\text{COL}(T_1) = \text{COL}(T_2)$
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Both cases will have some subcases.

Case I

$$\text{COL}(T_1) = \text{COL}(T_2)$$

Subcases For $\text{COL}(T_1) = \text{COL}(T_2)$

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There are three subcases:

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We assume $\text{COL}(T_1) = \text{COL}(T_2) = \mathbf{R}$.

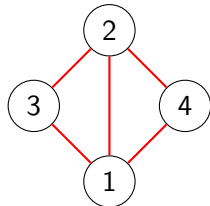
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Can assume the **red** triangles are 1, 2, 3 and 1, 2, 4.

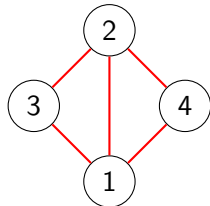
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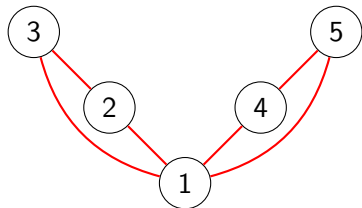
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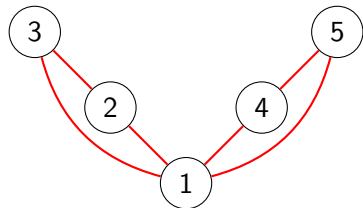
C_4 : 1 - 4 - 2 - 3 - 1. DONE.

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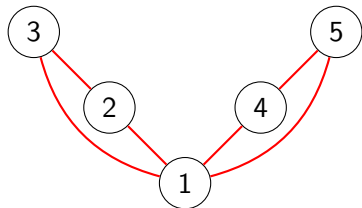


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If $\text{COL}(2, 4) = \mathbf{R}$ then $\mathbf{C}_4 : 2 - 4 - 1 - 3 - 2$.

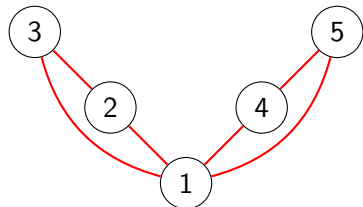
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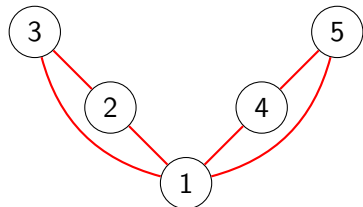


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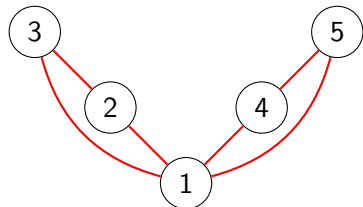
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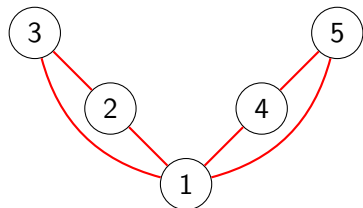
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Next slide considers the case

$\text{COL}(T_1) = \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



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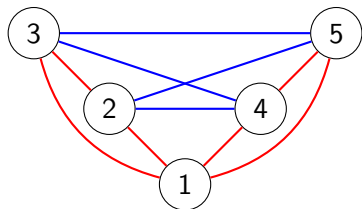
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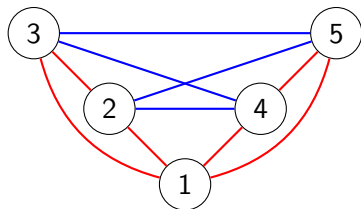
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$C_4 : 2 - 4 - 3 - 5 - 2$. DONE.

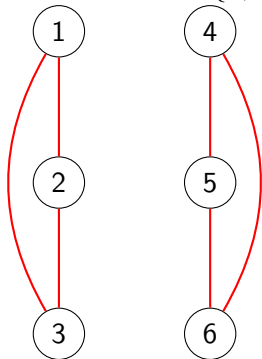
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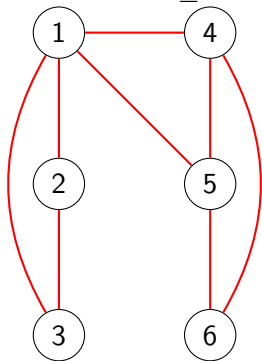


$\text{COL}(T_1) = \text{COL}(T_2)$, T_1, T_2 Disj, 2 R

Subcase A: $\exists \geq 2$ red edges between T_1 and T_2 .

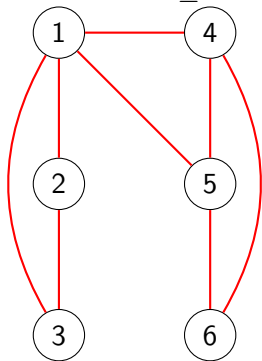
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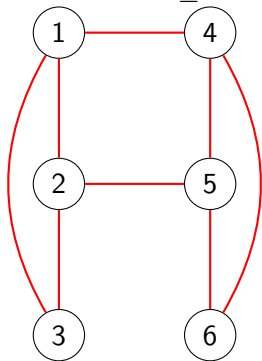
$C_4 : 1 - 4 - 6 - 5 - 1$

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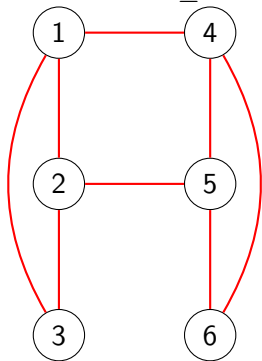
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$C_4 : 1 - 4 - 5 - 2 - 1$

$\text{COL}(T_1) = \text{COL}(T_2)$, T_1, T_2 Disj, $\leq 1R$

Subcase B: There is ≤ 1 red edges between T_1 and T_2 .

$\text{COL}(T_1) = \text{COL}(T_2)$, T_1, T_2 Disj, $\leq 1R$

Subcase B: There is ≤ 1 **red** edges between T_1 and T_2 .

Then there are ≥ 8 **blue** edges between T_1 and T_2 .

$\text{COL}(T_1) = \text{COL}(T_2)$, T_1, T_2 Disj, $\leq 1R$

Subcase B: There is ≤ 1 **red** edges between T_1 and T_2 .

Then there are ≥ 8 **blue** edges between T_1 and T_2 .

Easy Exercise: Show there is a C_4 .

Case II

$$\text{COL}(T_1) \neq \text{COL}(T_2)$$

Subcases For $\text{COL}(T_1) \neq \text{COL}(T_2)$

There are two subcases:

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There are two subcases:

1. T_1 and T_2 share a vertex.

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Subcases For $\text{COL}(T_1) \neq \text{COL}(T_2)$

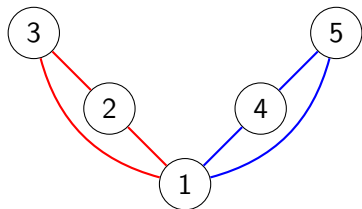
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We will assume $\text{COL}(T_1) = \mathbf{R}$ and $\text{COL}(T_2) = \mathbf{B}$.

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

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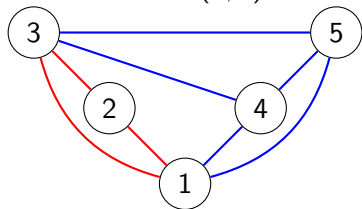


$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase: $\text{COL}(3, 4) = \text{COL}(3, 5) = \mathbf{B}$.

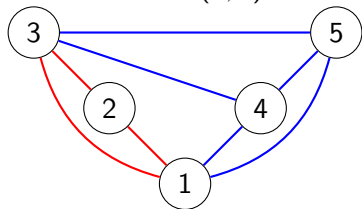
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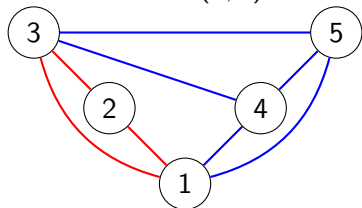
Subcase: $\text{COL}(3, 4) = \text{COL}(3, 5) = \mathbf{B}$.



$\mathbf{C}_4 : 3 - 5 - 4 - 1 - 3.$

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Subcase: $\text{COL}(3, 4) = \text{COL}(3, 5) = \mathbf{B}$.

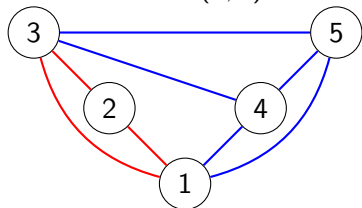


$\mathbf{C}_4 : 3 - 5 - 4 - 1 - 3$.

$\text{COL}(2, 5) = \text{COL}(2, 4) = \mathbf{B}$ is similar.

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase: $\text{COL}(3, 4) = \text{COL}(3, 5) = \mathbf{B}$.



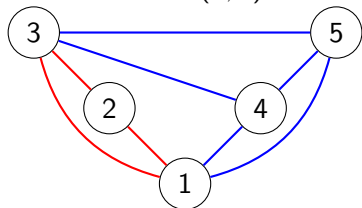
$C_4 : 3 - 5 - 4 - 1 - 3$.

$\text{COL}(2, 5) = \text{COL}(2, 4) = \mathbf{B}$ is similar.

$\text{COL}(5, 3) = \text{COL}(5, 2) = \mathbf{R}$ is similar.

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase: $\text{COL}(3, 4) = \text{COL}(3, 5) = \mathbf{B}$.



$C_4 : 3 - 5 - 4 - 1 - 3$.

$\text{COL}(2, 5) = \text{COL}(2, 4) = \mathbf{B}$ is similar.

$\text{COL}(5, 3) = \text{COL}(5, 2) = \mathbf{R}$ is similar.

$\text{COL}(4, 3) = \text{COL}(4, 2) = \mathbf{R}$ is similar.

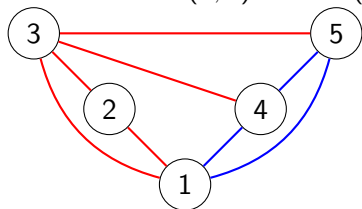
$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase $\text{COL}(3, 5) = \text{COL}(3, 4) = \mathbf{R}$.

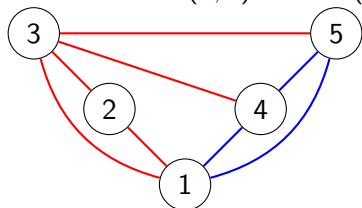
$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase $\text{COL}(3, 5) = \text{COL}(3, 4) = \mathbf{R}$.



$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

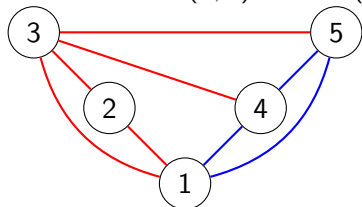
Subcase $\text{COL}(3, 5) = \text{COL}(3, 4) = \mathbf{R}$.



If $\text{COL}(2, 5) = \mathbf{R}$ then $C_4 : 2 - 5 - 3 - 1 - 2$.

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase $\text{COL}(3, 5) = \text{COL}(3, 4) = \mathbf{R}$.

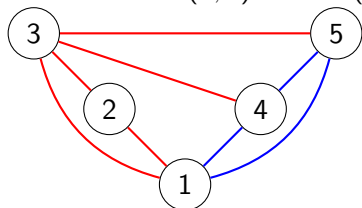


If $\text{COL}(2, 5) = \mathbf{R}$ then $C_4 : 2 - 5 - 3 - 1 - 2$.

If $\text{COL}(2, 4) = \mathbf{R}$ then $C_4 : 2 - 4 - 3 - 1 - 2$.

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase $\text{COL}(3, 5) = \text{COL}(3, 4) = \mathbf{R}$.



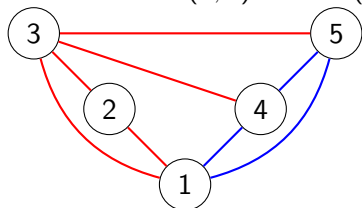
If $\text{COL}(2, 5) = \mathbf{R}$ then $\mathbf{C}_4 : 2 - 5 - 3 - 1 - 2$.

If $\text{COL}(2, 4) = \mathbf{R}$ then $\mathbf{C}_4 : 2 - 4 - 3 - 1 - 2$.

Hence $\text{COL}(2, 5) = \text{COL}(2, 4) = \mathbf{B}$ so $\mathbf{C}_5 : 2 - 5 - 1 - 4 - 2$.

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase $\text{COL}(3, 5) = \text{COL}(3, 4) = \mathbf{R}$.



If $\text{COL}(2, 5) = \mathbf{R}$ then $\mathbf{C}_4 : 2 - 5 - 3 - 1 - 2$.

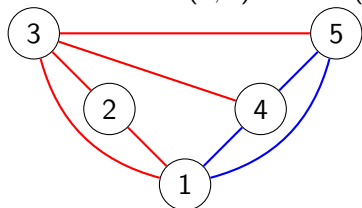
If $\text{COL}(2, 4) = \mathbf{R}$ then $\mathbf{C}_4 : 2 - 4 - 3 - 1 - 2$.

Hence $\text{COL}(2, 5) = \text{COL}(2, 4) = \mathbf{B}$ so $\mathbf{C}_5 : 2 - 5 - 1 - 4 - 2$.

$\text{COL}(2, 5) = \text{COL}(2, 4) = \mathbf{R}$ is similar.

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase $\text{COL}(3, 5) = \text{COL}(3, 4) = \mathbf{R}$.



If $\text{COL}(2, 5) = \mathbf{R}$ then $C_4 : 2 - 5 - 3 - 1 - 2$.

If $\text{COL}(2, 4) = \mathbf{R}$ then $C_4 : 2 - 4 - 3 - 1 - 2$.

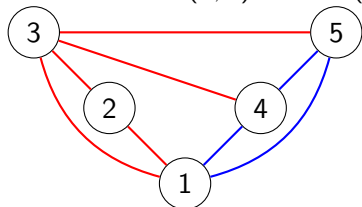
Hence $\text{COL}(2, 5) = \text{COL}(2, 4) = \mathbf{B}$ so $C_5 : 2 - 5 - 1 - 4 - 2$.

$\text{COL}(2, 5) = \text{COL}(2, 4) = \mathbf{R}$ is similar.

$\text{COL}(5, 3) = \text{COL}(5, 2) = \mathbf{B}$ is similar.

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

Subcase $\text{COL}(3, 5) = \text{COL}(3, 4) = \mathbf{R}$.



If $\text{COL}(2, 5) = \mathbf{R}$ then $\mathbf{C}_4 : 2 - 5 - 3 - 1 - 2$.

If $\text{COL}(2, 4) = \mathbf{R}$ then $\mathbf{C}_4 : 2 - 4 - 3 - 1 - 2$.

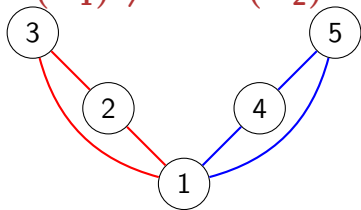
Hence $\text{COL}(2, 5) = \text{COL}(2, 4) = \mathbf{B}$ so $\mathbf{C}_5 : 2 - 5 - 1 - 4 - 2$.

$\text{COL}(2, 5) = \text{COL}(2, 4) = \mathbf{R}$ is similar.

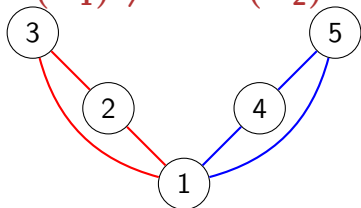
$\text{COL}(5, 3) = \text{COL}(5, 2) = \mathbf{B}$ is similar.

$\text{COL}(4, 3) = \text{COL}(4, 2) = \mathbf{B}$ is similar.

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

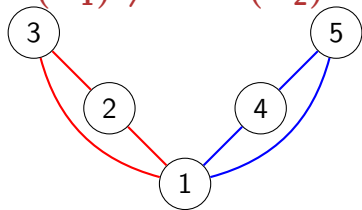


$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



The Negation of All of the Subcases

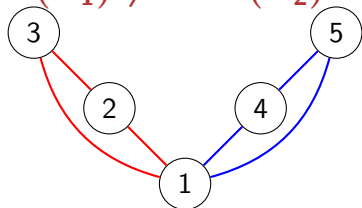
$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



The Negation of All of the Subcases

$\text{COL}(3,5) \neq \text{COL}(3,4)$

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

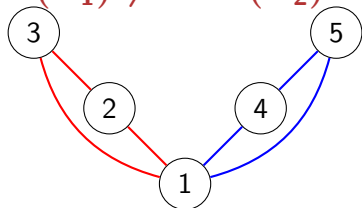


The Negation of All of the Subcases

$$\text{COL}(3,5) \neq \text{COL}(3,4)$$

$$\text{COL}(2,5) \neq \text{COL}(2,4)$$

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



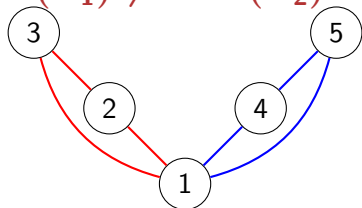
The Negation of All of the Subcases

$$\text{COL}(3,5) \neq \text{COL}(3,4)$$

$$\text{COL}(2,5) \neq \text{COL}(2,4)$$

$$\text{COL}(5,3) \neq \text{COL}(5,4)$$

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



The Negation of All of the Subcases

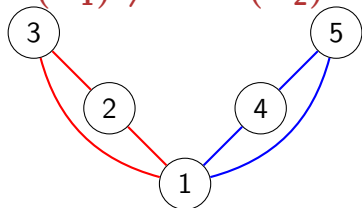
$$\text{COL}(3, 5) \neq \text{COL}(3, 4)$$

$$\text{COL}(2, 5) \neq \text{COL}(2, 4)$$

$$\text{COL}(5, 3) \neq \text{COL}(5, 4)$$

$$\text{COL}(4, 3) \neq \text{COL}(4, 2)$$

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



The Negation of All of the Subcases

$$\text{COL}(3, 5) \neq \text{COL}(3, 4)$$

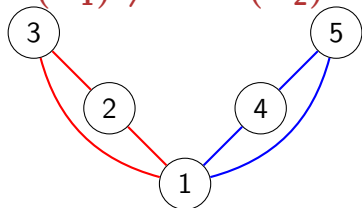
$$\text{COL}(2, 5) \neq \text{COL}(2, 4)$$

$$\text{COL}(5, 3) \neq \text{COL}(5, 4)$$

$$\text{COL}(4, 3) \neq \text{COL}(4, 2)$$

Only two colorings to check:

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



The Negation of All of the Subcases

$$\text{COL}(3,5) \neq \text{COL}(3,4)$$

$$\text{COL}(2,5) \neq \text{COL}(2,4)$$

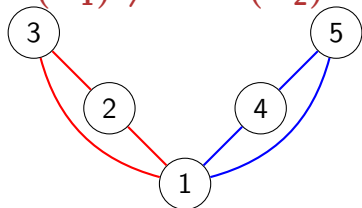
$$\text{COL}(5,3) \neq \text{COL}(5,4)$$

$$\text{COL}(4,3) \neq \text{COL}(4,2)$$

Only two colorings to check:

$$\text{COL}(3,5) = \mathbf{R} \rightarrow \text{COL}(3,4) = \mathbf{B} \rightarrow \text{COL}(4,2) = \mathbf{R} \rightarrow \text{COL}(2,5) = \mathbf{B}$$

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



The Negation of All of the Subcases

$$\text{COL}(3, 5) \neq \text{COL}(3, 4)$$

$$\text{COL}(2, 5) \neq \text{COL}(2, 4)$$

$$\text{COL}(5, 3) \neq \text{COL}(5, 4)$$

$$\text{COL}(4, 3) \neq \text{COL}(4, 2)$$

Only two colorings to check:

$$\text{COL}(3, 5) = \mathbf{R} \rightarrow \text{COL}(3, 4) = \mathbf{B} \rightarrow \text{COL}(4, 2) = \mathbf{R} \rightarrow \text{COL}(2, 5) = \mathbf{B}$$

$$\text{COL}(3, 5) = \mathbf{B} \rightarrow \text{COL}(3, 4) = \mathbf{R} \rightarrow \text{COL}(4, 2) = \mathbf{B} \rightarrow \text{COL}(2, 5) = \mathbf{R}$$

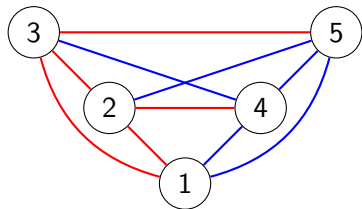
$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

$$\text{COL}(3, 5) = \mathbf{R} \rightarrow \text{COL}(3, 4) = \mathbf{B} \rightarrow \text{COL}(4, 2) = \mathbf{R} \rightarrow \text{COL}(2, 5) = \mathbf{B}$$

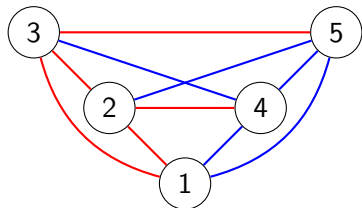
$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

$\text{COL}(3,5) = \mathbf{R} \rightarrow \text{COL}(3,4) = \mathbf{B} \rightarrow \text{COL}(4,2) = \mathbf{R} \rightarrow \text{COL}(2,5) = \mathbf{B}$



$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

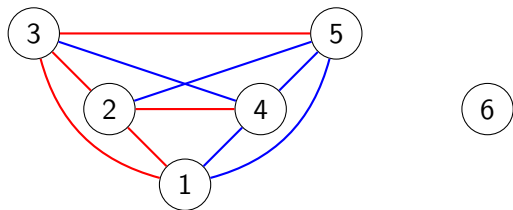
$\text{COL}(3,5) = \mathbf{R} \rightarrow \text{COL}(3,4) = \mathbf{B} \rightarrow \text{COL}(4,2) = \mathbf{R} \rightarrow \text{COL}(2,5) = \mathbf{B}$



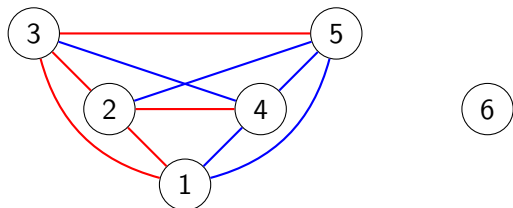
We now need to look at vertex 6.

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

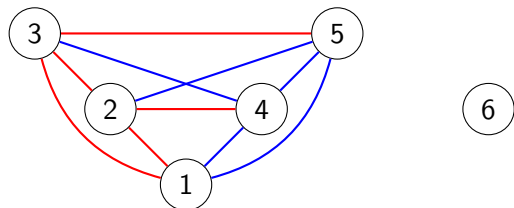


$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 3 - 2 - 6$

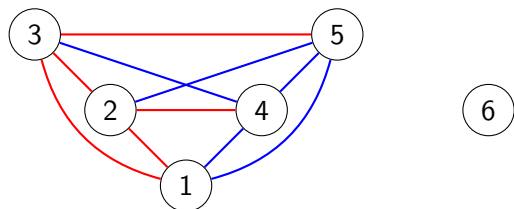
$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 3 - 2 - 6$

If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 5 - 2 - 6$

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

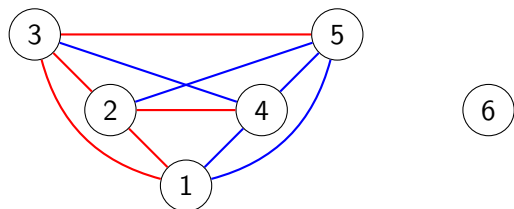


If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 3 - 2 - 6$

If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 5 - 2 - 6$

If $\text{COL}(1, 6) = \text{COL}(3, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 2 - 3 - 6$

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



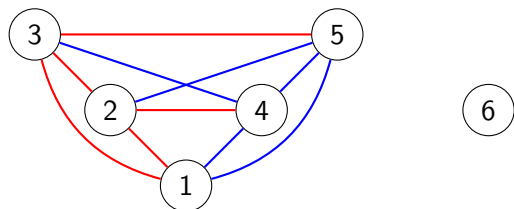
If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 3 - 2 - 6$

If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 5 - 2 - 6$

If $\text{COL}(1, 6) = \text{COL}(3, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 2 - 3 - 6$

If $\text{COL}(1, 6) = \text{COL}(3, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 4 - 3 - 6$

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 3 - 2 - 6$

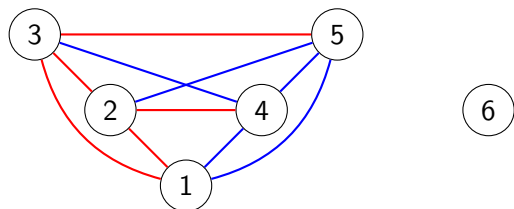
If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 5 - 2 - 6$

If $\text{COL}(1, 6) = \text{COL}(3, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 2 - 3 - 6$

If $\text{COL}(1, 6) = \text{COL}(3, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 4 - 3 - 6$

If $\text{COL}(1, 6) = \text{COL}(4, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 2 - 4 - 6$

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 3 - 2 - 6$

If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 5 - 2 - 6$

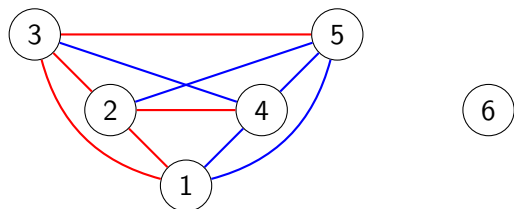
If $\text{COL}(1, 6) = \text{COL}(3, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 2 - 3 - 6$

If $\text{COL}(1, 6) = \text{COL}(3, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 4 - 3 - 6$

If $\text{COL}(1, 6) = \text{COL}(4, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 2 - 4 - 6$

If $\text{COL}(1, 6) = \text{COL}(4, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 5 - 4 - 6$

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 3 - 2 - 6$

If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 5 - 2 - 6$

If $\text{COL}(1, 6) = \text{COL}(3, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 2 - 3 - 6$

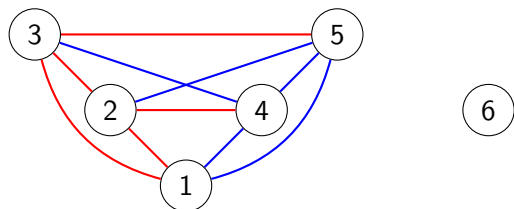
If $\text{COL}(1, 6) = \text{COL}(3, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 4 - 3 - 6$

If $\text{COL}(1, 6) = \text{COL}(4, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 2 - 4 - 6$

If $\text{COL}(1, 6) = \text{COL}(4, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 5 - 4 - 6$

If $\text{COL}(1, 6) = \text{COL}(5, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 3 - 5 - 6$

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 3 - 2 - 6$

If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 5 - 2 - 6$

If $\text{COL}(1, 6) = \text{COL}(3, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 2 - 3 - 6$

If $\text{COL}(1, 6) = \text{COL}(3, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 4 - 3 - 6$

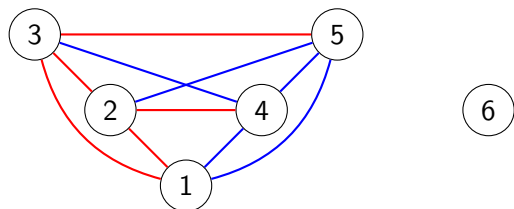
If $\text{COL}(1, 6) = \text{COL}(4, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 2 - 4 - 6$

If $\text{COL}(1, 6) = \text{COL}(4, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 5 - 4 - 6$

If $\text{COL}(1, 6) = \text{COL}(5, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 3 - 5 - 6$

If $\text{COL}(1, 6) = \text{COL}(5, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 4 - 5 - 6$

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 3 - 2 - 6$

If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 5 - 2 - 6$

If $\text{COL}(1, 6) = \text{COL}(3, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 2 - 3 - 6$

If $\text{COL}(1, 6) = \text{COL}(3, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 4 - 3 - 6$

If $\text{COL}(1, 6) = \text{COL}(4, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 2 - 4 - 6$

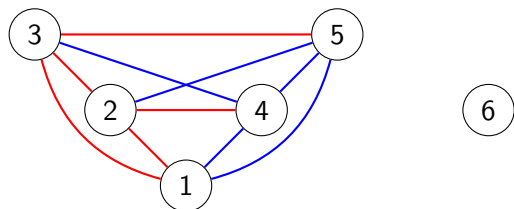
If $\text{COL}(1, 6) = \text{COL}(4, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 5 - 4 - 6$

If $\text{COL}(1, 6) = \text{COL}(5, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 3 - 5 - 6$

If $\text{COL}(1, 6) = \text{COL}(5, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 4 - 5 - 6$

$\text{COL}(1, 6) = \mathbf{R} : (\forall 2 \leq i \leq 5)[\text{COL}(i, 6) = \mathbf{B}]. \mathbf{6 - 5 - 1 - 4 - 6}$

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex



If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 3 - 2 - 6$

If $\text{COL}(1, 6) = \text{COL}(2, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 5 - 2 - 6$

If $\text{COL}(1, 6) = \text{COL}(3, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 2 - 3 - 6$

If $\text{COL}(1, 6) = \text{COL}(3, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 4 - 3 - 6$

If $\text{COL}(1, 6) = \text{COL}(4, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 2 - 4 - 6$

If $\text{COL}(1, 6) = \text{COL}(4, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 5 - 4 - 6$

If $\text{COL}(1, 6) = \text{COL}(5, 6) = \mathbf{R}$ then $\mathbf{C}_4 : 6 - 1 - 3 - 5 - 6$

If $\text{COL}(1, 6) = \text{COL}(5, 6) = \mathbf{B}$ then $\mathbf{C}_4 : 6 - 1 - 4 - 5 - 6$

$\text{COL}(1, 6) = \mathbf{R} : (\forall 2 \leq i \leq 5)[\text{COL}(i, 6) = \mathbf{B}]. \mathbf{6 - 5 - 1 - 4 - 6}$

$\text{COL}(1, 6) = \mathbf{B} : (\forall 2 \leq i \leq 5)[\text{COL}(i, 6) = \mathbf{R}]. \mathbf{6 - 1 - 2 - 3 - 6}$

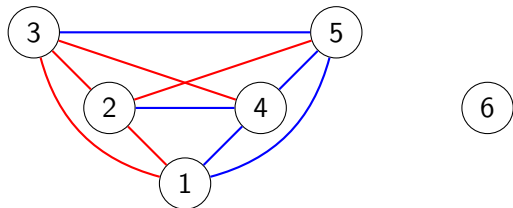
$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

$$\text{COL}(3, 5) = \mathbf{B} \rightarrow \text{COL}(3, 4) = \mathbf{R} \rightarrow \text{COL}(4, 2) = \mathbf{B} \rightarrow \text{COL}(2, 5) = \mathbf{R}$$

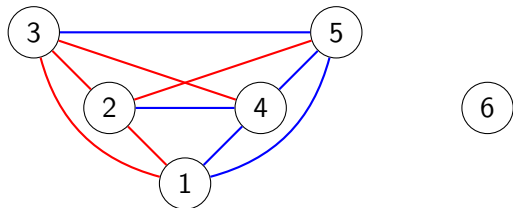
$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

$\text{COL}(3,5) = \mathbf{B} \rightarrow \text{COL}(3,4) = \mathbf{R} \rightarrow \text{COL}(4,2) = \mathbf{B} \rightarrow \text{COL}(2,5) = \mathbf{R}$



$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Share a Vertex

$\text{COL}(3, 5) = \mathbf{B} \rightarrow \text{COL}(3, 4) = \mathbf{R} \rightarrow \text{COL}(4, 2) = \mathbf{B} \rightarrow \text{COL}(2, 5) = \mathbf{R}$



We leave this case to the reader.

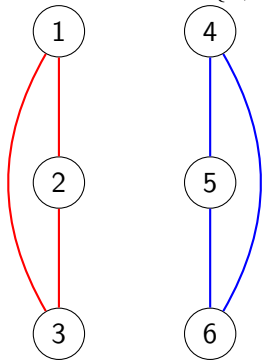
$\text{COL}(T_1) \neq \text{COL}(T_2)$ and T_1, T_2 Are Disj

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We assume $T_1 = \{1, 2, 3\}$ and $T_2 = \{4, 5, 6\}$.

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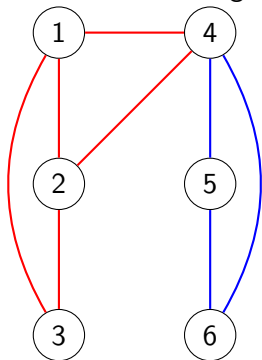


$\text{COL}(T_1) \neq \text{COL}(T_2)$, T_1, T_2 Disj, $\exists v, \text{deg}_R(v) \geq 2$

Subcase A: $\exists v$, Right Side, $\text{deg}_R(v) \geq 2$.

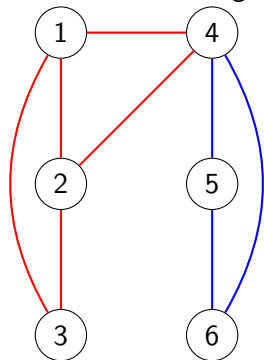
$\text{COL}(T_1) \neq \text{COL}(T_2)$, T_1, T_2 Disj, $\exists v, \text{deg}_R(v) \geq 2$

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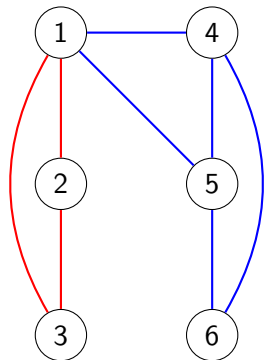
$C_4 : 4 - 2 - 3 - 1 - 4.$

$\text{COL}(T_1) \neq \text{COL}(T_2)$, T_1, T_2 Disj, $\exists v, \deg_{\mathbf{B}}(v) \geq 2$

Subcase B: $\exists v$, Left Side, $\deg_{\mathbf{B}}(v) \geq 2$.

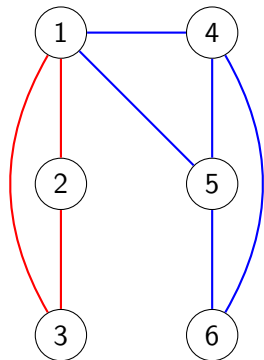
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$\text{COL}(T_1) \neq \text{COL}(T_2)$, T_1, T_2 Disj, $\exists v, \deg_{\mathbf{B}}(v) \geq 2$

Subcase B: $\exists v$, Left Side, $\deg_{\mathbf{B}}(v) \geq 2$.



$C_4 : 4 - 6 - 5 - 1 - 4.$

$\text{COL}(T_1) \neq \text{COL}(T_2)$, T_1, T_2 Disj, Subcase C

Subcase C:

$\text{COL}(T_1) \neq \text{COL}(T_2)$, T_1, T_2 Disj, Subcase C

Subcase C:

$\forall v$, left node, $\deg_{\mathbf{R}}(v) \geq 1$ and

$\text{COL}(T_1) \neq \text{COL}(T_2)$, T_1, T_2 Disj, Subcase C

Subcase C:

$\forall v$, left node, $\deg_{\mathbf{R}}(v) \geq 1$ and

$\forall v$, right node, $\deg_{\mathbf{B}}(v) \geq 1$.

$\text{COL}(T_1) \neq \text{COL}(T_2)$, T_1, T_2 Disj, Subcase C

Subcase C:

$\forall v$, left node, $\deg_{\mathbf{R}}(v) \geq 1$ and

$\forall v$, right node, $\deg_{\mathbf{B}}(v) \geq 1$.

Numb **edges** bet. $T_1, T_2 = \sum_{v \in \text{Left}} \deg_{\mathbf{R}}(v) \leq 3$.

$\text{COL}(T_1) \neq \text{COL}(T_2)$, T_1, T_2 Disj, Subcase C

Subcase C:

$\forall v$, left node, $\deg_{\mathbf{R}}(v) \geq 1$ and

$\forall v$, right node, $\deg_{\mathbf{B}}(v) \geq 1$.

Numb **edges** bet. $T_1, T_2 = \sum_{v \in \text{Left}} \deg_{\mathbf{R}}(v) \leq 3$.

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$\text{COL}(T_1) \neq \text{COL}(T_2)$, T_1, T_2 Disj, Subcase C

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Numb edges between T_1 and T_2 is $\leq 3 + 3 = 6$.

$\text{COL}(T_1) \neq \text{COL}(T_2)$, T_1, T_2 Disj, Subcase C

Subcase C:

$\forall v$, left node, $\deg_{\mathbf{R}}(v) \geq 1$ and

$\forall v$, right node, $\deg_{\mathbf{B}}(v) \geq 1$.

Numb **edges** bet. $T_1, T_2 = \sum_{v \in \text{Left}} \deg_{\mathbf{R}}(v) \leq 3$.

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Numb edges between T_1 and T_2 is $\leq 3 + 3 = 6$.

But there are 9 edges between T_1, T_2 .

$\text{COL}(T_1) \neq \text{COL}(T_2)$, T_1, T_2 Disj, Subcase C

Subcase C:

$\forall v$, left node, $\deg_{\mathbf{R}}(v) \geq 1$ and

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Numb edges between T_1 and T_2 is $\leq 3 + 3 = 6$.

But there are 9 edges between T_1, T_2 .

So can't occur.