Ramsey Theory on the Integers

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To

Eleanor

Emma and Sarah

–Bruce

To

Elisa

Quinn and Ava

–aaron

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Preface

Ramsey Theory on the Integers covers a variety of topics from the field of Ramsey theory, limiting its focus to the set of integers – an area that has seen a remarkable burst of research activity during the past twenty years.

The book has two primary purposes: (1) to provide students with a gentle, but meaningful, introduction to mathematical research – to give them an appreciation for the essence of mathematical research and its inescapable allure and also to get them started on their own research work; (2) to be a resource for all mathematicians who are interested in combinatorial or number theoretical problems, particularly "Erdős-type" problems.

Many results in Ramsey theory sound rather complicated and can be hard to follow; they tend to have a lot of quantifiers and may well involve objects whose elements are sets whose elements are sets (that is not a misprint). However, when the objects under consideration are sets of integers, the situation is much simpler. The student need not be intimidated by the words "Ramsey theory," thinking that the subject matter is too deep or complex – it is not! The material in this book is, in fact, quite accessible. This accessibility, together with the fact that scores of questions in the subject are still to be answered, makes Ramsey theory on the integers an ideal subject for a student's first research experience. To help students find suitable projects for their own research, every chapter includes a section of "Research Problems," where we present a variety of unsolved problems, along with a list of suggested readings for each problem.

Ramsey Theory on the Integers has several unique features. No other book currently available on Ramsey theory offers a cohesive study of Ramsey theory on the integers. Among several excellent books on Ramsey theory, probably the most well-known, and what may be considered *the* Ramsey theory book, is by Graham, Rothschild, and Spencer (*Ramsey Theory, 2^{nd} Edition* [**127**]). Other important books are by Graham (*Rudiments of Ramsey Theory* [**122**]), McCutcheon (*Elemental Methods in Ergodic Ramsey Theory* [**184**]), Nešetřil and Rödl (*Mathematics of Ramsey Theory* [**199**]), Prömel and Voigt (*Aspects of Ramsey Theory* [**207**]), Furstenberg (*Dynamical Methods in Ramsey Theory* [**111**]), and Winn (*Asymptotic Bounds for Classical Ramsey Numbers* [**274**]). These books, however, generally cover a broad range of subject matter of which Ramsey theory on the integers is a relatively small part. Furthermore, the vast majority of the material in the present book is not found in any other book. In addition, to the best of our knowledge, ours is the only Ramsey theory book that is accessible to the typical undergraduate mathematics major. It is structured as a textbook, with numerous (over 150) exercises, and the background needed to read the book is rather minimal: a course in elementary linear algebra and a 1-semester junior-level course in abstract algebra would be sufficient; an undergraduate course in elementary number theory or combinatorics would be helpful, but not necessary. Finally, *Ramsey Theory on the Integers* offers something new in terms of its potential appeal to the research community in general. Books offering a survey of solved and unsolved problems in combinatorics or number theory have been quite popular among researchers; they have also proven beneficial by serving as catalysts for new research in these fields. Examples include *Old and New Problems and Results in Combinatorial Number Theory* [**92**] by Erdős and Graham, *Unsolved Problems in Number Theory* [**135**] by Guy, and *The New Book of Prime Number Records* [**220**] by Ribenboim. With our text we hope to offer mathematicians an additional resource for intriguing unsolved problems. Although not nearly exhaustive, the present book contains perhaps the most substantial account of solved and unsolved problems in Ramsey theory on the integers.

This text may be used in a variety of ways:

- as an undergraduate or graduate textbook for a second course in combinatorics or number theory;
- in an undergraduate or graduate seminar, a capstone course for undergraduates, or an independent study course;
- by students working under an REU program, or who are engaged in some other type of research experience;
- by graduate students looking for potential thesis topics;
- by the established researcher seeking a worthwhile resource in its material, its list of open research problems, and its somewhat enormous (often a fitting word when discussing Ramsey theory) bibliography.

Chapter 1 provides preliminary material (for example, the pigeonhole principle) and a brief introduction to the subject, including statements of three classical theorems of Ramsey theory: van der Waerden's theorem, Schur's theorem, and Rado's theorem. Chapter 2 covers van der Waerden's theorem; Chapters 3–7 deal with various topics related to van der Waerden's theorem; Chapter 8 is devoted to Schur's theorem and a generalization; Chapter 9 explores Rado's theorem; and Chapter 10 presents several other topics involving Ramsey theory on the integers.

The text provides significant latitude for those designing a syllabus for a course. The only material in the book on which other chapters depend is that through Section 2.2. Thus, other chapters or sections may be included or omitted as desired, since they are essentially independent of one another (except for an occasional reference to a previous definition or theorem). We do, however, recommend that all sections included in a course be studied in the same order in which they appear in the book.

Each chapter concludes with a section of exercises, a section of research problems, and a reference section. Since the questions contained in the Research Problem sections are still open, we cannot say with certainty how difficult a particular one will be to solve; some may actually be quite simple and inconsequential. The problems that we deem most difficult, however, are labeled with the symbol $*$. The reference section of each chapter is organized by section numbers (including the exercise section). The specifics of each reference are provided in the bibliography at the end of the book.

The material covered in this book represents only a portion of the subject area indicated by the book's title. Many additional topics have been investigated, and we have attempted to include at least references for these in the reference sections. Yet, for every problem that has been thought of in Ramsey theory, there are many more which that problem will generate and, given the great variety of combinatorial structures and patterns that lie in the set of integers, countless new problems wait to be explored.

We would like to thank Dr. Edward Dunne and the members of the AMS production staff for their assistance in producing this book. We also thank Tom Brown, Scott Gordon, Jane Hill, Dan Saracino, Dan Schaal, Ralph Sizer, and the AMS reviewers for their helpful comments and advice, which greatly improved the manuscript. We also express our gratitude to Ron Graham and Doron Zeilberger for their support of this project. We owe a big debt to the pioneers and masters of the field, especially Ron Graham, Jarik Nešetřil, Joel Spencer, Neil Hindman, Tom Brown, Timothy Gowers, Hillel Furstenberg, Vitaly Bergelson, Vojtěch Rödl, Endre Szemerédi, László Lovász (we had to stop somewhere), and of course Bartel van der Waerden, Issai Schur, Richard Rado, and Frank Ramsey. To all of the others who have contributed to the field of Ramsey theory on the integers, we extend our sincere appreciation. Finally, we want to acknowledge that this book would not exist without the essential contributions of the late Paul Erdős. But beyond the content of his achievements, he has personally inspired the authors as mathematicians. Our professional lives would have had far less meaning and fulfillment without his work and his presence in our field. For that pervasive, though perhaps indirect, contribution to this text, we are in his debt.

Chapter 1

Preliminaries

Unsolved problems abound, and additional interesting open questions arise faster than solutions to the existing problems. - F. Harary

The above quote, which appeared in the 1983 article "A Tribute to F. P. Ramsey," is at least as apropos today as it was then. In this book alone, which covers only a modest portion of Ramsey theory, you will find a great number of open research problems. The beauty of Ramsey theory, especially Ramsey theory dealing with the set of integers, is that, unlike many other mathematical fields, very little background is needed to understand the problems. In fact, with just a basic understanding of some of the topics in this text, and a desire to discover new results, the undergraduate mathematics student will be able to experience the excitement and challenge of doing mathematical research.

Ramsey theory is named after Frank Plumpton Ramsey and his eponymous theorem, which he proved in 1928 (it was published posthumously in 1930). So, what is Ramsey theory? Although there is no universally accepted definition of Ramsey theory, we offer the following informal description:

> Ramsey theory is the study of the preservation of properties under set partitions.

In other words, given a particular set S that has a property P, is it true that whenever S is partitioned into finitely many subsets, one of the subsets must also have property P?

To illustrate further what sorts of problems Ramsey theory deals with, here are a few simple examples of Ramsey theory questions.

Example 1.1. Obviously, the equation $x + y = z$ has a solution in the set of positive integers (there are an infinite number of solutions); for example, $x = 1$, $y = 4$, $z = 5$ is one solution. Here's the question: is it true that whenever the set of positive integers is partitioned into a finite number of sets $S_1, S_2, \ldots, S_r$, then at least one of these sets will contain a solution to $x + y = z$? The answer turns out to be yes, as we shall see later in this chapter.

Example 1.2. Is it true that whenever the set $\{1, 2, \ldots, 100\}$ is partitioned into two subsets A and B, then at least one of the two subsets contains a pair of integers which differ by exactly two? To answer this question, consider the partition consisting of

$$A = \{1, 5, 9, 13, \ldots, 97\} \cup \{2, 6, 10, \ldots, 98\}$$

and

$$B = \{3, 7, 11, \ldots, 99\} \cup \{4, 8, 12, \ldots, 100\}.$$

We see that neither A nor B contains a pair of integers that differ by two, so that the answer to the given question is no.

Example 1.3. True or false: if there are 18 people in a group, then there must be either 4 people who are mutual acquaintances or 4 people who are mutual "strangers" (no two of whom have ever met)? (You will find the answer to this one later in the chapter.)

There is a wide range of structures and sets with which Ramsey theory questions may deal, including the real numbers, algebraic structures such as groups or vector spaces, graphs, points in the plane or in n dimensions, and others. This book limits its scope to Ramsey theory on the set of integers. (There is one exception – Ramsey's theorem itself – which is covered in this chapter.)

In this chapter we introduce the reader to some of the most well-known and fundamental theorems of Ramsey theory. We also present some of the basic terminology and notation that we will use.

1.1. The Pigeonhole Principle

Imagine yourself as a mailroom clerk in a mailroom with n slots in which to place the mail. If you have $n + 1$ pieces of mail to place into the n slots, what can we say about the amount of mail that will go into a slot? Well, we can't say much about the amount of mail a particular slot receives because, for example, one slot may get all of the mail. However, we can say that at least one slot must end up with at least two pieces of mail. To see this, imagine that you are trying to avoid having any slot with more than one piece of mail. By placing one piece of mail at a time into an unoccupied slot you can sort n pieces of mail. However, since there are more than n pieces of mail, you will run out of unoccupied slots before you are done. Hence, one slot must have at least two pieces of mail.

This simple idea is known as the pigeonhole principle, which can be stated this way:

If more than n pigeons are put into n pigeonholes, then some pigeonhole must contain at least two pigeons.

We now present the pigeonhole principle using somewhat more mathematical language.

Theorem 1.4 (Basic Pigeonhole Principle). *If an n-element subset is partitioned into r disjoint subsets where $n > r$, then at least one of the subsets contains more than one element.*

Example 1.5. In any class of 27 (or more) students, there must be two whose last name begins with the same letter.

Theorem 1.4 is a special case of the following more general principle.

Theorem 1.6 (Generalized Pigeonhole Principle). *If more than mr elements are partitioned into r sets, then some set contains more than m elements.*

Proof. Let S be a set with $|S| > mr$. Let $S = S_1 \cup S_2 \cup \cdots \cup S_r$ be any partition of S. Assume, for a contradiction, that $|S_i| \le m$ for all $i = 1, 2, \ldots, r$. Then

$$|S| = \sum_{i=1}^{r} |S_i| \le mr,$$

a contradiction. Hence, for at least one i, the set S_i contains more than m elements, i.e., $|S_i| \ge m + 1$. $\square$

We see that Theorem 1.4 is a special case of Theorem 1.6 by taking $m = 1$. There are other common formulations of the pigeonhole principle; you will find some of these in the exercises.

Although the pigeonhole principle is such a simple concept, and seems rather obvious, it is a very powerful result, and it can be used to prove a wide array of not-so-obvious facts.

Here are some examples.

Example 1.7. For each integer $n = 1, 2, \ldots, 200$, let $R(n)$ be the remainder when n is divided by 7. Then some value of $R(n)$ must occur at least 29 times. To see this, we can think of the 200 integers as the pigeons, and the seven possible values of $R(n)$ as the pigeonholes. Then, according to Theorem 1.6, since $200 > 28(7)$, one of the pigeonholes must contain more than 28 elements.

Example 1.8. We will show that within any sequence of $n^2 + 1$ integers there exists a monotonic subsequence of length $n+1$. (A sequence $\{x_i\}_{i=1}^{m}$ is called monotonic if it is either nondecreasing or nonincreasing). Let our sequence be $\{a_i\}_{i=1}^{n^2+1}$. For each $i \in \{1, 2, \ldots, n^2 + 1\}$, let ℓ_i be the length of the longest nondecreasing subsequence starting at a_i. If $\ell_i \ge n + 1$ for some i we are done, so we may assume that $\ell_i \le n$ for $1 \le i \le n^2 + 1$. Since each of the ℓ_i's has a value between 1 and n, by the pigeonhole principle there exists $j \in \{1, 2, \ldots, n\}$ so that $n + 1$ of the numbers ℓ_i equal j. Call these $\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_{n+1}}$, where $i_1 < i_2 < \cdots < i_{n+1}$. Next, look at the subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_{n+1}}$. We claim that this is a nonincreasing subsequence. To see this, assume, for a contradiction, that it is not nonincreasing. Then $a_{i_k} < a_{i_{k+1}}$ for some k. Hence, the longest nondecreasing subsequence starting at a_{i_k} would be of length greater than j, since there exists a subsequence of length j starting at $a_{i_{k+1}}$, a contradiction.

Next, we give another example for which the pigeonhole principle may not immediately appear to be applicable.

Example 1.9. Color each point in the xy-plane having integer coordinates either red or blue. We show that there must be a rectangle with all of its vertices the same color. Consider the lines $y = 0$, $y = 1$, and $y = 2$ and their intersections with the lines $x = i$, $i = 1, 2, \ldots, 9$. On each line $x = i$ there are three intersection points colored either red or blue. Since there are only $2^3 = 8$ different ways to color three points either red or blue, by the pigeonhole principle two of the vertical lines, say $x = j$ and $x = k \ne j$ must have the identical coloring (i.e. the color of (j, y) is the same as the color of (k, y) for $y = 0, 1, 2$). Using the pigeonhole principle again, we see that two of the points $(j, 0), (j, 1)$, and $(j, 2)$ must be the same color, say (j, y_1) and (j, y_2). Then the rectangle with vertices $(j, y_1), (j, y_2), (k, y_1)$, and (k, y_2) is the desired rectangle.

In the last example, we used colors as the "pigeonholes." Using colors to represent the subsets of a partition in this way is often convenient, and is quite typical in many areas of Ramsey theory.

1.2. Ramsey's Theorem

Ramsey's theorem can be considered a refinement of the pigeonhole principle, where we are not only guaranteed a certain number of elements in a pigeonhole, but we also have a guarantee of a certain relationship between these elements. It is a theorem that is normally stated in terms of the mathematical concept known as a graph. We will define what we mean by a graph very shortly, but before doing so, we consider the following example, known as the Party problem.

Example 1.10. We will prove the following: at a party of six people, there must exist either three people who have all met one another or three people who are mutual strangers (i.e., no two of whom have met). By the pigeonhole principle, we are guaranteed that for each person, there are three people that person has met or three people that person has never met. We now want to show that there are three people with a certain relationship *between* them, namely, three people who all have met one another, or three people who are mutual strangers. First, assign to each pair of people one of the colors red or blue, with a red "line" connecting two people who have met, and a blue "line" connecting two people who are strangers. Hence, we want to show that for any coloring of the lines between people using the colors red and blue, there is either a red triangle or a blue triangle (with the people as vertices). Next, pick out one person at the party, say person X. Since there are five other people at the party, by the pigeonhole principle X either knows at least three people, or is a stranger to at least three people. We may assume, without loss of generality, that X knows at least 3 people at the party. Call these people A, B, and C. So far we know that the lines connecting X to each of A, B, and C are red. If there exists a red line between any of A, B, and C then we are done, since, for example, a red line between A and B would give the red triangle ABX. If the lines connecting A, B, and C are all blue, then ABC is a blue triangle.

Concerning the Party problem, another question we might ask is this: is 6 the lowest number of party members for the property we seek to hold? That is, does there exist a way to have five people at the party and not have either of the types of "triangles" discussed in the above example? To see that we cannot have only five people at the party and expect the same result, place five people in a circle and assume that each person knows the two people next to him/her, but no one else (draw a sketch to see that there is no red triangle and no blue triangle).

The fact that there is a solution to the Party problem is a special case of what is known as Ramsey's theorem. In order to state Ramsey's theorem we will use a few definitions from graph theory.

Definition 1.11. A *graph* $G = (V, E)$ is a set V of points, called *vertices*, and a set E of pairs of vertices, called *edges*.

Definition 1.12. A *subgraph* $G' = (V', E')$ of a graph $G = (V, E)$ is a graph such that $V' \subseteq V$ and $E' \subseteq E$.

Definition 1.13. A *complete graph on n vertices*, denoted K_n, is a graph on n vertices, with the property that every pair of vertices is connected by an edge.

Definition 1.14. An *edge-coloring* of a graph is an assignment of a color to each edge of the graph. A graph which has been edge-colored is called a *monochromatic graph* if all of its edges are the same color.

We may now express the solution to the Party problem in graph-theoretical language. It says that for every 2-coloring, using the colors red and blue, of the edges of K_6 there must be either a red K_3 (a triangle) or a blue K_3; and furthermore, that there exists a 2-coloring of the edges of K_5 that fails to have this property.

We now state Ramsey's theorem for two colors.

Theorem 1.15 (Ramsey's Theorem for Two Colors). *Let $k, \ell \geq 2$. There exists a least positive integer $R = R(k, \ell)$ such that every edge-coloring of K_R, with the colors red and blue, admits either a red K_k subgraph or a blue K_l subgraph.*

Proof. First note that $R(k, 2) = k$ for all $k \geq 2$, and $R(2, \ell) = \ell$ for all $\ell \geq 2$ (this is easy). We proceed via induction on the sum $k + \ell$, having taken care of the case when $k + \ell = 5$. Hence, let $k + \ell \geq 6$, with $k, \ell \geq 3$. We may assume that both $R(k, \ell - 1)$ and $R(k - 1, \ell)$ exist. We claim that $R(k, \ell) \leq R(k - 1, \ell) + R(k, \ell - 1)$, which will prove the theorem.

Let $n = R(k - 1, \ell) + R(k, \ell - 1)$ and pick one vertex from K_n, call it v. Then there are $n - 1$ edges from v to the other vertices. Let A be the number of red edges and B be the number of blue edges coming out of v. Then, either $A \geq R(k - 1, \ell)$ or $B \geq R(k, \ell - 1)$, since if $A < R(k - 1, \ell)$ and $B < R(k, \ell - 1)$, then $A + B \leq n - 2$, contradicting the fact that $A + B = n - 1$. We may assume, without loss of generality, that $A \geq R(k - 1, \ell)$. Let V be the set of vertices connected to v by a red edge, so that $|V| \geq R(k - 1, \ell)$. By the inductive hypothesis, K_V contains either a red K_{k-1} subgraph or a blue K_ℓ subgraph. If it contains a blue K_ℓ subgraph, we are done. If it contains a red K_{k-1} subgraph, then by connecting v to each vertex of this red subgraph we have a red K_k subgraph (since v is connected to V by only red edges), and the proof is complete. □

The numbers $R(k, \ell)$ are known as the 2-color Ramsey numbers. The solution to the Party problem tells us that $R(3,3) = 6$. By Ramsey's theorem, we may extend the Party problem in various ways. For example, we know there exists a number n so that if there were n people at a party, then there would have to be either a group of four mutual acquaintances or a group of five mutual strangers. This number n is the Ramsey number $R(4,5)$.

There are other ways to extend the Party problem. For example, in Exercise 1.11 we consider the case where people either love, hate, or are indifferent to, each other. In this situation we want to find three people who all love one another, three people who all hate one another, or three people who are all indifferent toward one another. Exercise 1.11 states that 17 people at the party will suffice (in fact 17 is the least such number with this property, but you cannot conclude this from Exercise 1.11). This is an example of a 3-color Ramsey number. More generally, Ramsey's theorem for two colors can easily be generalized to $r \geq 3$ colors (this is left as Exercise 1.17), in which case the Ramsey numbers are denoted by $R(k_1, k_2, \ldots, k_r)$. In case $k_i = k$ for $i = 1, \ldots, r$, we use the simpler notation $R_r(k)$. Thus, for example, in the "love-hate-indifferent" problem, we have $R(3,3,3) = R_3(3) = 17$.

The existence of the Ramsey numbers has been known since 1930. However, they are notoriously difficult to compute; the only known values are $R(3,3) = 6$, $R(3,4) = 9$, $R(3,5) = 14$, $R(3,6) = 18$, $R(3,7) = 23$, $R(3,8) = 28$, $R(3,9) = 36$, $R(4,4) = 18$, $R(4,5) = 25$, and $R(3,3,3) = 17$. (The fact that $R(4,4) = 18$ answers the question posed in Example 1.3.)

Obviously, Ramsey theory is named after Frank Ramsey. However, his famous theorem is the only result of Ramsey's in the field named after him. Unfortunately, Ramsey died of complications due to jaundice in 1930, a month before his 27th birthday, but not before he left his mark.

1.3. Some Notation

In this section we go over some notation that we will frequently use.

We shall denote the set of integers by $\mathbb{Z}$, and the set of positive integers by $\mathbb{Z}^+$. Most of our work will be confined to the set of integers. Hence, when speaking about an "interval" we will mean a set of the form $\{a, a+1, \ldots, b\}$, where $a < b$ are integers. Usually we will denote this interval more simply by $[a, b]$.

When dealing with two sets X and Y, we will sometimes use the set $S = X - Y$, which we define to be the set of elements in X that are not in Y. Also, for S a set and a a real number, $a + S$ and aS will denote $\{a + s : s \in S\}$ and $\{as : s \in S\}$, respectively.

Sometimes we will find it convenient to use symbols such as 0,1, or 2 to stand for different "colors" rather than actual color names such as red or blue. We make this more formal in the following definition.

Definition 1.16. An *r-coloring* of a set S is a function $\chi : S \to C$, where $|C| = r$.

Typically, we will use $C = \{0, 1, \ldots, r-1\}$ or $C = \{1, 2, \ldots, r\}$. We can think of an r-coloring χ of a set S as a partition of S into r subsets $S_1, S_2, \ldots, S_r$, by associating the subset S_i with the set $\{x \in S : \chi(x) = i\}$.

The next definition will be used extensively.

Definition 1.17. A coloring χ is *monochromatic* on a set S if χ is constant on S.

Example 1.18. Let $\chi : [1,5] \to \{0,1\}$ be defined by $\chi(1) = \chi(2) = \chi(3) = 1$ and $\chi(4) = \chi(5) = 0$. Then χ is a 2-coloring of [1,5] that is monochromatic on $\{1,2,3\}$ and on $\{4,5\}$.

We will often find it convenient to represent a particular 2-coloring of an interval as a string of 0's and 1's. For example, the coloring in Example 1.18 could be represented by the string 11100. We may also abbreviate this coloring by writing $1^3 0^2$. We may extend this notation to r-colorings for $r \geq 3$ by using strings with symbols belonging to the set $\{0, 1, 2, \ldots, r-1\}$. For example, define the 3-coloring χ on the interval [1,10] by $\chi(i) = 0$ for $1 \leq i \leq 5$, $\chi(i) = 1$ for $6 \leq i \leq 9$, and $\chi(10) = 2$. Then we may write $\chi = 0000011112$ or, equivalently, $\chi = 0^5 1^4 2$.

Sometimes we will want to describe the magnitude of functions asymptotically. For this purpose we mention two very commonly used symbols, called "Big-O" and "little-o."

Let $f(n)$ and $g(n)$ be functions that are nonzero for all n. We say that $f(n) = O(g(n))$ if there exist constants $c, m > 0$, independent of n, such that $0 < \left|\frac{f(n)}{g(n)}\right| \leq c$ for all $n > m$. In other words, $\lim_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| \leq c$, if the limit exists. We say that $f(n) = o(g(n))$ if for all $c > 0$ there exists a constant $m > 0$, independent of n, such that $\left|\frac{f(n)}{g(n)}\right| < c$ for all $n > m$. In other words, $\lim_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| = 0$.

If f and g are nonzero functions such that $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists and is equal to ℓ, where $|\ell| \neq \infty$ and $\ell \neq 0$, then $f(n) = O(g(n))$. If $\ell = 0$ we have $f(n) = o(g(n))$. If $\ell = \infty$, then we have $g(n) = o(f(n))$ by taking the reciprocal of the argument of the limit. Intuitively, if $f(n) = O(g(n))$ and $g(n) = O(f(n))$, then $f(n)$ and $g(n)$ have a similar growth rate; and if $f(n) = o(g(n))$, then $f(n)$ is insignificant compared to $g(n)$, for large n.

If $f(n)$ and $g(n)$ are functions with the same growth rate, i.e., if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$, we may write $f(n) \sim g(n)$.

An example to explain these concepts is in order.

Example 1.19. Let $f(n) = \frac{n^2}{22} + 5n$ and $g(n) = n^2$. Then $f(n) = O(g(n))$, or, equivalently, $f(n) = O(n^2)$. We may also describe $f(n)$'s rate of growth by $f(n) = \frac{n^2}{22}(1 + o(1))$. To see this, we have

$$\frac{n^2}{22}(1 + o(1)) = \frac{n^2}{22} + o(1)\frac{n^2}{22}.$$

Now, since $\frac{5n}{n^2/22} = \frac{110}{n}$ and $\lim_{n\to\infty} \frac{110/n}{1} = 0$, we have $5n = o(1)\frac{n^2}{22}$. We may also write $f(n) \sim \frac{n^2}{22}$ to describe the growth rate of $f(n)$.

We will also use the following functions. For x a real number, we use $\lfloor x \rfloor$ to denote the greatest integer n such that $n \leq x$ (this is often called the "floor" function). The least integer function of a real number x, defined as the least integer n such that $n \geq x$, is denoted by $\lceil x \rceil$ (this is often referred to as the "ceiling function.")

1.4. Three Classical Theorems

Somewhat surprisingly, Ramsey's theorem was not the first, nor even the second, theorem in the area now known as Ramsey theory. The results that are generally accepted to be the earliest Ramsey-type theorems are due, in chronological order, to Hilbert, Schur, and van der Waerden. All of these results, which preceded Ramsey's theorem, deal with colorings of the integers, the theme of this book. Interestingly, even though Ramsey's theorem is a theorem about graphs, we will see later that it can be used to give some Ramsey-type results about the integers.

In this section we introduce three classical theorems concerning Ramsey theory on the integers. We will talk much more about each of these theorems in later chapters.

We start with a reminder of what an arithmetic progression is.

Definition 1.20. A k-term *arithmetic progression* is a sequence of the form $a, a+d, a+2d, \ldots, a+(k-1)d$, where $a \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$.

We now state van der Waerden's theorem, which was proved in 1927.

Theorem 1.21 (Van der Waerden's Theorem). *For all positive integers k and r, there exists a least positive integer $w(k; r)$ such that for every r-coloring of $[1, w(k; r)]$ there is a monochromatic arithmetic progression of length k.*

The numbers $w(k; r)$ are known as the van der Waerden numbers. Let's look at a simple case. Let $k = r = 2$. Hence, we want to find the minimum integer $w = w(2; 2)$ so that no matter how we partition the interval $[1, w] = \{1, 2, \ldots, w\}$ into two subsets (i.e., 2-color $[1, w]$), we must end up with at least one of the two subsets containing a pair of elements $a, a+d$, where $d \geq 1$ (i.e., we must end up with a monochromatic 2-term arithmetic progression). Consider a 2-coloring of $\{1, 2\}$ where 1 and 2 are assigned different colors. Obviously, under a such a coloring, $\{1, 2\}$ does not contain a 2-term arithmetic progression that is monochromatic. Thus, $w(2; 2)$ is not equal to 2 (not every 2-coloring of $[1, 2]$ yields the desired monochromatic sequence). Does 3 work? That is, does every 2-coloring of $[1, 3]$ yield a monochromatic

2-term arithmetic progression. The answer is yes, by a simple application of the pigeonhole principle, since any 2-element set of positive integers is a 2-term arithmetic progression. Thus, we have shown that $w(2;2) = 3$.

Finding $w(2;2)$ was rather simple. All the van der Waerden numbers $w(2;r)$ are just as easy to find (we leave this as an exercise in a later chapter). For $k \geq 3$, the evaluation of these numbers very quickly becomes much more difficult. In fact, the only known van der Waerden numbers are $w(3;2) = 9$, $w(3;3) = 27$, $w(3;4) = 76$, $w(4;2) = 35$ and $w(5;2) = 178$. Besides trying to find exact values of the van der Waerden numbers, there is another open question that has been one of the most difficult, and most appealing, problems in Ramsey theory. Namely, finding a reasonably good estimate of $w(k;r)$ in terms of k and r. We shall talk more about such questions, and the progress that has been made on them, in Chapter 2.

Van der Waerden's theorem has spawned many results in Ramsey theory. For this reason, and because the notion of an arithmetic progression is such a natural and simple concept, a large portion of this book is dedicated to various offshoots, refinements, extensions, and generalizations of van der Waerden's theorem.

The next two main results deal with solutions to equations and systems of equations. Let $\mathcal{E}$ represent a given equation or system of equations. We call $(x_1, x_2, \ldots, x_k)$ a *monochromatic solution to* $\mathcal{E}$ if $x_1, x_2, \ldots, x_k$ are all the same color and they satisfy $\mathcal{E}$.

We next theorem we present, proved by Issai Schur in 1916, is one of the earliest results in Ramsey theory.

Theorem 1.22 (Schur's Theorem). *For any $r \geq 1$, there exists a least positive integer $s = s(r)$ such that, for any r-coloring of $[1,s]$, there exists a monochromatic solution to $x + y = z$.*

The numbers $s(r)$ are called the Schur numbers. As a simple example, we look at $s(2)$. Here we want the least positive integer s so that whenever $[1,s]$ is 2-colored, there will exist integers x, y, z (not necessarily distinct) satisfying $x + y = z$. Notice that $s(2)$ must be greater than four, because if we take the 2-coloring χ of [1,4] defined by $\chi(1) = \chi(4) = 0$ and $\chi(2) = \chi(3) = 1$, then it is not possible to find x, y, and z all of the same color satisfying $x + y = z$. Meanwhile, every 2-coloring of $[1,5]$ (there are $2^5 = 32$ of them) does yield such a monochromatic triple (this is proved in Example 8.5). Thus, $s(2) = 5$.

As it turns out, the only Schur numbers that are currently known are $s(1) = 2$, $s(2) = 5$, $s(3) = 14$, and $s(4) = 45$. We will learn much more about Schur's theorem in Chapter 8.

The third classical theorem we mention is Rado's theorem, which is a generalization of Schur's theorem. In fact, Richard Rado was a student of Schur. The idea of Rado's theorem may be described as follows. Thinking of Schur's theorem as a theorem about the homogeneous linear equation $x + y - z = 0$, we ask the following more general question. Which systems, $\mathcal{L}$, of homogeneous linear equations with integer coefficients have the following property: for every $r \geq 1$, there exists a least positive integer $n = n(\mathcal{L}; r)$ such that every r-coloring of $[1,n]$ yields a monochromatic solution to $\mathcal{L}$?

In a series of articles published in the 1930's, Rado completely answered this question. Since Rado's theorem, in its most general form, is a bit complicated to describe, we will postpone stating the general theorem until Chapter 9, which is devoted to Rado's theorem. Instead, we mention here the special case of Rado's theorem in which the system consists of only a single equation.

We first need the following definition.

Definition 1.23. For $r \geq 1$, a linear equation $\mathcal{E}$ is called *r-regular* if there exists $n = n(\mathcal{E}; r)$ such that for every r-coloring of $[1,n]$ there is a monochromatic solution to $\mathcal{E}$. It is called *regular* if it is r-regular for all $r \geq 1$.

Example 1.24. Using Definition 1.23, Schur's theorem can be stated as "the equation $x + y = z$ is regular."

We now state Rado's theorem for a single equation.

Theorem 1.25 (Rado's Single Equation Theorem). *Let $\mathcal{E}$ represent the linear equation $\sum_{i=1}^{n} c_i x_i = 0$, where $c_i \in \mathbb{Z} - \{0\}$ for $1 \leq i \leq n$. Then $\mathcal{E}$ is regular if and only if some nonempty subset of the c_i's sums to 0.*

Example 1.26. The equation $x + y = z$, i.e., $x + y - z = 0$, satisfies the requirements of Theorem 1.25. Hence, as noted before and proved by Schur, $x + y = z$ is regular.

Example 1.27. It follows from Rado's theorem that the equation $3x_1 + 4x_2 + 5x_3 - 2x_4 - x_5 = 0$ is regular, since the sum of the first, fourth, and fifth coefficients is 0.

1.5. A Little More Notation

The three classical theorems mentioned above all have a somewhat similar flavor. That is, they have the following general form: there exists a positive integer $n(r)$ such that for every r-coloring of $[1, n(r)]$ there is a monochromatic set belonging to a particular family of sets. In one case, the family of sets was the k-term arithmetic progressions; in another case the family consisted of all solutions to a certain equation; and so on. Throughout this book we will be looking at this type of problem, and so it will be worthwhile to have a general notation that can be used for any such problem.

Let $\mathcal{F}$ be a certain family of sets, and let k and r be positive integers. We denote by $R(\mathcal{F}, k; r)$ the least positive integer, if it exists, such that for any r-coloring of $[1, R(\mathcal{F}, k; r)]$, there is a monochromatic k-term member of $\mathcal{F}$. In the case where no such integer exists, we say $R(\mathcal{F}, k; r) = \infty$. Because our discussion will often be confined to the situation in which the number of colors is two, we often denote the function $R(\mathcal{F}, k; 2)$ more simply as $R(\mathcal{F}, k)$. If the length of the sequence is understood (as in Schur's theorem), we write $R(\mathcal{F}; r)$.

For certain Ramsey-type functions we deal with, it will be convenient to use a notation other than $R(\mathcal{F}, k; r)$. For example, later in the book we will encounter a type of sequence called a descending wave, for which we will use the notation $DW(k; r)$ rather than something like $R(DW, k; r)$ (where DW would represent the family of all descending waves). Similarly, since the notation $w(k; r)$ is so standard, we will use $w(k; r)$ instead of $R(AP, k; r)$ and $w(k)$ instead of $R(AP, k)$, where AP is the family of all arithmetic progressions.

Finally, we remark that the notation $R(k, \ell)$ is reserved for the classical Ramsey numbers defined in Section 1.2 (note the absence of a family $\mathcal{F}$ here).

Throughout this book we will be considering various collections, $\mathcal{F}$, of sets of integers and, as with the three classical theorems of Section 1.4, wanting to know if, for a specified value of r and a particular set $M \subseteq \mathbb{Z}$, every r-coloring of M yields a monochromatic member of $\mathcal{F}$. For the case in which M is the set of positive integers, we have the following definition.

Definition 1.28. Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{Z}^+$, and let $r \geq 1$. If for every r-coloring of $\mathbb{Z}^+$ and all $k \geq 1$, there is a monochromatic k-element member of $\mathcal{F}$, then we say that $\mathcal{F}$ is *r-regular*. If $\mathcal{F}$ is r-regular for all r, we say that $\mathcal{F}$ is *regular*.

Sometimes we will replace the phrase "for all $k \geq 1$, there is a monochromatic k-element member of $\mathcal{F}$" by "there are arbitrarily large members of $\mathcal{F}$."

Example 1.29. Let $\mathcal{F} = AP$, the collection of all arithmetic progressions. By van der Waerden's theorem, $\mathcal{F}$ is regular since for every finite coloring of $\mathbb{Z}^+$, there exists, for every $k \geq 1$, a monochromatic k-term arithmetic progression.

Whereas Definition 1.28 pertains to *all* colorings of a set, we will also want to consider whether or not a particular coloring of a set M yields a monochromatic member of the collection $\mathcal{F}$. For this we have the next definition.

Definition 1.30. Let $\mathcal{F}$ be a family of subsets of $\mathbb{Z}$ and let k be a positive integer. Let $r \geq 1$. An r-coloring of a set $M \subseteq \mathbb{Z}$ is called *$(\mathcal{F}, k; r)$-valid* if there is no monochromatic k-element member of $\mathcal{F}$ contained in M.

When the number of colors is understood, we may simply say that a coloring is (F, k)-valid. Also, when there is no possible confusion as to the meaning of $\mathcal{F}$ or the value of k, we may simply say that a coloring is *valid*.

As an example, if $\mathcal{F}$ is the family of sets of even numbers, then the 2-coloring of $[1, 10]$ represented by the binary sequence 1110001110 is

$(\mathcal{F}, 4)$-valid since there is no monochromatic 4-term sequence belonging to $\mathcal{F}$ (i.e., there do not exists four even numbers that have the same color).

Let's consider another example.

Example 1.31. Let $\mathcal{F}$ be the family of all subsets of $\mathbb{Z}^+$. We will determine a precise formula for $R(\mathcal{F}, k; r)$. First, let χ be any r-coloring of $[1, r(k-1)+1]$. By the generalized pigeonhole principle, since we are partitioning a set of $r(k-1)+1$ elements into r sets, there must be, for some color, more than $k-1$ elements of that color. Thus, under χ, there is a monochromatic k-element member of $\mathcal{F}$. Since χ is an arbitrary r-coloring, we have that $R(\mathcal{F}, k; r) \leq r(k-1)+1$. On the other hand, there do exist r-colorings of the interval $[1, r(k-1)]$ that are $(\mathcal{F}, k)$-valid. Namely, assign exactly $k-1$ members of the interval to each of the colors. Then no color will have a k-element member of $\mathcal{F}$. Thus, $R(\mathcal{F}, k; r) \geq r(k-1)+1$, and hence $R(\mathcal{F}, k; r) = r(k-1)+1$.

The fact that in Example 1.31 the numbers $R(\mathcal{F}, k; r)$ always exist is not very surprising. After all, $\mathcal{F}$ is so plentiful that it is easy to find a monochromatic member. When the family of sets we are considering is not as "big," the behavior of the associated Ramsey function is much less predictable. For certain $\mathcal{F}$ we will find that $R(\mathcal{F}, k; r) < \infty$ for all k and r, while for others this will only happen (for all k) provided r does not exceed a certain value. There will even be cases where $R(\mathcal{F}, k; r)$ never exists except for a few small values of k and r.

We will encounter many different results in this book, but the common thread will be an attempt to find answers, to whatever extent we can, to the following two questions.

1. For which $\mathcal{F}$, k, and r does the function $R(\mathcal{F}, k; r)$ exist?
2. If $R(\mathcal{F}, k; r)$ exists, what can we say about its magnitude?

1.6. Exercises

1.1 A bridge club has 10 members. Every day, four members of the club get together and play one game of bridge. Prove that after two years, there is some particular set of four members that has played at least four games of bridge together.

1.2 Prove that if the numbers $1, 2, \ldots, 12$ are randomly positioned around a circle, then some set of three consecutively positioned numbers must have a sum of at least 19.

1.3 Prove the following versions of the pigeonhole principle.
a) If $a_1, a_2, \ldots, a_n$, c are real numbers such that $\sum_{i=1}^{n} a_i \geq c$, then there is at least one value of i such that $a_i \geq \frac{c}{n}$.
b) If $a_1, a_2, \ldots, a_n$ are integers, and c is a real number such that $\sum_{i=1}^{n} a_i \geq c$, then there is at least one value of i such that $a_i \geq \left\lceil \frac{c}{n} \right\rceil$.

1.4 With regard to Example 1.8, show that, given a sequence of only n^2 numbers, there need not be a monotonic subsequence of length $n+1$.

1.5 Let $r \geq 2$. Show that there exists a least positive integer $M = M(k; r)$ so that any r-coloring of M integers admits a monochromatic monotonic k-term subsequence. Determine $M(k; r)$. (Note that from Example 1.8, $M(k; 1) = k^2 + 1$.)

1.6 Let $r \geq 3$. Let χ be any r-coloring of the set

$$S = \{(x, y) \in \mathbb{R}^2 : x, y \in \mathbb{Z}\}$$

(the members of S are sometimes called lattice points). Show that, under χ, there must exist a rectangle with all vertices the same color.

1.7 Explain how the Party problem fits the description of Ramsey theory offered on page 1.

1.8 Since $R(3, 3) = 6$, we know that any 2-coloring of K_6 must admit at least one monochromatic triangle. In fact, any 2-coloring of K_6 must admit at least two monochromatic triangles. Prove this fact.

1.9 Show that any 2-coloring of K_7 must admit at least four monochromatic triangles.

1.10 Generalize Exercises 1.8 and 1.9 above to K_n. (Hint: Let r_i, $i = 1, 2, \ldots, n$, be the number of red edges connected to vertex i. Show that the number of monochromatic triangles is thus $\binom{n}{3} - \frac{1}{2}\sum_{i=1}^{n} r_i(n-1-r_i)$. Minimize this function to deduce the result.)

1.11 Show that any 3-coloring of K_{17} must admit at least one monochromatic triangle, via an argument similar to the one showing $R(3,3) \le 6$, and using the fact that $R(3,3) = 6$.

1.12 Explain why $R(k,\ell) = R(\ell,k)$.

1.13 Prove that $R(k,\ell) < R(k-1,\ell)+R(k,\ell-1)$, if both $R(k-1,\ell)$ and $R(k,\ell-1)$ are even.

1.14 Show that $R(k,\ell) \le \binom{k+\ell-2}{k-1}$ by showing that the recurrence $R(k,\ell) = R(k-1,\ell) + R(k,\ell-1)$ is satisfied by a certain binomial coefficient.

1.15 We can determine a lower bound for $R(k,k)$ by using the probabilistic method (largely due to Erdős). Show that $R(k,k) > \frac{k}{e\sqrt{2}}2^{\frac{k}{2}}$ for large k via the following steps.
a) Randomly color the edges of K_n either red or blue, i.e., each edge is colored red with probability $\frac{1}{2}$. Show that for a given set of k vertices of K_n, the probability that the complete graph on these k vertices is monochromatic equals $\frac{2}{2^{\binom{k}{2}}}$.
b) Let p_k be the probability that a monochromatic K_k subgraph exists in our random coloring. Show that

$$p_k \le \sum_{i=1}^{\binom{n}{k}} 2^{1-\binom{k}{2}} = \binom{n}{k}2^{1-\binom{k}{2}}.$$

c) Use (b) to show that if $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$.
d) Stirling's formula for the asymptotic behavior of $n!$ says that $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Use Stirling's formula to finish the problem.

1.16 Consider the following way to color the edges of K_n. Number the vertices of K_n in a counterclockwise fashion from 1 to n. Next, partition the numbers $\{1,2,\ldots,n-1\}$ into two subsets. Call these sets R and B for red and blue. Now, each edge has two vertices, say i and j. Calculate $|j-i|$ for that edge. If $|j-i| \in R$, then color the edge connecting i and j red. If $|j-i| \in B$, then color the edge connecting i and j blue. Such a coloring is called a *difference coloring.* Since $R(3,4) = 9$, we know that there is an edgewise 2-coloring of K_8 with no red K_3 and no blue K_4. One such coloring is a difference coloring defined as follows. Color an edge red if $|j-i| \in \{1,4,7\}$, and blue if $|j-i| \in \{2,3,5,6\}$. Show that this coloring does indeed prove that $R(3,4) > 8$, i.e., that there is no red K_3 and no blue K_4.

1.17 Prove Ramsey's theorem for r colors, where $r \ge 3$.

1.18 Show that $s(3) \ge 14$, where $s(r)$ is the r-color Schur number.

1.19 Let $r \ge 1$. Show that for any integer a, there exists an integer $M = M(a;r)$ such that, for any r-coloring of $[1,M]$, there is a monochromatic solution to $x + ay = z$. Deduce Schur's theorem from this result.

1.7. Research Problems

Note: in this chapter and the next we present some problems which, although understandable and rather simple to state, are considered to be extremely difficult to solve. We include them primarily for illustrative purposes (and because they are intriguing problems). We suggest that the research problems from Chapters 3 through 10 are more suitable for beginning research in Ramsey theory.

***1.1** For $n \ge 3$, define $g(n)$ to be the least positive integer with the following property. Whenever the set of lines through $g(n)$ points satisfy both (a) no two lines are parallel, and (b) no three lines intersect in the same point, then the set of $g(n)$ points contains the vertices of a convex n-gon. Prove or disprove: $g(n) = 2^{n-2}+1$. It is known to hold for $n = 3,4,5$. It is also known that the existence of $g(n)$ is equivalent to Ramsey's theorem.
References: **[42]**, **[90]**, **[103]**

***1.2** Prove or disprove the following conjecture proposed by Paul Erdős and V. Sós: $R(3,n+1) - R(3,n) \to \infty$ as $n \to \infty$.
References: **[12]**, **[89]**, **[155]**

***1.3** Determine $\lim_{n\to\infty} R(n,n)^{1/n}$ if it exists. It is known that if this limit exists, then it is between $\sqrt{2}$ and 4. (The lower

bound comes from Exercise 1.15 and the upper bound is deduced from Exercise 1.14 using Stirling's formula, which is given in Exercise 1.15.)
References: [**99**], [**255**], [**256**]

1.8. References

Harary's quote appears in [**138**], which also contains a biographical sketch of Ramsey.

§**1.1.** Example 1.8 is due to Erdős and Szekeres [**101**].

§**1.2.** Theorem 1.15 is proved in [**218**]. There is a much more general form of Ramsey's theorem (not confined to edge-colorings); for a proof, see, for example, [**127**]. Erdős and Szekeres rediscovered Ramsey's theorem, in an equivalent form, in [**101**] (see Research Problem 1.1). The Ramsey number $R(4,5) = 25$, discussed after Theorem 1.15, is the most recently discovered Ramsey number. See [**187**] for details. The Ramsey numbers $R(3,4)$, $R(3,5)$, $R(4,4)$, and $R(3,3,3)$ were discovered by Greenwood and Gleason [**132**]. Kéry determined $R(3,6)$ in [**154**]. Graver and Yackel [**130**] determine $R(3,7)$ by matching the upper bound given by Kalbfleisch in [**153**]. McKay and Min [**186**] determined $R(3,8)$ by matching the lower bound given by Grinstead and Roberts [**133**]. Grinstead and Roberts [**133**] determined $R(3,9)$ by matching the lower bound given in [**153**]. For a survey of the best bounds to date on small Ramsey numbers see [**216**].

§**1.4.** Hilbert's result is in [**142**]. A proof using more modern language is in [**124**, p. 1368]. Recent work regarding Hilbert's theorem can be found in [**57**] and [**134**]. Van der Waerden's theorem and its original proof are in [**270**]. Schur's theorem and its original proof can be found in [**247**]. Rado's theorem was proved in the series of papers [**215**], [**214**], and [**213**]. For a summary of Rado's theorem see [**211**].

§**1.7.** The original work on $g(n)$ from Research Problem 1.1 can be found in [**90**].

Additional References: Another brief account of the life and work of Ramsey can be found in [**188**]. The book [**217**] contain an account of Ramsey's work as it pertains to philosophy and logic. A brief history of Ramsey theory is given by Spencer in [**258**].

Chapter 2

Van der Waerden's Theorem

Perhaps the most fundamental Ramsey-type theorem on the integers is van der Waerden's theorem concerning arithmetic progressions. Loosely, it says that for any given coloring of $\mathbb{Z}^+$, monochromatic arithmetic progressions cannot be avoided.

Let's consider arithmetic progressions of length three. We wish to find the least positive integer w such that regardless of how the integers $1, 2, \ldots, w$ are colored, using two colors, there will be a monochromatic 3-term arithmetic progression. This number w, denoted by $w(3;2)$ (or $w(3)$), is called a van der Waerden number. Using the notation introduced in Chapter 1, we may also denote it by $R(AP, 3; 2)$, where AP is the family of all arithmetic progressions. Before finding w, we describe the standard methodology for finding the exact value of any particular Ramsey-type number $R(\mathcal{F}, k; r)$. The goal is to show that some number serves both as a lower bound and an upper bound for the minimum number we desire.

(a) To establish that a certain value v is a lower bound for a specific Ramsey-type number $R(\mathcal{F}, k; r)$, it suffices to find *some* r-coloring of $[1, v-1]$ that yields no monochromatic k-element member of $\mathcal{F}$.

(b) To establish that v serves as an upper bound for $R(\mathcal{F}, k; r)$, it is necessary to show that *every* r-coloring of $[1, v]$ yields a monochromatic k-element member of $\mathcal{F}$.

Back to the determination of w: we will establish that $w = 9$ by using the above method to prove that $w \geq 9$ and $w \leq 9$.

According to (a), to show that $w \geq 9$ it suffices to exhibit a 2-coloring of $[1, 8]$ with no monochromatic 3-term arithmetic progression. One such coloring is the following: color 1, 4, 5, and 8 red, and color 2, 3, 6, and 7 blue. It is easy to check that this coloring avoids 3-term monochromatic arithmetic progressions.

To show that $w \leq 9$, we must show that every 2-coloring of $[1, 9]$ admits a monochromatic 3-term arithmetic progression. Assume, for a contradiction, that there exists a 2-coloring of $[1, 9]$ with no monochromatic 3-term arithmetic progression. Using red and blue as the colors, consider the possible ways in which the integers 3 and 5 may be colored. Can both 3 and 5 be red? If they were, then since $(1, 3, 5)$ cannot be monochromatic, 1 must be blue. Likewise, since neither $(3, 4, 5)$ nor $(3, 5, 7)$ can be red, 4 and 7 must be blue. This situation is not possible because now $(1, 4, 7)$ is blue. Thus we may conclude that 3 and 5 cannot both be red. The same argument, with the colors reversed, shows that 3 and 5 cannot both be blue. Hence, 3 and 5 are of different colors. Similarly, 5 and 7 cannot have the same color, and 4 and 6 cannot have the same color (explain why).

Without loss of generality, we assume the color of 3 is red. By the observations above, this leaves

$$\chi_1 = (\text{red, red, blue, blue, red})$$

and

$$\chi_2 = (\text{red, blue, blue, red, red})$$

as the only possible colorings of $(3, 4, 5, 6, 7)$. If χ_1 is the coloring of $(3, 4, 5, 6, 7)$ then, because of $(2, 3, 4)$, the color of 2 must be blue. Then, because of $(2, 5, 8)$, 8 must be red. Because of $(1, 4, 7)$, 1 is blue. Finally, because of $(1, 5, 9)$, 9 is red. From this we have that $(7, 8, 9)$ is red, contradicting our assumption. Since χ_2 is the reverse of χ_1, a symmetric argument will show that χ_2 also leads us to a contradiction. Thus, every 2-coloring of $[1, 9]$ yields a monochromatic arithmetic progression of length 3.

Now that we know that w exists (i.e., that w is finite), it is natural to ask whether the analogous least positive integer exists if we use more than two colors and/or require longer arithmetic progressions. The answer, as it turns out, is yes. This fact is known as van der Waerden's theorem. We start by presenting what is usually called the *finite* version of van der Waerden's theorem.

Theorem 2.1 (Van der Waerden's Theorem). *Let $k, r \geq 2$ be integers. There exists a least positive integer $w = w(k; r)$ such that for all $n \geq w$, for every r-coloring of $[1, n]$ there is a monochromatic arithmetic progression of length k.*

Van der Waerden's theorem is one of the most fundamental results in the area of Ramsey theory. However, because its proof can be somewhat difficult to follow, we postpone the proof until later in this chapter.

To help reinforce what van der Waerden's theorem says, we consider an application.

Example 2.2. Let a, b, k, and r be fixed positive integers. We use van der Waerden's theorem to show that every r-coloring of the set $\{a, a+b, a+2b, \ldots a+(w(k; r)-1)b\}$ admits a monochromatic k-term arithmetic progression. Let

$$\chi : \{a, a+b, \ldots a+(w(k; r)-1)b\} \rightarrow \{0, 1, \ldots, r-1\}$$

be any r-coloring, and define $\chi' : [1, w(k; r)] \rightarrow \{0, 1, \ldots, r-1\}$ by $\chi'(j) = \chi(a+(j-1)b)$. By van der Waerden's theorem, χ' admits a monochromatic k-term arithmetic progression, say

$$\{c, c+d, \ldots, c+(k-1)d\}.$$

Hence $\{a+cb, a+(c+d)b, \ldots, a+(c+(k-1)d)b\}$ is monochromatic under χ, by the definition of χ'. Rewriting this set, we have

$$\{(a+bc), (a+bc)+bd, (a+bc)+2(bd), \ldots, (a+bc)+(k-1)(bd)\},$$

the desired monochromatic k-term arithmetic progression.

As we shall see, there are several other forms of van der Waerden's theorem that, although it is not at all obvious at first glance, are equivalent to Theorem 2.1 (the finite version). One alternate form is the statement that under any finite coloring of $\mathbb{Z}^+$ there exist arbitrarily long monochromatic arithmetic progressions. That this is equivalent to Theorem 2.1 requires some explanation.

Note that the existence of arbitrarily long monochromatic arithmetic progressions under a given coloring of $\mathbb{Z}^+$ does *not* imply that infinitely long monochromatic arithmetic progressions exist. What it does say is that for each *finite* number k we can find a monochromatic arithmetic progression of length k. We look at an example.

Example 2.3. Consider the following 2-coloring of $\mathbb{Z}^+$, with the colors 0 and 1:

$$\underbrace{1}_{1}\underbrace{00}_{2}\underbrace{1111}_{4}\underbrace{00\ldots0}_{8}\underbrace{11\ldots1}_{16}00\ldots,$$

i.e., for $j \geq 0$, the interval $I_j = [2^j, 2^{j+1} - 1]$ is colored 1 if j is even, and colored 0 if j is odd. It is clear that for any k, there exists a monochromatic arithmetic progression of length k (take k consecutive integers in I_k, which has length 2^k). Thus, under this coloring there are arbitrarily long monochromatic arithmetic progressions.

We next show that there is no monochromatic arithmetic progression of infinite length. Assume that $A = \{a, a+d, a+2d, \ldots\}$ is an infinitely long arithmetic progression. Then there is some n such that $2^n > d$ and $A \cap I_n$ is not empty. Since $d < 2^n$, we know that $A \cap I_{n+1}$ is also not empty. Since the color of I_n is different from the color of I_{n+1}, A is not monochromatic.

Now that we have clarified what we mean by arbitrarily long arithmetic progressions, we next show why the following two statements are equivalent: (a) every finite coloring of $\mathbb{Z}^+$ admits arbitrarily long monochromatic arithmetic progressions; and (b) $w(k;r)$ exist for all k and r. This is the subject of the next section.

2.1. The Compactness Principle

The compactness principle, also known as Rado's selection principle, in its full version, is beyond the scope of this book. We will use a simpler version of this principle and refer to it simply as the compactness principle.

The compactness principle, in very general terms, is a way of going from the infinite to the finite. It gives us a "finite" Ramsey-type statement provided the corresponding "infinite" Ramsey-type statement is true. For example, we may conclude the "finite" version of van der Waerden's theorem for two colors:

> For all $k \geq 2$, there exists a least integer $n = w(k)$ such that for every 2-coloring of $[1, m]$, $m \geq n$, there is a monochromatic k-term arithmetic progression,

from the "infinite" version:

> For every 2-coloring of $\mathbb{Z}^+$, there are, for every $k \geq 2$, monochromatic k-term arithmetic progressions.

An alternative way to state the above "infinite" version of van der Waerden's theorem (using 2 colors) is to say that every 2-coloring of $\mathbb{Z}^+$ admits arbitrarily long monochromatic arithmetic progressions.

Before stating the compactness principle, we remark here that this principle does not give us any bound for the minimum number in the "finite" version; it only gives us its existence. The proof we give may seem somewhat familiar, as it is essentially what is known as Cantor's diagonal argument, the standard argument used to prove that the set of real numbers is uncountable.

Theorem 2.4 (The Compactness Principle). *Let $r \geq 2$ and let $\mathcal{F}$ be a family of finite subsets of $\mathbb{Z}^+$. Assume that for every r-coloring of $\mathbb{Z}^+$ there is a monochromatic member of $\mathcal{F}$. Then there exists a least positive integer $n = n(\mathcal{F}; r)$ such that, for every r-coloring of $[1, n]$, there is a monochromatic member of $\mathcal{F}$.*

Proof. Let $r \geq 2$ be fixed and assume that every r-coloring of $\mathbb{Z}^+$ admits a monochromatic member of $\mathcal{F}$. Assume, for a contradiction, that for each $n \geq 1$ there is an r-coloring

$$\chi_n : [1, n] \to \{0, 1, \ldots, r-1\}$$

with no monochromatic member of $\mathcal{F}$. We proceed by constructing a specific r-coloring, χ, of $\mathbb{Z}^+$.

Among $\chi_1(1), \chi_2(1), \ldots$ there must be some color that occurs an infinite number of times. Call this color c_1 and let $\chi(1) = c_1$. Now let $\mathcal{T}_1$ be the collection of all colorings χ_j with $\chi_j(1) = c_1$. Within the set of colors $\{\chi_j(2) : \chi_j(2) \in \mathcal{T}_1\}$, there must be some color c_2 that occurs an infinite number of times. Let $\chi(2) = c_2$ and let $\mathcal{T}_2$ be the collection of all colorings $\chi_j \in \mathcal{T}_1$ with $\chi_j(2) = c_2$. Continuing in this fashion, we can find, for each $i \geq 2$, some color c_i such that the family of colorings

$$\mathcal{T}_i = \{\chi_j \in \mathcal{T}_{i-1} : \chi_j(i) = c_i\}$$

is infinite. We define $\chi(i) = c_i$ for $i = 2, 3, \ldots$. The resulting coloring $\chi : \mathbb{Z}^+ \to \{0, 1, \ldots, r-1\}$ has the property that for every $k \geq 1$, $\mathcal{T}_k$ is the collection of colorings χ_i with $\chi(x) = \chi_i(x)$ for $x = 1, 2, \ldots, k$.

By assumption, χ admits a monochromatic member of $\mathcal{F}$, say S. Let $m = \max\{s : s \in S\}$. By construction, for every $\tau \in \mathcal{T}_m$ we have S monochromatic under τ. This contradicts our assumption that all of the χ_n's avoid monochromatic members of $\mathcal{F}$. □

It is clear that for $m > n$, if χ is an r-coloring of $[1, n]$ that yields a monochromatic member of a specific family $\mathcal{F}$, and χ' is any extension of χ to $[1, m]$, then χ' also yields a monochromatic member of $\mathcal{F}$. Thus, the conclusion of Theorem 2.4 could be replaced by the stronger-sounding (but equivalent) wording "there exists a least positive integer $n = n(\mathcal{F}; r)$ such that for every $m \geq n$ and every r-coloring of $[1, m]$, there is a monochromatic member of $\mathcal{F}$."

Note also that the converse of Theorem 2.4 is true. This is because if it is true that $n(\mathcal{F}; r)$ exists, then certainly it is the case that every r-coloring of $\mathbb{Z}^+$ yields a monochromatic member of $\mathcal{F}$.

By taking $\mathcal{F}$ to be the family of all arithmetic progressions (of all finite lengths), we see, by the compactness principle, that if for each k, every r-coloring of $\mathbb{Z}^+$ admits a monochromatic arithmetic progression of length k, then $w(k; r)$, as defined in Theorem 2.1, exists.

We remark here that we will often use the compactness principle or its converse without explicitly stating that we are doing so. Thus, we may use either the finite or infinite version of a particular Ramsey-type theorem according to convenience.

2.2. Alternate Forms of van der Waerden's Theorem

In the last section we encountered two equivalent forms of van der Waerden's theorem (the "finite" and "infinite" versions). There are, in fact, several equivalent forms of van der Waerden's theorem. We state some of these in the following theorem.

Theorem 2.5. *The following statements are equivalent.*

(i) For $k \geq 2$, any 2-coloring of $\mathbb{Z}^+$ admits a monochromatic arithmetic progression of length k.

(ii) For $k \geq 2$, $w(k; 2)$ exists.

(iii) For $k, r \geq 2$, $w(k; r)$ exists.

(iv) Let $r \geq 2$. For any r-coloring of $\mathbb{Z}^+$ and any finite subset $S = \{s_1, s_2, \ldots, s_n\} \subseteq \mathbb{Z}^+$, there exist integers $a, d \geq 1$ such that $a + dS = \{a + s_1 d, a + s_2 d, \ldots, a + s_n d\}$ is monochromatic.

(v) For $k, r \geq 2$, any r-coloring of $\mathbb{Z}^+$ admits a monochromatic arithmetic progression of length k.

(vi) For $k \geq 2$, any infinite set of positive integers, $S = \{s_i\}_{i \geq 0}$, for which $c = \max\{|s_{i+1} - s_i| : i \geq 0\}$ exists, must contain an arithmetic progression of length k.

Proof. (i) ⇒ (ii) by the compactness principle.

Next, we prove (ii) ⇒ (iii) by induction on r. Statement (ii) is the initial case ($r = 2$) of (iii), so now let $r \geq 3$ and assume that $w(k; r-1)$ exists for all k. We will show that $w(k; r)$ exists for all k. Let $m = w(w(k; r-1); 2)$ and let χ be an arbitrary r-coloring of $[1, m]$, where the colors are $red, blue_1, blue_2, blue_3, \ldots, blue_{r-1}$. We prove $w(k; r) \leq m$ by showing that χ admits a monochromatic k-term arithmetic progression.

If we were to view the colors $blue_1, blue_2, \ldots, blue_{r-1}$ all as an identical shade of blue (so that we are only able to distinguish between the colors red and blue), then we would have a 2-coloring of $[1, m]$ and hence, under this coloring, there is a "monochromatic" arithmetic progression of length $w(k; r-1) \geq k$. If this progression is red then we are done, so assume that it is "blue." However, our "blue" progression actually consists of the $r-1$ colors $\{blue_1, blue_2, \ldots, blue_{r-1}\}$. Hence, under χ, we have an $(r-1)$-coloring of some arithmetic progression $\{a, a+d, \ldots, a+w(k; r-1)d\}$. By Example 2.2, we are done.

To show that (iii) $\Rightarrow$ (iv), let $\max(S) = s_n$ and $w = w(s_n + 1; r)$. Under any r-coloring of $[1, w]$ we have a monochromatic arithmetic progression $\{a, a+d, a+2d, \ldots, a+s_n d\}$ for some $a, d \geq 1$. Since $a + dS$ is a subset of this monochromatic arithmetic progression, we are done.

To prove that (iv) $\Rightarrow$ (v), let k be fixed and take $S = \{1, 2, \ldots, k\}$, so that $a + dS = \{a+d, a+2d, \ldots, a+kd\}$. Hence, by (iv), we are guaranteed that under any r-coloring of $\mathbb{Z}^+$ there exists a k-term monochromatic arithmetic progression.

Next, we show that (v) $\Rightarrow$ (vi). Let S and c be defined as in (vi). Let $T_0 = S$ and define the sets T_j, $j = 1, 2, \ldots, c-1$, as

$$T_j = \{s + j : s \in S\} - \bigcup_{i=1}^{j-1} T_i,$$

i.e., just translate S by j, and make sure that T_j is disjoint from the previous T_i's. Since $|s_{i+1} - s_i| \leq c$ for all $i \geq 0$, $T_1, T_2, \ldots, T_c$ is a partition of $\mathbb{Z}^+$. Color each $z \in \mathbb{Z}^+$ with color j if $z \in T_j$. This defines a c-coloring of $\mathbb{Z}^+$. By (v), there is an arithmetic progression $\{a+d, a+2d, \ldots, a+kd\} \subseteq T_{j_0}$ for some j_0. Hence,

$$\{(a-j_0)+d, (a-j_0)+2d, \ldots, (a-j_0)+kd\} \subseteq T_0 = S,$$

which proves (vi).

We complete the proof by showing that (vi) $\Rightarrow$ (i). Fix k. Consider any 2-coloring of $\mathbb{Z}^+$ using the colors red and blue. For one of the colors, say red, there must be an infinite number of positive integers $r_1 < r_2 < r_3 < \cdots$ with that color. If there exists an i such that $r_{i+1} - r_i > k$, then we have k consecutive blue integers, which gives us a monochromatic arithmetic progression of length k. If no such i exists, then for all $i \geq 0$, $|r_{i+1} - r_i| \leq k$ (so that $\max\{|r_{i+1} - r_i| : i \geq 0\}$ exists), and, by (vi), we have a red arithmetic progression of length k. □

It is interesting to note that (ii) and (iii) are equivalent when, *a priori*, (ii) seems to be weaker than (iii). In fact, in many instances in this book, we will find statements which hold for 2 colors that don't hold for an arbitrary number of colors. As can be seen from the proof that (ii) implies (iii), the fact that we are dealing with arithmetic progressions is vital.

2.3. Computing van der Waerden Numbers

Thus far we have focused on the existence of $w(k; r)$. The next natural step is to determine, if feasible, the values of $w(k; r)$. Below we present all known (to date) nontrivial values of $w(k; r)$. (We leave it as Exercise 2.2 to prove the trivial values $w(2; r) = r + 1$.)

$$\begin{array}{lll} w(3;2) = 9; & w(3;3) = 27; & w(4;2) = 35; \\ w(3;4) = 76; & w(5;2) = 178. & \end{array}$$

So why aren't more values known? Consider $w(3; 5)$, which is obviously greater than $w(3, 4) = 76$. Say we want to check whether every 5-coloring of $[1, 100]$ admits a monochromatic 3-term arithmetic progression. Using brute force, there are 5^{100} colorings to consider. Assuming (incorrectly) that it takes only one computer step to check a coloring for a monochromatic triple in arithmetic progression, we may need $5^{100} \approx 8 \times 10^{69}$ computer steps. At a trillion computer steps per second, if we had a trillion worlds, each with a trillion cities, each with a trillion computer labs, each with a trillion computers we could use, it would take 250,000 years to run this many steps. Then, if we should find success, we would need to check the 5-colorings of $[1, 99]$, since so far we would know only that $w(3, 5) \leq 100$.

So, we can see that the brute force method is not at all feasible. However, there are algorithms for finding values of $w(k; r)$ that are somewhat more efficient. We present three such algorithms below. We encourage the reader to try each of the algorithms, by hand, to calculate $w(3;2)$. In the algorithms, we use 1,2,...,r as the colors. Recall that we say a coloring is valid (in this circumstance) if it admits no monochromatic k-term arithmetic progression.

In Algorithm 1, g_j represents an r-coloring and $g_j(i)$ is the color of i under g_j. The final value of n equals $w(k;r)$.

Algorithm 1.

STEP 1: Set $n=1$, $k=1$, and $g_1(1)=1$ (at this point g_1 is defined only on $\{1\}$)
STEP 2: Set $S=\{g_1,g_2,\ldots,g_k\}$ and $k=|S|$
STEP 3: Increment n by 1
STEP 4: Set $S=\emptyset$ and $j=0$
STEP 5: Set $i=0$ and increment j by 1
STEP 6: Increment i by 1
STEP 7: Set $g_j(n)=i$
STEP 8: If $g_j:[1,n]\to\{1,2,\ldots,r\}$ is valid then set $S=S\cup\{g_j\}$
STEP 9: If $i<r$, go to STEP 6
STEP 10: If $j<k$, go to STEP 5
STEP 11: If $S\neq\emptyset$, go to STEP 2
STEP 12: STOP and output n

This algorithm is more efficient than brute force because we know that all colorings in S avoid monochromatic arithmetic progressions of length k. Furthermore, the set S is built up along the way so as to weed out those colorings that do contain a k-term monochromatic arithmetic progression.

Even though Algorithm 1 is more efficient than the brute force method, its usefulness is limited to only two or three nontrivial (i.e., $k\geq 3$) values of $w(k;r)$; this is due to memory restrictions, as the set S tends to have many members before the algorithm starts to efficiently weed out invalid colorings.

The next algorithm is also more efficient than the brute force method, without the memory restrictions of Algorithm 1. In Algorithm 2, $g(i)$ represents the color of i. The final value of w equals $w(k;r)$.

Algorithm 2.

STEP 1: Set $i=1$, $g(1)=1$, $w=r+1$
STEP 2: Increment i by 1 and set $g(i)=1$
STEP 3: If $g:[1,i]\to\{1,2,\ldots r\}$ is valid, go to STEP 2
STEP 4: If $g(i)=r$, go to STEP 6
STEP 5: Increment $g(i)$ by 1 and then go to STEP 3
STEP 6: If $i>w$, set $w=i$
STEP 7: Decrement i by 1
STEP 8: If $i>1$, go to STEP 4
STEP 9: STOP and output w

Although Algorithm 2 does not have the memory problem of Algorithm 1, it still has the problem of being very time-intensive. It has been used successfully in finding all the known values of $w(k;r)$. With the ever-increasing speed of new computers, one would wonder why we still do not know the exact values of, say, $w(6;2)$ or $w(4;3)$. Even with these somewhat more efficient algorithms, the number of steps required as a function of $[1,w]$, the interval being colored, still grows at an exponential rate. For example, how might we make a rough comparison between the time it would take to calculate $w(5;2)=178$, and the time it would take to calculate $w(6,2)$? It has been observed that the (admittedly few) known values of $w(k;2)$ approximate $\frac{3n!}{2}$, so perhaps a reasonable approximation for $w(6,2)$ is 1080. Even if $w(6;2)=720$ (it is known that $w(6;2)\geq 696$), the number of different 2-colorings of $[1,720]$ is 2^{720}, whereas the number of 2-colorings of $[1,178]$ is 2^{178}. To get some idea of how the number of computer steps required to find $w(5;2)$ would compare to the number required to find $w(6,2)$, it would not be unreasonable to compare 2^{720} to 2^{178}; the ratio of these two numbers is 2^{542}(!). Looking at how the computing times needed to calculate $w(3;2)$, $w(4;2)$, and $w(5;2)$ have compared may also help us understand why the calculation of $w(6;2)$ has, thus far, been out of reach. For example, in one study, the computing times it took to calculate $w(3;2)$, $w(4;2)$, and $w(5;2)$ were instantaneous, 3 seconds, and more than a week, respectively.

The third algorithm we present is essentially a refinement of Algorithm 2, employing what we call the "culprit method." Given a coloring χ of some interval $[1, m]$ and an integer $k \geq 2$, we define a *culprit* of a positive integer n to be any integer $a + (k-2)d$ with $a, d \geq 1$, such that $a, a+d, a+2d, \ldots a+(k-2)d$ is monochromatic and $a + (k-1)d = n$. In other words, it is the $(k-1)^{\text{st}}$ term of an arithmetic progression whose k^{th} term is n, with the first $k-1$ terms all of the same color. To illustrate how the culprit can be used to cut down on the number of calculations, we consider an example.

Example 2.6. Assume that we are calculating $w(4;2)$ (which equals 35) by means of Algorithm 2 . Say that the program is running and we are at the following point in the computations: $w = 30$ (this means that so far we have discovered a valid coloring of [1,29] but not one of [1,30]); $i = 26$ and we are about to assign $g(i) = 1$; the integers $1, 2, 4, 6, 9, 10, 11, 13, 17, 18, 20, 22, 23$ all have color 1; and the integers $3, 5, 7, 8, 12, 14, 15, 16, 19, 21, 24, 25$ all have color 2. According to Algorithm 2, since 26 cannot be assigned either color to yield a valid coloring of $[1, 26]$, we proceed to STEP 7, and then try various colors on $23, 24$, and 25; should that fail, we will go back to 22, etc. However, the least culprit of 26 with color 1 is 18 (that is, as long as $2, 10, 18$ have color 1, we will not be able to assign the color 1 to the integer 26); and the least culprit of 26 with color 2 is 19 (as long as $5, 12, 19$ have color 2, we will not be able to assign the color 2 to the integer 26). Therefore, to try various color assignments on any of the integers in the set $\{20, 21, 22, 23, 24, 25\}$ at this point would be a waste of time, because we will not find any valid colorings of $[1, 26]$ until we change the way $[1, 19]$ is colored. Thus, we would save time by moving 19 into the set with color 2 at this point, rather than moving 23 into that set, which is the next move in Algorithm 2.

Algorithm 3 takes advantage of the culprit to give a more efficient algorithm. We use $cul_j(i)$ to represent the minimum of all culprits of i having color j.

Algorithm 3.

STEP 1: Set $i = 1$, $g(1) = 1$, $w = r + 1$
STEP 2: Increment i by 1 and set $g(i) = 1$
STEP 3: If $i > w$, set $w = i$
STEP 4: If $g : [1, i] \to \{1, 2, ..., r\}$ is valid, go to STEP 2
STEP 5: If $g(i) = r$, go to STEP 7
STEP 6: Increment $g(i)$ by 1 and then go to STEP 4
STEP 7: Set $i = \max\{cul_j(i) : 1 \leq j \leq r\}$
STEP 8: If $g(i) < r$, go to STEP 6
STEP 9: Decrement i by 1
STEP 10: If $i = 1$, STOP and output w
STEP 11: Go to STEP 8

Algorithm 3 is somewhat more efficient that Algorithm 2 but, it seems, not efficient enough to allow (at least thus far) the discovery of new van Waerden numbers.

We conclude this section by considering what are sometimes called "mixed" van der Waerden numbers and a table of all known (to date) computed values. Instead of requiring an arithmetic progression of length k to be of one of the colors (so that k is a constant that is independent of the color), the mixed van der Waerden numbers allow the required length to vary with the color. For example, if the colors are red and blue, then $w(4)$ represents the least positive integer such that, for every 2-coloring of $[1, w(4)]$, there is either a 4-term red arithmetic progression or a 4-term blue arithmetic progression. What if we "mix" the lengths so that we want the least positive integer n such that, for every 2-coloring of $[1, n]$, there is either a 4-term red arithmetic progression or a 5-term blue arithmetic progression? We know that $w(4) = 35$ and $w(5) = 178$. So we must have $35 \leq n \leq 178$ (why?). More generally, we have the following as a corollary of van der Waerden's theorem. We leave the proof as Exercise 2.4.

Corollary 2.7. *Let $r \geq 2$, and let $k_i \geq 1$ for $1 \leq i \leq r$. Then there exists a least positive integer $w = w(k_1, k_2, ..., k_r; r)$ such that, for every r-coloring $\chi : [1, w] \to [1, r]$, there exists, for some $i \in [1, r]$, a k_i-term arithmetic progression with color i.*

Let's consider an example.

Example 2.8. According to Corollary 2.7, $w(3,4,2;3)$ is the least positive integer n such that for every 3-coloring (using the colors 1, 2, and 3) of $[1,n]$, there is a 3-term arithmetic progression with color 1 *or* a 4-term arithmetic progression with color 2 *or* a 2-term arithmetic progression with color 3. It is easy to see that $w(3,4,2;3) = w(4,3,2;2)$ and, in fact, that any re-ordering of the components of $(k_1, k_2, \ldots, k_r)$ in Corollary 2.7 will have no effect on the value of n.

In Table 2.1 we present all known (to date) mixed van der Waerden numbers (including the classical van der Waerden numbers) $w(k_1, k_2, \ldots, k_r; r)$, where $k_i \geq 3$ for at least two values of i. The table gives, for $r = 2,3,4$, each r-tuple $(k_1, k_2, \ldots, k_r)$ with the k_i's ordered in nonincreasing order (as mentioned in Example 2.8, the order in which the k_i's appear is irrelevant).

r	k_1	k_2	k_3	k_4	w
2	3	3	-	-	9
2	4	3	-	-	18
2	4	4	-	-	35
2	5	3	-	-	22
2	5	4	-	-	55
2	5	5	-	-	178
2	6	3	-	-	32
2	6	4	-	-	73
2	7	3	-	-	46
2	7	4	-	-	109
2	8	3	-	-	58
2	9	3	-	-	77
2	10	3	-	-	97
2	11	3	-	-	114
2	12	3	-	-	135
3	3	3	2	-	14

r	k_1	k_2	k_3	k_4	w
3	3	3	3	-	27
3	4	3	2	-	21
3	4	3	3	-	51
3	4	4	2	-	40
3	4	4	3	-	84
3	5	3	2	-	32
3	5	3	3	-	77
3	5	4	2	-	71
3	6	3	2	-	40
4	3	3	2	2	17
4	3	3	3	2	40
4	3	3	3	3	76
4	4	3	2	2	25
4	4	3	3	2	60
4	4	4	2	2	53
4	5	3	2	2	43

Table 2.1: Mixed van der Waerden numbers

2.4. Bounds on van der Waerden Numbers

Since we have seen that finding exact values for van der Waerden numbers is an extremely difficult problem, we turn to the problem of finding bounds on these numbers.

As we will see, there is a wide gap between the best known upper bounds on $w(k)$ and the best known lower bounds. We begin with lower bounds.

The best known lower bound on $w(k)$, due to Berlekamp, is presented in the following theorem. We do not include the proof, which requires a knowledge of field extensions.

Theorem 2.9. *Let p be prime. Then $w(p+1) \geq p2^p$.*

The person most responsible for the development of Ramsey theory as a branch of mathematics, in fact for the much broader area of mathematics known as combinatorial number theory, is Paul Erdős. The extremely insightful Erdős (also the most prolific mathematician of the twentieth century) conjectured that $\lim_{k\to\infty} \frac{w(k)}{2^k} = \infty$ (i.e., that $w(k)$ grows significantly faster than 2^k). Whether this conjecture is true is uncertain, although Theorem 2.9 lends credence to it. Along these lines, we have the following theorem.

Theorem 2.10. *Let $\epsilon > 0$. There exists $k_0 = k(\epsilon)$ such that for all $k \geq k_0$,*

$$w(k) \geq \frac{2^k}{k^\epsilon}.$$

We omit the proof of Theorem 2.10, but will prove Theorem 2.18, which gives an asymptotic lower bound for $w(k)$. The proof makes use of a graph-theoretic result due to W.M. Schmidt (Lemma 2.17). The proof of Theorem 2.18 provides us with a method for obtaining lower bounds that may be applied to many of the other types of sequences mentioned in this book (these may be fruitful avenues of research to explore).

Before stating Schmidt's lemma, we need to give some definitions.

Definition 2.11. A *hypergraph* $\Gamma = (V, \mathcal{E})$ is a set of vertices V and a collection $\mathcal{E}$ of subsets of V such that, for every $E \in \mathcal{E}$, $|E| \geq 2$. The members of $\mathcal{E}$ are called *hyperedges*.

Note that if for every $E \in \mathcal{E}$ we have $|E| = 2$, then the hypergraph $(V, \mathcal{E})$ reduces to a graph as defined in Definition 1.11.

Example 2.12. Let $V = \{1, 2, 3, 4\}$ and consider the collection $\mathcal{E} = \{\{1, 2, 4\}, \{3, 4\}, \{2, 3, 4\}\}$. Then $(V, \mathcal{E})$ is a hypergraph with 3 hyperedges. If $\mathcal{G} = \{\{1, 2\}, \{1, 3\}, \{4\}\}$, then $(V, \mathcal{G})$ is not a hypergraph.

Definition 2.13. Let $\Gamma = (V, \mathcal{E})$ be a hypergraph. We say that Γ has *Property B* if there exists $V_1 \subseteq V$ such that for every $E \in \mathcal{E}$ we have $E \cap V_1 \neq \emptyset$ and $E \cap V_1 \neq E$.

We consider an example.

Example 2.14. Let $\Gamma_1 = (V, \mathcal{E}_1)$ with

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

and

$$\mathcal{E}_1 = \{\{1, 2, 3, 9\}, \{1, 4, 5, 6\}, \{3, 7, 8\}\}.$$

Then, by letting $V_1 = \{1, 3\}$, as in Definition 2.13, we see that Γ_1 has Property B. Let $\Gamma_2 = (V, \mathcal{E}_2)$, where $\mathcal{E}_2$ is the collection of all 3-term arithmetic progressions contained in $[1, 9]$. We show, by contradiction, that Γ_2 does not have Property B. Assuming Γ_2 does have Property B, there is a $V_2 \subseteq [1, 9]$ such that $V_2 \cap E \neq \emptyset$ and $V_2 \cap E \neq E$ for every $E \in \mathcal{E}_2$. Consider the partition of $[1, 9]$ into the two sets V_2 and $V - V_2$. By the fact that $w(3; 2) = 9$, we know there is some 3-term arithmetic progression E_0 such that either $E_0 \subseteq V_2$ or $E_0 \subseteq V - V_2$. If $E_0 \subseteq V_2$, then $E_0 \cap V_2 = E_0$, and if $E_0 \subseteq V - V_2$, then $E_0 \cap V_2 = \emptyset$. This contradicts our assumption about V_2. Hence Γ_2 does not have Property B.

Definition 2.15. Let $\Gamma = (V, \mathcal{E})$ be a hypergraph. For $k \geq 2$, denote by $\mu(k)$ the minimal number of hyperedges $\mathcal{E} = \{E_1, E_2, \ldots, E_{\mu(k)}\}$ such that $|E_i| = k$ for $1 \leq i \leq \mu(k)$ and Γ does not have Property B.

Example 2.16. In Example 2.14 we saw that the set of all 3-term arithmetic progressions in [1,9] is a hypergraph on $\{1, 2, \ldots, 9\}$ not having Property B. The number of 3-term arithmetic progressions in [1,9] is

$$\left\lfloor \frac{9-1}{2} \right\rfloor + \left\lfloor \frac{9-2}{2} \right\rfloor + \left\lfloor \frac{9-3}{3} \right\rfloor + \cdots + \left\lfloor \frac{9-7}{2} \right\rfloor = 16.$$

Therefore $\mu(3) \leq 16$.

We now state Schmidt's lemma, the proof of which we omit.

Lemma 2.17 (Schmidt's Lemma). *For* $k \geq 1$,

$$\mu(k) \geq \frac{2^k}{1 + 2k^{-1}}.$$

With the help of Lemma 2.17 we are able to obtain the following lower bound on $w(k)$.

Theorem 2.18. $w(k; 2) \geq \sqrt{k}2^{\frac{k+1}{2}}(1 - o(1))$.

Proof. Let $m \geq w(k)$. Let $\mathcal{E}$ be the collection of all k-term arithmetic progressions that are contained in $[1, m]$. Just as we saw in Example 2.14 that Γ_2 (for 3-term arithmetic progressions) does not have Property B, the hypergraph $\Gamma = (\{1, 2, \ldots, m\}, \mathcal{E})$ does not have Property B (we leave the details to the reader as Exercise 2.9).

To use Schmidt's lemma, we need to know something about the size of $\mathcal{E}$. For $E = \{a, a + d, a + 2d, ..., a + (k - 1)d\}$ to belong to $\mathcal{E}$, we must have $d \leq \frac{m-1}{k-1}$. We also must have $a \leq m - (k - 1)d$. Since each k-term arithmetic progression is completely determined by the values of a and d, we have

$$\begin{aligned} |\mathcal{E}| &= \sum_{d=1}^{\lfloor \frac{m-1}{k-1} \rfloor} (m - (k-1)d) \\ &= m\left\lfloor \tfrac{m-1}{k-1} \right\rfloor - \tfrac{k-1}{2}\left\lfloor \tfrac{m-1}{k-1} \right\rfloor \left(\left\lfloor \tfrac{m-1}{k-1} \right\rfloor + 1\right) \\ &\geq \frac{m^2}{2k}. \end{aligned}$$

Therefore, if $m = w(k)$, then since Γ does not have Property B, by Schmidt's lemma we have

$$\frac{m^2}{2k} \geq \mu(k) \geq \frac{2^k}{1 + 2k^{-1}}.$$

This implies that

$$m \geq \sqrt{\frac{k2^{k+1}}{1 + 2k^{-1}}},$$

and therefore

$$w(k) \geq \sqrt{k}2^{\frac{k+1}{2}}(1 - o(1)).$$

□

We next state two theorems on lower bounds for $w(k;r)$, where r may be greater than two.

Theorem 2.19. *Let $p \geq 5$ and q be primes. Then*

$$w(p+1;q) \geq p(q^p - 1) + 1.$$

Theorem 2.20. *For all $r \geq 2$, $w(k;r) > \frac{r^k}{ekr}(1 + o(1))$.*

In the following table we present several of the best lower bounds (known to date), along with all of the known values (to date), for specific van der Waerden numbers.

$k \setminus r$	2	3	4	5
3	9	27	76	?
4	35	≥ 292	≥ 1048	≥ 2254
5	178	≥ 1210	≥ 10437	≥ 24045
6	≥ 696	≥ 8886	≥ 90306	≥ 93456
7	≥ 3703	≥ 43855	≥ 119839	?

Table 2.2: Lower bounds and values for $w(k;r)$

We now move on to upper bounds for the van der Waerden numbers. The main result in this direction is a remarkable theorem due to Timothy Gowers. In 1998, Gowers received a Fields Medal (the mathematical equivalent of the Nobel Prize). His work related to finding an upper bound on the van der Waerden numbers played a significant part in his winning of this prestigious award.

Before stating Gowers' result, we give some historical perspective on the search for an upper bound. First we define some very fast-growing functions.

For a function f, and $n \in \mathbb{Z}^+$, denote by $f^{(n)}$ the composition of f with itself n times (i.e., $f^{(n)}(x) = \underbrace{f(f(\ldots(f}_{n \text{ times}}(x))\ldots)))$. Let f_1 be the function $f_1(k) = 2k$, $k \geq 1$, and let f_2 be the function $f_2(k) = 2^k$,

$k \geq 1$. Notice that $f_2(k) = f_1^{(k)}(1)$. Similarly, define $f_3(k) = f_2^{(k)}(1)$; and, more generally, for $i \in \mathbb{Z}^+$ define the functions

$$f_{i+1}(k) = f_i^{(k)}(1),$$

$k \geq 1$. As the subscript increases, the growth of these functions accelerates at a phenomenal rate. For example,

$$f_3(k) = \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{k \text{ 2's}}.$$

In considering the magnitude of the functions $f_i(k)$, for reasons that are apparent, we will refer to the functions $f_2(k)$ and $f_3(k)$ as $exp_2(k)$ and $tower(k)$, respectively.

Now consider $f_4(k)$. We call this function the *wow* function due to its incredible rate of growth, and we denote it by $wow(k)$. To see how quickly $wow(k)$ grows, consider $wow(k)$ for $k = 1, 2, 3, 4$. We easily have $wow(1) = 2, wow(2) = 4$, and $wow(3) = tower(4) = 65536$. For $k = 4$, we have

$$wow(4) = f_4(4) = f_3(f_3(f_3(f_3(1)))) = f_3(65536) = tower(65536),$$

a tower of 65536 2's.

Generalizing from $f_3(3)$ and $f_4(4)$, we now consider a function on $\mathbb{Z}^+$ known as the Ackermann function. The Ackermann function is defined as $ack(k) = f_k(k)$. This function is named after a similar function derived in 1928 by Wilhelm Ackermann, a high school math teacher, who received his Ph.D. under the direction of David Hilbert. This function is perhaps the fastest growing function you will ever encounter. To get an idea of how fast it grows, consider $wow(5)$ and $ack(5)$. We have

$$wow(5) = tower(tower(65536)),$$

for which calling it enormous would be an understatement. Meanwhile, we have

$$ack(5) = wow(wow(tower(65536))).$$

Given the incredibly immense size of $wow(5)$, grasping the magnitude of $wow(tower(65536))$, let alone $ack(5)$, is no easy accomplishment. Comparing $tower(5) = 2^{65536}$, a number with 19719 digits, to $exp_2(5) = 2^5 = 32$, may give some insight into how $wow(5)$ and $ack(5)$ compare.

We introduced the above functions to show the enormity of the historical upper bounds on $w(k)$. It turns out that the original proof of van der Waerden's theorem gives

$$w(k) \leq ack(k),$$

which tells us, in particular, that $w(4) \leq tower(65536)$. Comparing this to the actual value $w(4) = 35$ certainly gives some reason to suspect that this upper bound might not be the best possible.

For many years, the much-revered mathematician Ronald Graham offered a \$1000 prize to anyone who could prove or disprove that $w(k) \leq tower(k)$. The idea of offering monetary prizes for what are considered difficult problems was championed by Paul Erdős. Graham and Erdős were good friends, and since Erdős' passing in 1996, Graham has kept this "prize money" tradition alive by honoring all of Erdős' prize problems and adding some of his own.

Since there is a very significant difference in magnitude between the functions $tower(k)$ and $ack(k)$, showing that $w(k) \leq tower(k)$ would seem to be quite a feat – some new method of proof would need to be used.

In 1987, the eminent logician Saharon Shelah, while not answering Graham's question, used an argument fundamentally different from earlier proofs of van der Waerden's theorem, to prove that

$$w(k) \leq wow(k),$$

quite an improvement over the previous upper bound of $ack(k)$.

Shelah's result is one of the most significant results in Ramsey theory. In fact, although Shelah did not answer Graham's "prize" question completely, Graham gave him half of the award money anyway.

When we consider the magnitude of $wow(k)$, for example that $wow(4) = tower(65536)$, and compare it to the best lower bounds known for $w(k)$, it would be reasonable to think that this upper bound is not the best possible. In fact, Gowers' amazing result showed that the bound could be substantially improved. Here is Gowers' bound.

Theorem 2.21. *For $k \geq 2$,*

$$w(k;2) \leq 2^{2^{2^{2^{2^{k+9}}}}}.$$

Gowers' bound is of a much smaller magnitude than $tower(k)$, thereby settling Graham's question. As a consequence of Gowers' result, Graham asked in 1998 whether or not $w(k) < 2^{k^2}$ and currently offers \$1000 for an answer.

Remark 2.22. Gowers actually proved a more general result than what we state as Theorem 2.21. Namely, letting $f(k;r) = r^{2^{2^{k+9}}}$, he showed that

$$w(k;r) \leq 2^{2^{f(k;r)}}.$$

Recently, a new upper bound for $w(3;r)$ for $r \geq 5$ has been obtained. We present this result without proof.

Theorem 2.23. *Let $r \geq 5$. Then*

$$w(3;r) < \left(\frac{r}{4}\right)^{3^r}.$$

Even with Gowers' upper bound and the upper bound from Theorem 2.23, the best known upper and lower bounds on van der Waerden numbers are still nowhere near each other!

2.5. The Erdős and Turán Function

By van der Waerden's theorem we know that, for n large enough, whenever $[1,n]$ is partitioned into a finite number of subsets, at least one of the subsets must contain a k-term arithmetic progression. In an effort to find out more about the van der Waerden numbers, Erdős and Turán defined a function that approaches the problem from what could be considered the opposite direction. That is, we may ask the following question: given a positive integer n, what is the maximum size of a subset of $[1,n]$ that does not contain an arithmetic progression of length k? Erdős and Turán defined the following function.

Definition 2.24. For $k \geq 2$ and $n \geq 3$, let $\mathcal{S}$ be the collection of sets $S \subseteq [1,n]$ such that S does not contain an arithmetic progression of length k. Then

$$\nu_k(n) = \max\{|S| : S \in \mathcal{S}\}.$$

The function ν_2 is trivial, since $\nu_2(n) = 1$ for all n. In the following examples we consider $\nu_n(k)$ for some small values of k and n.

Example 2.25. Let $k = 3$ and $n = 8$. Since $w(3) = 9$, we know that there exists a 2-coloring of $[1,8]$ with no 3-term monochromatic arithmetic progression. One such coloring is 11001100. Hence, taking $S = \{1,2,5,6\}$, we have $\nu_3(8) \geq 4$. Furthermore, there are only two other colorings, up to renaming the colors, of $[1,8]$ that avoid monochromatic 3-term arithmetic progressions, each of which has four integers of each color.

Hence, for any $T \subseteq [1,8]$ with $|T| \geq 5$, if we color the elements of T with one color and the elements of $R = [1,8] - T$ with the other color, we are guaranteed to have a monochromatic 3-term arithmetic progression. If $|T| \geq 6$, then (since $|R| \leq 2$) that arithmetic progression resides in T. This leaves us to consider $|T| = 5$ (so that $|R| = 3$). If R contains the monochromatic triple, then R is one of the following: $\{1,2,3\}$, $\{1,3,5\}$, $\{1,4,7\}$, $\{2,3,4\}$, $\{2,4,6\}$, $\{2,5,8\}$, $\{3,4,5\}$, $\{3,5,7\}$, $\{4,5,6\}$, $\{4,6,8\}$, $\{5,6,7\}$, $\{6,7,8\}$. It is easy to check that for each of these sets, T also contains a 3-term arithmetic progression. Hence, $\nu_3(8) \leq 4$, and we have established that $\nu_3(8) = 4$.

Example 2.26. We show here that $\nu_3(26) \geq 9$. Since $w(3;3) = 27$, there exists a 3-coloring of $[1,26]$ that avoids 3-term monochromatic arithmetic progressions. Clearly, there is some color c such that at least nine elements have color c. Since there is no 3-term arithmetic progression with color c, $\nu_3(26) \geq 9$.

The above examples illustrate the relationship between the Erdős and Turán function and the van der Waerden numbers. One important reason for studying $\nu_k(n)$ is that having an upper bound on these numbers would lead to an upper bound on $w(k;r)$. To be more precise, we have the following theorem.

Theorem 2.27. *Let $k \geq 3$, $r \geq 2$, and assume $\nu_k(m) \leq f(k,m)$, with $f(k,m) \leq \frac{m}{r}$. Then*

$$w(k,r) \leq rf(k,m) + 1.$$

Proof. If $\nu_k(m) \leq f(k,m)$, then for any r-coloring of $[1, rf(k,m)+1]$, the most used color, say c, would occur more than $f(k,m)$ times. Since $f(k,m) \geq \nu_k(m)$, by the definition of $\nu_k(m)$ there must be a k-term arithmetic progression with color c. □

To end this section, we state the following theorem; its proof is left to the reader as Exercise 2.13.

Theorem 2.28. *Let $n, k \geq 3$. Let $r(n)$ be the minimum number of colors required to color $[1,n]$ so that no monochromatic k-term arithmetic progression exists. Then*

$$\nu_k(n) \geq \left\lceil \frac{n}{r(n)} \right\rceil.$$

One of the most significant results in Ramsey theory, now known as Szemerédi's theorem, involves a conjecture made by Erdős and Turán in 1936. They conjectured that for every k,

$$\lim_{n\to\infty} \frac{\nu_k(n)}{n} = 0.$$

In 1952, Roth proved their conjecture for $k = 3$. In 1969, Szemerédi showed it holds for $k = 4$. Then, using an ingenious and very complex proof, Szemerédi fully settled the conjecture, establishing its truth, in 1975. Unfortunately his proof, which makes use of van der Waerden's theorem, does not yield any useful bounds for $w(k)$.

2.6. Proof of van der Waerden's Theorem

For completeness, we restate the finite version of van der Waerden's theorem.

Van der Waerden's Theorem. *Let $k, r \geq 2$ be integers. There exists a least positive integer $w = w(k;r)$ such that for every r-coloring of $[1,w]$ there is a monochromatic arithmetic progression of length k.*

The proof we give of van der Waerden's theorem is constructed from two important lemmas. Before presenting these lemmas, we need some preliminaries.

The following proposition essentially tells us that the guarantee of a monochromatic arithmetic progression in an interval is unaffected by translation (adding a constant integer) and/or dilation (multiplying by a positive integer) of the interval.

Proposition 2.29. *Let k, r, m, a, and b be positive integers. Every r-coloring of $[1,m]$ yields a monochromatic k-term arithmetic progression if and only if every r-coloring of*

$$S = \{a, a+b, a+2b, \ldots, a+(m-1)b\}$$

yields a monochromatic arithmetic progression.

Remark. The "only if" portion of Proposition 2.29 immediately implies the result of Example 2.2.

We leave the proof of Proposition 2.29 to the reader as Exercise 2.14.

As an aside, it is worth mentioning that Proposition 2.29 may be easily extended to other sequences besides arithmetic progressions. Specifically, the following more general statement may be proved in essentially the same way. We leave the proof to the reader as Exercise 2.15.

Proposition 2.30. *Let $\mathcal{F}$ be a collection of sets and let $a,b \in \mathbb{Z}^+$. Assume the following: $S \in \mathcal{F}$ if and only if $a+bS \in \mathcal{F}$. Let $r \in \mathbb{Z}^+$. Then every r-coloring of $[1,n]$ yields a monochromatic member of $\mathcal{F}$ if and only if every r-coloring of*

$$a + b[0, n-1] = \{a, a+b, a+2b, \ldots, a+(n-1)b\}$$

yields a monochromatic member of $\mathcal{F}$.

We now give two definitions that are crucial to our proof of van der Waerden's theorem.

Definition 2.31. Let $r, m, n \geq 1$. Let γ be an r-coloring of $[1, n+m]$. Define $\chi_{\gamma,m}$ to be the r^m-coloring of $[1,n]$ as follows: for $j \in [1,n]$, let $\chi_{\gamma,m}(j)$ be the m-tuple $(\gamma(j+1), \gamma(j+2), \ldots, \gamma(j+m))$. We call $\chi_{\gamma,m}$ a coloring *derived from* γ, or a *derived coloring.*

Note that $\chi_{\gamma,m}$ of Definition 2.31 is, in fact, an r^m-coloring since there are r^m possible m-tuples $(\gamma(j+1), \gamma(j+2), \ldots, \gamma(j+m))$, for $1 \leq j \leq n$. Also note that the above definition states that $i,j \in [1,n]$ have the same color under $\chi_{\gamma,m}$ precisely when $[i+1, i+m]$ and $[j+1, j+m]$ are colored in the same fashion under γ. Consider the following example.

Example 2.32. Take $r = 2$, $m = 3$, and $n = 18$ in Definition 2.31. Define $\gamma : [1, 21] \to \{0, 1\}$ by the coloring

$$011000111100000111111.$$

To describe the 2^3-coloring χ_3 of $[1,18]$ derived from γ, we make, for convenience, the following correspondence between the set of all 3-tuples of two colors, $T = \{(i,j,k) : i,j,k \in \{0,1\}\}$ (there are 8 of them), and the colors $0, 1, \ldots, 7$:

$$\begin{array}{llll} (0,0,0) \leftrightarrow 0; & (1,0,0) \leftrightarrow 1; & (0,1,0) \leftrightarrow 2; & (0,0,1) \leftrightarrow 3; \\ (1,1,0) \leftrightarrow 4; & (1,0,1) \leftrightarrow 5; & (0,1,1) \leftrightarrow 6; & (1,1,1) \leftrightarrow 7. \end{array}$$

Note that any one-to-one correspondence between T and a set of eight "colors" would suffice.

Since $\chi_{\gamma,3}(1)$ corresponds to $(\gamma(2), \gamma(3), \gamma(4)) = (1,1,0)$, we see that $\chi_{\gamma,3}(1) = 4$. Similarly, $\chi_{\gamma,3}(2)$ corresponds to $(1,0,0)$, so that $\chi_{\gamma,3}(2) = 1$. Evaluating the other 16 elements of $[1,18]$, we find that the derived coloring under $\chi_{\gamma,3}$ is

$$410367741000367777.$$

Definition 2.33. We say that a triple $(k,t;r)$ is *refined* if there exists a positive integer $m = m(k,t;r)$ such that for every r-coloring of $[1,m]$, there exist positive integers $z, x_0, x_1, \ldots, x_t$ such that each of the sets

$$T_s = \left\{ b_s + \sum_{i=0}^{s-1} c_i x_i : c_i \in [1,k] \right\},$$

$0 \leq s \leq t$, is monochromatic, where

$$b_s = z + (k+1) \sum_{i=s}^{t} x_i.$$

The above definition may seem rather cumbersome, but it will make the proof of van der Waerden's theorem easier to digest. So, before moving on, let's look at an example. Consider $k = 2$ and $t = 2$

in Definition 2.33. The sets are

$$T_0 = \{b_0\},$$
$$T_1 = \{b_1 + x_0, b_1 + 2x_0\},$$
$$T_2 = \{b_2 + x_0 + x_1, b_2 + 2x_0 + x_1, b_2 + x_0 + 2x_1, b_2 + 2x_0 + 2x_1\},$$

where $b_s = z + 3\sum_{i=s}^{2} x_i$ for $s = 0, 1, 2$. Clearly, T_0 is monochromatic since it contains only a single point. Hence, in order for $(2, 2; r)$ to be refined, there must exist $m = m(2, 2; r)$ such that, under any r-coloring χ of $[1, m]$, we have

$$\chi(b_1 + x_0) = \chi(b_1 + 2x_0)$$

and

$$\chi(b_2+x_0+x_1) = \chi(b_2+2x_0+x_1) = \chi(b_2+x_0+2x_1) = \chi(b_2+2x_0+2x_1)$$

for some positive integers z, x_0, x_1, x_2.

A subtle point of Definition 2.33, perhaps made clear by the above example, is that, even if $(k, t; r)$ is a refined triple, we do not necessarily have that the T_s are all monochromatic *of the same color*, but rather that each T_s, individually, is monochromatic.

The following remark gives a glimpse as to how Definition 2.33 is used in the proof of van der Waerden's theorem.

Remark 2.34. By taking $c_0 = c_1 = \cdots = c_{s-1} = j$ for $j = 1, 2, \ldots, k$ in Definition 2.33, we have the arithmetic progression

$$\{a + jd : j = 1, 2, \ldots, k\} \subseteq T_s,$$

where $a = b_s$ and $d = \sum_{i=0}^{s-1} x_i$.

We are now ready to present the lemmas that, when combined, will prove van der Waerden's theorem. Our approach will be proof by induction, showing that the existence of $w(k; r)$ implies the existence of $w(k + 1; r)$. In order to do this, we will be using refined triples.

Lemma 2.35. *Let $k \geq 1$. If $w(k; r)$ exists for all $r \geq 1$, then $(k, t; r)$ is refined for all $r, t \geq 1$.*

Proof. Let $r \geq 1$. The proof is by induction on t, starting with $t = 1$. To prove that $(k, 1; r)$ is refined, we first show that we may take $m = m(k, 1; r)$ (in Definition 2.33) to be $3w(k; r) + k + 1$. Let χ be an arbitrary r-coloring of $[1, 3w(k; r) + k + 1]$. Since we are assuming that $w(k; r)$ exists, applying Proposition 2.29, the interval $[w(k; r) + k + 2, 2w(k; r) + k + 1]$ must admit a monochromatic k-term arithmetic progression $S = \{a + d, a + 2d, \ldots, a + kd\}$.

Using the notation of Definition 2.33, let $z = a - (k + 1)$, $x_0 = d$, and $x_1 = 1$. This gives $T_0 = \{a + (k + 1)d\}$ and $T_1 = S$, which are both contained in $[1, m]$ and are both (individually) monochromatic (T_0 contains only one point), thereby proving that $(k, 1; r)$ is a refined triple.

Now let $t \geq 1$ and assume that $(k, t; r)$ is refined. We will show that $(k, t + 1; r)$ is refined. Let $m = m(k, t; r)$ be as in Definition 2.33 and let $n = 2w(k; r^m)$.

We claim that we may take $m(k, t + 1; r) = n + m$. Let γ be an r-coloring of $[1, n + m]$. Let $\chi = \chi_{\gamma,m}$ be the r^m-coloring of $[1, n]$ derived from γ (see Definition 2.31). By the definition of n, and since $w(k; r^m)$ exists, there must be an arithmetic progression

$$\{a + d, a + 2d, \ldots, a + (k + 1)d\} \subseteq [1, n]$$

with the first k terms monochromatic under χ_m. By the definition of χ, the k intervals $I_j = [a+jd+1, a+jd+m]$, $1 \leq j \leq k$, have identical colorings under γ. Since $(k, t; r)$ is refined, there exist $z, x_0, x_1, \ldots, x_t$ so that the T_i's (as in Definition 2.33) are monochromatic under γ. Therefore, each I_j contains the monochromatic sets

$$\begin{aligned} S_s(j) &= T_s + (a + jd) \\ &= \{y + a + jd : y \in T_s\} \\ &= \left\{(b_s + a + jd) + \sum_{i=0}^{s-1} c_i x_i : c_i \in [1, k]\right\} \end{aligned}$$

for $s = 0, 1, \ldots, t$ (where b_s is as in Definition 2.33). Furthermore, since the intervals have the same coloring under γ, $S_s(u)$ and $S_s(v)$ must have the same coloring under γ for $1 \leq u, v \leq k$. Hence, by construction, the set

$$Q_s = \left\{(b_s + a) + \sum_{i=0}^{s-1} c_i x_i + jd : j, c_i \in [1, k]\right\}$$

is monochromatic under γ for each $s = 0, 1, \ldots, t$.

We shall provide integers $z', x'_0, x'_1, \ldots, x'_{t+1}$, which produce monochromatic sets T'_s (analogous to the T_i's in Definition 2.33), for $s = 0, 1, \ldots, t+1$, to show that $(k, t+1; r)$ is refined. Let

$$\begin{aligned} z' &= z + a, \\ x'_i &= x_i \quad \text{for } 0 \le i \le s-1, \\ x'_s &= d, \\ x'_i &= x_i \quad \text{for } s+1 \le i \le t, \text{ and} \\ x'_{t+1} &= x_s. \end{aligned}$$

Since for each $s = 0, 1, \ldots, t$ we have $T'_{s+1} = Q_s$, it follows that, under γ, T'_{s+1} is monochromatic for $0 \le s \le t$. Since $T'_0 = \{b_0 + a + (k+1)d\}$ consists of a single point (whose value is less than $n+m$), it is trivially monochromatic. Thus we have satisfied the conditions required to prove that $(k, t+1; r)$ is refined, thereby proving the lemma. □

Lemma 2.36. *If $(k, t; r)$ is refined for all $r, t \ge 1$, then $w(k+1; r)$ exists for all $r \ge 1$.*

Proof. Let r be given and let χ be any r-coloring of $\mathbb{Z}^+$. By assumption, $(k, t; r)$ is refined for, in particular, $t = r$. Using the notation of Definition 2.33, there exist $z, x_0, x_1, \ldots, x_r$ such that each of the sets $T_0, T_1, \ldots, T_r$ is monochromatic under χ. By the pigeonhole principle, two of these sets must be the same color. Let T_v and T_w, $v < w$, be two such sets. We have

$$T_v = \left\{ z + (k+1)\sum_{i=v}^{r} x_i + \sum_{i=0}^{v-1} c_i x_i : c_i \in [1,k] \right\}$$

and

$$T_w = \left\{ z + (k+1)\sum_{i=w}^{r} x_i + \sum_{i=0}^{w-1} c_i x_i : c_i \in [1,k] \right\}.$$

Letting $a = z + \sum_{i=0}^{v-1} x_i + (k+1)\sum_{i=w}^{r} x_i$, we rewrite these as

$$T_v = \left\{ a + (k+1)\sum_{i=v}^{w-1} x_i + \sum_{i=0}^{v-1} (c_i - 1) x_i : c_i \in [1,k] \right\}$$

and

$$T_w = \left\{ a - \sum_{i=0}^{v-1} x_i + \sum_{i=0}^{w-1} c_i x_i : c_i \in [1,k] \right\}.$$

Taking $c_0 = c_1 = \cdots = c_{v-1} = 1$ in T_w, we have

$$T'_w = \left\{ a + \sum_{i=v}^{w-1} c_i x_i : c_i \in [1,k] \right\} \subseteq T_w.$$

Letting $d = \sum_{i=v}^{w-1} x_i$, we have $a + (k+1)d \in T_v$ and, from Remark 2.34, $\{a+d, a+2d, \ldots, a+kd\} \subseteq T'_w$. Hence, we have found a monochromatic arithmetic progression of length $k+1$, thereby proving the existence of $w(k+1; r)$. □

We now put the previous two lemmas together to prove van der Waerden's theorem.

Proof of van der Waerden's Theorem. Clearly $w(1; r)$ exists for all $r \ge 1$, since we need only a single point. Lemma 2.35 shows that $(1, t; r)$ is refined for all $r, t \ge 1$. This in turn, by Lemma 2.36, gives us the existence of $w(2; r)$ for all $r \ge 1$. Hence, by repeated application of Lemmas 2.35 and 2.36 we have the existence of $w(k; r)$ for any given k, for all $r \ge 1$. □

2.7. Exercises

2.1 Show that within $[1, n]$ there are $\frac{n^2}{4} + O(n)$ arithmetic progressions.

2.2 Prove that $w(2; r) = r + 1$ for all $r \ge 1$.

2.3 Show that the following 3-colorings yield no monochromatic 3-term arithmetic progression. Note that $w(3; 3) = 27$, so that the following colorings are maximal.
a) 00110 01212 20200 10112 02212 1
b) 21001 01221 12210 10012 20020 2

2.4 Show that $w(k_1, k_2, \ldots, k_r; r)$, i.e., the mixed van der Waerden numbers, always exist.

2.5 For each of the following van der Waerden-like functions, explain why it is reasonable to say that "the function is not worth studying."
a) For positive integers k_1 and k_2, define $f(k_1, k_2)$ to be the least positive integer n with the property that, for every 2-coloring of $[1, n]$, there is a color i such that there is either

a k_1-term arithmetic progression with color i or a k_2-term arithmetic progression with color i.
b) For positive integers k_1 and k_2, define $g(k_1, k_2)$ to be the least positive integer n such that for every 2-coloring (using the colors red and blue) of $[1, n]$, there is a k_1-term arithmetic progression with color red and a k_2-term arithmetic progression with color blue.

2.6 Prove that, for all $k \geq 2$, $w(k, 2; 2) = 2k - 1$ if k is even, and $w(k, 2; r) = 2k$ if k is odd.

2.7 Prove that, for all $k \geq 2$, $w(k, 2, 2; 3) = 3k$ if and only if $k \equiv \pm 1 \pmod 6$.

2.8 We can easily obtain a lower bound for $w(k; 2)$ by using the probabilistic method. Show that $w(k; 2) > 2^{k/2}$ for large k via the following steps.
a) Randomly color the integers in $[1, n]$ either red or blue. Let A be an arbitrary arithmetic progression of length k within $[1, n]$. Show that the probability that A is monochromatic is $\frac{2}{2^k} = 2^{1-k}$.
b) Use Exercise 2.1 to deduce that the probability that a monochromatic arithmetic progression exists is at most

$$\sum_{i=1}^{\frac{n^2}{4}+O(n)} 2^{1-k} = \frac{n^2 + O(n)}{2^{k+1}}.$$

c) Argue that if $\frac{n^2+O(n)}{2^{k+1}} < 1$ then $w(k; 2) > n$.
d) Conclude that $n = 2^{k/2}$ satisfies the first inequality in (c) for k large enough.

2.9 Fill in the details of the proof of Theorem 2.18.

2.10 Let $r = 2^n$. Show that for $3 \leq n \leq 12$, the bound given by Theorem 2.23 is better than Gowers' bound (Remark 2.22), but that for $n \geq 13$, Gowers' bound is better. (A computer will be helpful for this exercise.)

2.11 Find $\nu_3(7)$ and $\nu_3(9)$.

2.12 Exhibit a 2-coloring of $[1, 14]$ which shows that $\nu_4(14) \geq 7$.

2.13 Prove Theorem 2.28.

2.14 Prove Proposition 2.29 (hint: see Example 2.2).

2.15 Prove Proposition 2.30.

2.8. Research Problems

***2.1** Write a more efficient algorithm for determining $w(k; r)$.
References: [**25**], [**261**]

***2.2** Find $w(3; 5)$, $w(4; 3)$, or $w(6; 2)$, or improve the known lower bounds.
References:[**71**], [**210**], [**261**]

2.3 Determine the value of a new "mixed" van der Waerden number $w(k_1, k_2, \ldots, k_r; r)$ (where at least two of the k_i's are greater than 2), or obtain a lower bound, by using a computer program.
References: [**25**], [**55**], [**71**], [**74**]

2.4 Find the rate of growth, or improve the known bounds, on the mixed van der Waerden numbers $w(k, k_0)$, as a function of k, for any fixed $k_0 \geq 3$. Try similar problems using more than two colors, where all but one of the k_i's $(1 \leq i \leq r)$ are fixed.
References: [**25**], [**55**], [**71**],[**74**]

2.5 Investigate the set of maximal length valid colorings for various van der Waerden (or mixed van der Waerden) numbers.
References: [**25**], [**55**], [**71**], [**74**], [**210**], [**261**]

2.6 The first-named author has conjectured the following concerning mixed van der Waerden numbers:

$$\begin{aligned} w(k,k) &\geq w(k+1, k-1) \\ &\geq w(k+2, k-2) \\ &\geq \cdots \geq w(2k-2, 2) \end{aligned}$$

for all $k \geq 3$. Prove or disprove this. Try extending this to mixed van der Waerden numbers where the number of colors is greater than two.
References: [**25**], [**55**], [**71**],[**74**]

***2.7** Let $A = \{a_1, a_2, \ldots\}$ be a sequence of positive integers such that $\sum_{i=1}^{\infty} \frac{1}{a_i} = \infty$. Erdős conjectured that A must contain

arbitrarily long arithmetic progressions. Prove or disprove this.
Reference: [**275**]

*__2.8__ Erdős conjectured (in 1961) that $\lim_{k\to\infty} \frac{w(k;2)}{2^k} = \infty$. Prove or disprove this.
References: [**34**], [**263**]

*__2.9__ Improve on the known upper bound for $w(k;2)$.
References: [**121**], [**141**]

*__2.10__ Improve on the known lower bounds for $w(k;r)$.
Reference: [**127**]

*__2.11__ Find the asymptotic value of $w(3;r)$.
Reference: [**141**]

*__2.12__ Improve on the known upper bound for $w(4;r)$.
References: [**121**], [**141**]

*__2.13__ Find an upper bound on $\nu_k(n)$ for general k and n.
References: [**229**], [**264**], [**265**]

2.9. References

§**2.1**. The statement and proof of the full version of the compactness principle can be found in [**127**].

§**2.2**. Theorem 2.5 is taken primarily from [**123**, p. 1000], with statement (iv) from [**72**] and statement (vi) originally from [**56**]. More proofs of the equivalence of different forms of van der Waerden's theorem are given in [**205**] and [**209**], the latter of which also contains several corollaries of the theorem.

§**2.3**. The algorithms, and information about the computer time needed to find some exact values, may be found in [**25**] and [**261**]. The references for the values (except $w(3,3;2) = 9$) given in Table 2.1 are as follows:

$w(4,3;2)$, $w(4,4;2)$, $w(5,3;2)$, $w(5,4;2)$, $w(6,3;2)$, $w(7,3;2)$, and $w(3,3,3;3)$ are from [**71**]; $w(5;2) = 178$ is from [**261**];

$w(6,4;2)$, $w(8,3;2)$, $w(9,3;2)$, $w(10,3;2)$, $w(4,3,3;3)$, and $w(3,3,3,3;4)$ are from [**25**];

$w(11,3;2)$, $w(12,3;2)$, $w(4,4,3;3)$, $w(5,3,3;3)$, and $w(4,3,3,2;4)$ are from [**74**]; $w(7,4;2)$ is from [**24**];

and the remaining values are from [**55**].

The mixed van der Waerden numbers such that all but one entry equals 2 are investigated in [**74**]. Bounds on numbers very similar to the mixed van der Waerden numbers can be found in [**137**].

§**2.4**. Theorem 2.9 is from [**34**]. Theorem 2.10 is from [**263**]. Erdős and Rado [**95**] showed, in 1952, that

$$w(k;r) > r^{k+1} - c\sqrt{(k+1)\ln(k+1)}.$$

A series of improvements (but not as strong as Theorem 2.10) were subsequently made in [**191**], [**246**], and [**5**]. The bound

$$w(k;r) > \frac{kr^k}{e(k+1)^2},$$

which is sometimes better than that of Theorem 2.9, was proven by Everts [**105**]. Other results concerning Property B are found in [**4**], [**6**], [**7**], [**8**], [**9**], [**20**], [**46**], [**85**], [**86**], [**87**], [**104**], [**107**], and [**150**]. The function $\mu(k)$ (Definition 2.15) was defined by Erdős and Hajnal [**93**]. Schmidt's lemma is proven in [**245**], and later in [**99**], which also includes a simpler proof of a weaker bound. Erdős [**86**] gives an upper bound for $\mu(k)$. The proof of Theorem 2.18 is from [**99**]. The proof of Theorem 2.19 can be easily derived from a similar proof in [**127**, p. 96]. Theorem 2.20 is an application of the Lovász Local Lemma [**94**]. For a clear proof of the Lovász Local Lemma see [**255**]. For a proof of Theorem 2.20 see [**127**]. The lower bounds of Table 2.2 are from [**210**]. The given development of the *tower*, *wow*, and *ack* functions can be found in [**127**]. Gowers' results (Theorem 2.21 and Remark 2.22) can be found in [**121**]. Theorem 2.23 is from [**141**].

§**2.5**. The Erdős and Turán function is defined in [**102**]. A proof of Roth's 1952 result can be found in [**229**]. Szemerédi's 1969 result and proof can be found in [**264**]. A proof of the conjecture of Erdős and Turán by Szemerédi is found in [**265**]. Furstenberg [**112**] gives a proof of this result using ergodic theory. A shorter proof is given in [**267**]. Bounds on $\nu_3(n)$ are found in [**26**], [**192**], and [**227**]. Rankin [**219**] obtains a lower bound on $\nu_k(n)$ for $k > 3$. Some relatively early results on the Erdős and Turán function may be found in [**230**] and [**231**]. Pomerance [**204**] and Riddell [**221**] consider analogues of Szemerédi's theorem in the set of lattice points in the plane (points having both coordinates in $\mathbb{Z}$).

§**2.6**. For a brief history of van der Waerden's theorem, see [**68**], which contains many references. The original proof of van der Waerden's theorem, which was referred to as Baudet's conjecture before it became a theorem, can be found in [**270**], with a recounting of its discovery in [**271**]. The proof presented here (namely, Lemmas 2.35 and 2.36) is derived from [**17**] and is similar to the one found in [**127**]. The shortest proof (to date) is found in [**126**]. A topological proof can be found in [**116**], while an algebraic proof can be found in [**30**]. Other proofs can be found in [**78**], [**189**], [**206**], [**249**], and [**266**].

§**2.7**. The colorings given in Exercise 2.3 can be found in [**71**]. Exercise 2.9 uses what is known as the probabilistic method, which is the subject of a book by Erdős and Spencer [**99**].

§**2.8**. Wróblewski [**275**] proved the following result: there exists a set of positive integers $\{x_1 < x_2 < \cdots\}$ with no 3-term arithmetic progression such that $\sum_{i=1}^{\infty} \frac{1}{x_i} > 3.00849$ (which may help to explain why Research Problem 2.7 is considered difficult).

Additional References: For biographies of Paul Erdős, see [**148**] and [**243**]. An algorithm that gives partitions of $\mathbb{Z}^+$ into an infinite number of subsets, none of which contain a 3-term arithmetic progression, is discussed in [**118**]. Earlier related work is covered in [**119**]. In [**259**], Spencer showed (among other things) that for each $k \geq 1$, there exists a set S of positive integers so that for every finite coloring of S, there is a monochromatic k-term arithmetic progression, but such that S is "sparse" enough that it itself does not contain any $(k+1)$-term arithmetic progressions. An alternate proof, using a direct construction, is given in [**198**].

Chapter 3

Supersets of AP

As we discussed in Chapter 1, we can define a function analogous to the van der Waerden function $w(k;r)$ by substituting for AP (the set of arithmetic progressions) some other set of sequences. That is, if $\mathcal{F}$ is some specific collection of sequences, denote by $R(\mathcal{F},k;r)$ the least positive integer m (if it exists) such that every r-coloring of $[1,m]$ yields a monochromatic k-term member of $\mathcal{F}$.

Of course, there is no guarantee that $R(\mathcal{F},k;r)$ exists, unless we choose the collection $\mathcal{F}$ wisely. As a simple example, let $r=2$ and let $\mathcal{F}$ be the family of all sequences of positive integers that begin with the pattern $i, i+1$ for some i. In order for $R(\mathcal{F},2;2)$ to exist, there must exist a number $n = R(\mathcal{F},2;2)$ such that for every 2-coloring of $[1,n]$, there is a monochromatic pair $i, i+1$. This is not true since, for example, the coloring of $\mathbb{Z}^+$ where all odd numbers are colored blue and all even numbers are colored red does not yield such a pair.

Let us assume that $\mathcal{F}$ is some collection of sequences such that $R(\mathcal{F},k;2)$ exists for all k. If $\mathcal{G}$ is a family of sequences such that $\mathcal{F} \subseteq \mathcal{G}$, then since we are assured of finding k-term monochromatic members of $\mathcal{F}$ in $[1, R(\mathcal{F},k;2)]$, we are also assured of finding k-term monochromatic members of $\mathcal{G}$ in $[1, R(\mathcal{F},k;2)]$. In fact, if $\mathcal{G}$ is larger than $\mathcal{F}$, then it may happen that we do not have to go out as far as the integer $R(\mathcal{F},k;2)$ in order to guarantee a monochromatic k-term member of $\mathcal{G}$. Summarizing, we have the following fact.

Theorem 3.1. *Let $\mathcal{F}$ be a collection of sequences and let k be a positive integer such that $R(\mathcal{F}, k; 2)$ exists. If $\mathcal{F} \subseteq \mathcal{G}$, then $R(\mathcal{G}, k; 2)$ exists and $R(\mathcal{G}, k; 2) \leq R(\mathcal{F}, k; 2)$.*

Before proceeding, we remind the reader that we may denote $R(F, k; 2)$ more simply as $R(F, k)$.

As we saw in Chapter 2, $R(AP, k)$ exists for all k; these are just the van der Waerden numbers $w(k)$. We noted in Chapter 2 that the determination of the rate of growth of $w(k)$ is a very difficult problem. Can we alter the problem somewhat so as to make a solution easier to obtain? Theorem 3.1 suggests a method: replace AP with a different (appropriately chosen) collection $\mathcal{G}$ such that $AP \subseteq \mathcal{G}$.

If $S \subseteq T$, T is called a *superset* of S.

In this chapter we will look at several different choices of $\mathcal{G}$ such that $\mathcal{G}$ is a superset of AP, and study the corresponding function $R(\mathcal{G}, k)$. In some cases we will see that there are interesting relationships between $R(\mathcal{G}, k)$ and $w(k)$. The supersets of AP that we present here are by no means exhaustive, and we encourage the reader to consider other choices of $\mathcal{G}$.

3.1. Quasi-progressions

An arithmetic progression $\{a, a+d, a+2d, \dots\}$ can be thought of as an increasing sequence of positive integers such that the gap between each adjacent pair of integers is the constant d. One way of generalizing the notion of an arithmetic progression is to allow a little more variance in this gap. For example, we can require only that the gaps be either d or $d+1$ for some positive integer d; or, that the gaps belong to the set $\{d, d+1, d+2\}$ for some $d \in \mathbb{Z}^+$. This prompts the next definition.

Definition 3.2. Let $k \geq 2$ and $n \geq 0$. A k-term *quasi-progression of diameter* n is a sequence of positive integers $\{x_1, x_2, \dots, x_k\}$ such that there exists a positive integer d with the property that

$$d \leq x_i - x_{i-1} \leq d + n$$

for $i = 2, 3, \dots, k$.

We call the integer d of Definition 3.2 a *low-difference* for the quasi-progression $X = \{x_1, x_2, \dots, x_k\}$; and, when helpful to the discussion, we will say that X is a (k, n, d)*-progression.*

Example 3.3. Let A be any arithmetic progression with gap d. Then A is a quasi-progression of diameter 0. Note that A may also be considered a quasi-progression with diameter 1 (since all gaps belong to $\{d, d+1\}$ for some d). Any 1-element or 2-element sequence may be considered to have diameter 0.

Example 3.4. Let $B = \{1, 3, 5, 8\}$. We see that B is a 4-term quasi-progression with diameter 1 and low-difference 2; hence it is a $(4, 1, 2)$-progression. Note that B may also be considered a $(4, 2, 2)$-progression since all of the differences $x_i - x_{i-1}$ belong to the set $\{2, 3, 4\}$. Moreover, B is a $(4, n, 2)$-progression for all $n \geq 1$. Lastly, note that B is a $(4, 2, 1)$-progression since all of the gaps belong to $\{1, 2, 3\}$ – in fact, it is a $(4, n, 1)$-progression for each $n \geq 2$.

We see from Examples 3.3 and 3.4 that the diameter n and low-difference d of a quasi-progression are not unique, but that the possible values for the low-difference depend on the choice of diameter. In particular, any quasi-progression of diameter $n \geq 0$ is also a quasi-progression of diameter m for all $m \geq n$. Obviously, for each $n \geq 0$, the set of quasi-progressions of diameter n is a superset of AP.

Although for a given sequence there are many choices for n and d, it will ordinarily be most advantageous for us to use the minimum value of n that works, and then to choose the minimum value for d corresponding to this n. Hence, we usually think of progression A of Example 3.3 as having low-difference d and diameter 0; and of progression B of Example 3.4 as having low-difference 2 and diameter 1.

We next mention a convenient notation for the function analogous to $w(k)$, where we are concerned with quasi-progressions rather than arithmetic progressions.

Notation. For positive integers n and k, denote by $Q_n(k)$ the least positive integer m such that for every 2-coloring of $[1, m]$ there is a monochromatic k-term quasi-progression of diameter n.

In the following theorem we list some rather obvious properties of $Q_n(k)$. We state these without proof and ask the reader to verify them.

Theorem 3.5. *Let $n \geq 0$ and $k \geq 2$.*

(i) $Q_n(k)$ exists.

(ii) $Q_0(k) = w(k)$.

(iii) $Q_0(k) \geq Q_1(k) \geq Q_2(k) \geq \cdots$.

Our goal is to find out as much as we can about the magnitude of the function $Q_n(k)$. We would hope that, for $n \geq 1$, a "reasonable" upper bound for $Q_n(k)$ is more easily obtainable than has been the case for the van der Waerden numbers. From Theorem 3.5(iii), we see that the larger the value of n, the more likely that we will be able to obtain a "not-so-big" upper bound on the function $Q_n(k)$.

We first look at the easiest case (where the diameter is at least $k-1$). In this case we are able to give the exact value of $Q_n(k)$.

We first give a lower bound for $Q_n(k)$ that holds for all $n \geq 1$ and all $k \geq 2$.

Theorem 3.6. *Let $k \geq 2$ and $n \geq 1$. Then $Q_n(k) \geq 2k-1$.*

Proof. Consider the 2-coloring $\chi : [1, 2k-2] \to \{0, 1\}$ defined by $\chi([1, k-1]) = 1$ and $\chi([k, 2k-2]) = 0$. This coloring admits no monochromatic k-element set. In particular, it yields no k-term monochromatic quasi-progression of diameter n. Therefore, we have $Q_n(k) \geq 2k-1$. □

We next show that the lower bound given in Theorem 3.6 also serves as an upper bound when $n = k-1$.

Theorem 3.7. $Q_{k-1}(k) \leq 2k-1$ *for all* $k \geq 2$.

Proof. Let χ be an arbitrary 2-coloring of $[1, 2k-1]$. Clearly there is some k-element set $X = \{x_1 < x_2 < \cdots < x_k\}$ that is monochromatic under χ. If for some j, $2 \leq j \leq k$, we have $x_j - x_{j-1} > k$, then

$$x_k - x_1 = \sum_{i=2}^{k}(x_i - x_{i-1}) > k + (k-2) = 2k-2,$$

which is not possible. Thus, X is a monochromatic $(k, k-1, 1)$-progression, since $1 \leq x_i - x_{i-1} \leq k$ for $2 \leq i \leq k$. This shows that $Q_{k-1}(k) \leq 2k-1$. □

Note that by Theorems 3.5(iii), 3.6, and 3.7 we have the following corollary.

Corollary 3.8. *For all $k \geq 2$ and all $n \geq k-1$, $Q_n(k) = 2k-1$.*

As we alluded to earlier, it would be most desirable to obtain results in which the diameters are as small as possible. Although no "nice" upper bound is known for the function $Q_1(k)$, we look next at a lower bound for this function.

Theorem 3.9. $Q_1(k) \geq 2(k-1)^2 + 1$ *for* $k \geq 2$.

Proof. Define the 2-coloring χ of $[1, 2(k-1)^2]$ by the string

$$\underbrace{00\ldots0}_{k-1}\underbrace{11\ldots1}_{k-1}\underbrace{00\ldots0}_{k-1}\underbrace{11\ldots1}_{k-1}\ldots\underbrace{00\ldots0}_{k-1}\underbrace{11\ldots1}_{k-1},$$

where each of the $(2k-2)$-element blocks $\underbrace{00\ldots0}_{k-1}\underbrace{11\ldots1}_{k-1}$ appears $k-1$ times. To prove the theorem, it suffices to show that under this coloring there is no k-term monochromatic quasi-progression of diameter 1.

By way of contradiction, let $m = 2(k-1)^2$, and assume that $X = \{x_1, x_2, \ldots, x_k\} \subseteq [1, m]$ is a quasi-progression of diameter 1 that is monochromatic under χ. By the symmetry of χ, without any loss of generality, we may assume that $\chi(X) = 1$. Since each monochromatic block of color 1 has $k-1$ elements, there is some i, $2 \leq i \leq k$, where x_i and x_{i-1} belong to two different such blocks. For this i, we have $x_i - x_{i-1} \geq k$. Since X has diameter 1, this implies that X has a low-difference of at least $k-1$. Thus, each of the blocks of $k-1$ consecutive 1's contains no more than one member of X. Hence, X must have length at most $k-1$, a contradiction. □

In the following theorem, we use a generalization of the coloring used in the proof of Theorem 3.9 to obtain a lower bound for $Q_{k-i}(k)$ in terms of i and k.

Theorem 3.10. *Let $1 \le i < k$ and let $m = 1 + \lfloor \frac{k-2}{i} \rfloor$. Then*

$$Q_{k-i}(k) \ge 2\left(\left\lfloor \frac{k-1}{m} \right\rfloor (k-1-im) + i(k-1)\right) + 1. \tag{3.1}$$

Before presenting the proof, first notice that this is indeed a generalization of Theorem 3.9, since if $i = k-1$, we have $m = 1$, and thus Theorem 3.10 gives $Q_1(k) \ge 2((k-1)(0) + (k-1)^2) + 1$. At the other extreme, if we take $i = 1$, then $m = k-1$, so that $Q_{k-1}(k) \ge 2((k-1-(k-1)) + k-1) + 1$, giving us Theorem 3.6.

Proof of Theorem 3.10. Let $s = 2\left(\lfloor \frac{k-1}{m} \rfloor (k-1-im) + i(k-1)\right)$. Define the 2-coloring χ of $[1, s]$ by the string

$$1^y(0^{k-1}1^{k-1}0^{k-1}1^{k-1}\dots0^{k-1}1^{k-1})0^y,$$

where within the parentheses each of the blocks $0^{k-1}1^{k-1}$ occurs $\lfloor \frac{k-1}{m} \rfloor$ times, and where $y = i\left(k-1-m\lfloor \frac{k-1}{m} \rfloor\right)$. Note that this is, in fact, a string of length s. To establish (3.1) it is sufficient to show that, under χ, $[1, s]$ contains no monochromatic k-term quasi-progression of diameter $k-i$. We proceed by contradiction.

Assume that $X = \{x_1, x_2, \dots, x_k\} \subseteq [1, s]$ is a quasi-progression of diameter $k-i$ that is monochromatic under χ. By the symmetry of χ, we may assume that $\chi(X) = 1$. Note that since $m\lfloor \frac{k-1}{m} \rfloor \ge k-m$ (why?),

$$\begin{aligned} y &= i\left(k-1-m\left\lfloor \tfrac{k-1}{m} \right\rfloor\right) \\ &\le i(k-1-(k-m)) \\ &= i\left\lfloor \tfrac{k-2}{i} \right\rfloor \\ &\le k-2. \end{aligned}$$

Hence, there is no block of more than $k-1$ consecutive 1's. Thus, for some $j \in \{2, 3, \dots, k\}$, we have $x_j - x_{j-1} \ge k$, which implies that X cannot have a low-difference that is less than i.

Since the low-difference of X is at least i, the first block of 1's (having length y), contains at most $\frac{y}{i} = k-1-m\lfloor \frac{k-1}{m} \rfloor$ members of X. Similarly, in any block of $k-1$ consecutive 1's, there are at most $1 + \lfloor \frac{k-2}{i} \rfloor = m$ members of X. There are $\lfloor \frac{k-1}{m} \rfloor$ blocks of $k-1$ consecutive 1's, so that, accounting for all blocks of 1's, we see that X has at most

$$k-1-m\left\lfloor \frac{k-1}{m} \right\rfloor + m\left\lfloor \frac{k-1}{m} \right\rfloor = k-1$$

elements, a contradiction. □

Computing some actual values of Ramsey-type functions can be quite helpful in forming conjectures about the magnitude (or rate of growth) of the functions. As we saw in Chapter 2, for the classical van der Waerden numbers $w(k; r)$, the computations can be quite prohibitive. In dealing with supersets of AP, however, we often find the computations much more reasonable. In fact, in some circumstances we have enough computed data to help us form "educated" conjectures. Such is the situation with certain special cases of Theorem 3.10. For these cases, computer-generated values of $Q_{k-i}(k)$ suggest that, whenever $i \le \frac{k}{2}$, the right-hand side of (3.1) is the precise value of $Q_{k-i}(k)$ (see Table 3.1 at the end of this section). We single out these special cases in the following corollary.

Corollary 3.11. *Let $1 \le i \le k-1$.*

(i) If $k \equiv 0 \pmod{i}$, then $Q_{k-i}(k) \ge 2ik - 4i + 3$.

(ii) If $k \equiv 1 \pmod{i}$, then $Q_{k-i}(k) \ge 2ik - 2i + 1$.

Proof. We will prove (ii), and leave the proof of (i) to the reader as Exercise 3.2. Letting $k = ti + 1$, then using the notation of Theorem 3.10, we have $m = 1 + \lfloor \frac{ti-1}{i} \rfloor = t$, and $\lfloor \frac{k-1}{m} \rfloor = i$. Therefore, (3.1) becomes

$$Q_{k-i}(k) \ge 2(t(k-1-it) + i(k-1)) + 1 = 2i(k-1) + 1.$$

□

We see that Corollary 3.11(ii) implies Theorem 3.9, simply by letting $i = k-1$. Also notice that by letting $i = 1$ in Corollary 3.11(i), we get that $Q_{k-1}(k) \ge 2k-1$, as given by Theorem 3.6. So far, $i = 1$ is the only case we have presented for which we have a precise formula for $Q_{k-i}(k)$. In the next theorem, we give the formula for the only other nontrivial value of i for which an exact formula for $Q_{k-i}(k)$ is known.

Theorem 3.12. *Let $k \geq 2$. Then*

$$Q_{k-2}(k) = \begin{cases} 4k-5 & \text{if } k \text{ is even,} \\ 4k-3 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. Letting $i = 2$ in Corollary 3.11, we obtain $Q_{k-2}(k) \geq 4k-5$ when k is even, and $Q_{k-2}(k) \geq 4k-3$ when k is odd. Hence we need only establish these as upper bounds for $Q_{k-2}(k)$.

To obtain the upper bounds, let $\chi : \mathbb{Z}^+ \to \{0,1\}$ be any 2-coloring. We will show that if k is even then there is a monochromatic k-term quasi-progression with diameter $k-2$ in $[1, 4k-5]$, and that if k is odd then there exists such a progression in $[1, 4k-3]$.

First note that the upper bound obviously works for $k = 2$, since $Q_0(2) = w(2) = 3$. So we may assume that $k \geq 3$. Notice that from the proof of Theorem 3.7, $[1, 2k-3]$ contains a monochromatic $(k-1, k-2, 1)$-progression $\{x_1 < x_2 < \cdots < x_{k-1}\}$, say of color 1. Now if $x_{k-1} + j$ has color 1 for some $j \in \{1, 2, \ldots, k-1\}$, then $[1, 4k-5]$ will contain the k-term monochromatic quasi-progression $\{x_1, x_2, \ldots, x_{k-1}\} \cup \{x_{k-1} + j\}$, having diameter $k-2$, so in this case we are done. Thus, we may assume that $[x_{k-1}+1, x_{k-1}+k-1]$ has color 0.

If there is some integer b with color 0 that satisfies

$$x_{k-1} - (k-2) \leq b < x_{k-1},$$

then $\{b\} \cup [x_{k-1}+1, x_{k-1}+k-1]$ forms a monochromatic $(k, k-2, 1)$-progression, and we are done. Hence, we may further assume that $\chi([x_{k-1} - (k-2), x_{k-1} - 1]) = 1$. For ease of notation, we let

$$a = x_{k-1} - (k-2),$$

so that $\chi([a, a+k-2]) = 1$.

Note that $a \leq k-1$, since $x_{k-1} \leq 2k-3$. Hence we may assume that $\chi(x) = 0$ for all $x < a$, for otherwise for some $x_0 < a$ the set $\{x_0\} \cup [a, a+k-2]$ would be a monochromatic k-term quasi-progression with diameter $k-2$ (having color 1), and we would be done. Likewise, we assume that all of $[a+k-1, a+2k-3]$ has color 0 (why?), and hence that all of $[a+2k-2, a+3k-4]$ has color 1. Finally, we may assume that $[a+3k-3, a+4k-5]$ has color 0.

To complete the proof, we consider three cases. In each case we find a monochromatic quasi-progression with the desired properties.

Case 1. k is even. In this case, the set

$$\left\{a + 2i : 0 \leq i \leq \frac{k-2}{2}\right\} \bigcup \left\{a + 2i : k-1 \leq i \leq \frac{3k-4}{2}\right\}$$

is monochromatic of color 1, has length k, and (since $a \leq k-1$) is contained in $[1, 4k-5]$. It obviously is a quasi-progression having diameter $k-2$ and low-difference 2.

Case 2. k is odd and $a = 1$. If $\chi(4k-3) = 0$, then the interval $[3k-2, 4k-3]$ is a monochromatic k-term quasi-progression with diameter $k-2$. If, on the other hand, $\chi(4k-3) = 1$, then the sequence $\{1, 4, 6, 8, \ldots, k-1, 2k-1, 2k+2, 2k+4, 2k+6, \ldots, 3k-3, 4k-3\}$ is a k-term quasi-progression with color 1, having diameter $k-2$.

Case 3. k is odd and $2 \leq a \leq k-1$. Let

$$A_1 = \{a - 1 - 2i : 0 \leq i \leq \lfloor \tfrac{a}{2} \rfloor - 1\},$$
$$A_2 = \{a+k-1\} \cup \{a+k+2, a+k+4, \ldots, a+2k-5, a+2k-3\},$$
$$A_3 = \{a + 3k - 3 + 2i : 0 \leq i \leq \lfloor \tfrac{k-a}{2} \rfloor\}.$$

Then $A_1 \cup A_2 \cup A_3$ is contained in $[1, 4k-3]$ and forms a k-term quasi-progression of diameter $k-2$. We leave the details as Exercise 3.3. □

We next turn our attention to the problem of finding upper bounds for $Q_n(k)$.

The main result we have on upper bounds for $Q_n(k)$ relies on the following lemma. The proof of the lemma is fairly complicated, and we choose not to include it here.

Lemma 3.13. *For $3 \leq j \leq \frac{n}{2}$, let $E(n,j)$ be the least positive integer such that for every 2-coloring of $[1, E(n,j)]$ there is a monochromatic $(n+j-1, n, \ell)$-progression for some $\ell \leq j-1$. Then*

$$E(n, j+1) \leq E(n,j) + 2\lambda(n+j-1), \tag{3.2}$$

where $\lambda = \left\lceil \frac{n+j-2}{\lfloor \frac{n+j}{j} \rfloor} \right\rceil + 1.$

With the help of Lemma 3.13, we are now able to obtain a bound on $Q_n(k)$, provided k does not exceed $\frac{3n}{2}$.

Theorem 3.14. *Let $3 \le t \le \frac{n}{2}$. Then*

$$Q_n(n+t) \le \frac{1}{2n}t^4 + \frac{4}{3}t^3 + (n+2)t^2 + 8nt. \tag{3.3}$$

Proof. First, we note that by the definition of $E(n,j)$ we have $Q_n(n+t) \le E(n,t+1)$. To complete the theorem, we will show that $E(n,t+1)$ is bounded above by the right-hand side of (3.3). Applying (3.2) $t-2$ times, we obtain

$$\begin{aligned} E(n,t+1) &\le E(n,3) + 2\sum_{r=3}^{t}(n+r-1)\left(\left\lceil \frac{n+r-2}{\lfloor \frac{n+r}{r} \rfloor}\right\rceil + 1\right) \\ &\le E(n,3) + 2\sum_{r=3}^{t}(n+r-1)\left(\frac{n+r-3}{\lfloor \frac{n+r}{r} \rfloor} + 2\right) \\ &\le E(n,3) + 2\sum_{r=3}^{t}(n+r-1)\left(\frac{nr+r^2-3r}{n+1} + 2\right). \end{aligned}$$

By the proof of Theorem 3.12, for every 2-coloring of $[1, 4(n+2)-3]$ there is a monochromatic $(n+2)$-term quasi-progression of diameter n and low-difference at most 2. Thus, $E(n,3) \le 4n+5$. This gives

$$\begin{aligned} E(n,t+1) \le\ & 4n+5+4\sum_{r=3}^{t}(n+r-1) \\ &+ \frac{2}{n+1}\sum_{r=1}^{t-2}(n+r+1)(r^2+r(n+1)+2n-2). \end{aligned}$$

Hence,

$$\begin{aligned} E(n,t+1) \le\ & 4n+5+4nt+2t^2 \\ &+\frac{2}{n}\left(\sum_{r=1}^{t-2}(r^3+r^2(2n+2)+r(n^2+4n-1)+2n^2-2)\right). \end{aligned}$$

Simplifying, we get

$$E(n,t+1) \le 4n+5+\frac{2}{n}\left(\frac{t^4}{4}+\frac{2}{3}nt^3+\frac{(n^2+4n-1)(t-1)(t-2)}{2}\right)$$
$$+\frac{2}{n}\left(2(t-1)(n^2-1)+2n^2t+nt^2\right). \tag{3.4}$$

To complete the proof, we note that since $3 \le t \le \frac{n}{2}$,

$$13+4t^2+2n+\frac{2}{n} \le \frac{t^2}{n}+2tn+8t+\frac{2t}{n}$$

(which may be shown in a straightforward manner by induction on n). Hence,

$$4n+5+(t-1)(t-2)(n+4-\frac{1}{n})+4(t-1)(n-\frac{1}{n}) \le nt^2+4nt,$$

which implies that the right hand side of (3.4) is not greater than $\frac{t^4}{2n}+\frac{4}{3}t^3+nt^2+8nt+2t^2$. Since $E(n,t+1)$ is bounded above by this expression, the proof is complete. □

As mentioned before, we seek an upper bound on $Q_n(k)$ for the smallest possible value of n. The best result that Theorem 3.14 provides in this regard is for the case in which n is two-thirds of the value of k. We single out this bound in the next corollary, which is a direct consequence of Theorem 3.14; its proof is left as Exercise 3.4.

Corollary 3.15.

$$Q_{\lceil \frac{2k}{3} \rceil}(k) \le \frac{43}{324}k^3(1+o(1)).$$

We note that Theorem 3.10 gives a quadratic lower bound for $Q_{\lfloor \frac{2k}{3} \rfloor}(k)$, namely $\frac{2}{3}k^2(1+o(1))$, while the upper bound of Corollary 3.15 is a cubic. It would be interesting to know the actual rate of growth for this Ramsey-type function.

We end this section with a table. For a given pair (k,i) the corresponding entry in the table is the value (or the best known lower bound) for $Q_{k-i}(k)$. We only include those pairs where $i \ge 3$, since we have presented exact formulas for $i=1$ and $i=2$.

$k \setminus i$	3	4	5	6	7	8
4	19	-	-	-	-	-
5	29	33	-	-	-	-
6	27	49	67	-	-	-
7	37	65	73	≥ 124	-	-
8	39	51	93	?	≥ 190	-
9	45	65	≥ 115	?	?	≥ 287
10	55	67	83	?	?	?
11	57	75	101	?	?	?
12	63	83	≥ 103	?	?	?
13	73	97	≥ 108	?	?	?
14	75	99	≥ 119	?	?	?
15	81	?	≥ 133	?	?	?
16	91	?	?	?	?	?
17	93	?	?	?	?	?

Table 3.1: Values and lower bounds for $Q_{k-i}(k)$

3.2. Generalized Quasi-progressions

We now direct our attention to a type of sequence that generalizes the notion of a quasi-progression. The idea of this generalization is that we allow the "diameter" to vary with the terms of the sequence. Specifically, we give the following definition.

Definition 3.16. Let $\delta : \{2, 3, \dots\} \to [0, \infty)$ be a function. A k-term *generalized quasi-progression with diameter function* δ(or a GQ_δ-*progression*) is a sequence $\{x_1, x_2, \dots, x_k\}$ such that, for some positive integer d, we have $d \leq x_i - x_{i-1} \leq d + \delta(i)$ for all $i = 2, 3, \dots, k$.

We see that generalized quasi-progressions have "diameters" that may vary, and that a quasi-progression of diameter n may be considered a generalized quasi-progression, where δ is the constant function n. As we did with quasi-progressions, we will refer to d as the low-difference of the progression.

Analogous to the notation $Q_n(k)$, we use the following notation for the Ramsey-type function associated with generalized quasi-progressions.

Notation. Let $\delta : \{2, 3, \dots\} \to [0, \infty)$ be a function and k a positive integer. Denote by $GQ_\delta(k)$ the least positive integer such that for every 2-coloring of $[1, GQ_\delta(k)]$ there is a monochromatic k-term GQ_δ-progression.

Example 3.17. Let δ be the function defined as the constant function $\delta(x) = 1$ for all x. Then in this case $GQ_\delta(k)$ has the same meaning as $Q_1(k)$. Likewise, if δ is the zero function, then $GQ_\delta(k)$ is simply the classical van der Waerden function $w(k)$. Moreover, for any constant function $\delta = c$ with $c \geq 0$, $GQ_\delta(k)$ coincides with $Q_c(k)$.

When $\delta = \delta(x)$ is a specified function of x, it is often convenient to use the notation $GQ_{\delta(x)}$. For example if $\delta(x) = x^2$, then we may refer to a GQ_δ-progression as a GQ_{x^2}-progression and denote the associated Ramsey-type function by $GQ_{x^2}(k)$.

Example 3.18. Let δ be the function $\delta(x) = x - 1$ for all $x \geq 2$. Then $\{x_1, x_2, \dots, x_k\}$ is a GQ_{x-1}-progression provided there is a positive integer d such that $x_i - x_{i-1} \in [d, d + i - 1]$ for $2 \leq i \leq k$. For example, $\{1, 3, 6, 7, 10, 16\}$ is a GQ_{x-1}-progression where $d = 1$, and $\{4, 6, 8, 10, 16\}$ is a GQ_{x-1}-progression where $d = 2$. Thus, unlike a quasi-progression, where the gaps between consecutive members of the progression (even an infinite progression) must belong to an interval $[d, d+n]$ of fixed length (since n is fixed), the set of allowable gaps in a GQ_{x-1}-progression can grow as the terms of the sequence increase (so that if the progression is infinite, there is no upper bound on the entire set of possible gaps).

It is clear that the smaller the values of the function δ are, the closer to the classical van der Waerden numbers the function $GQ_\delta(k)$ is. Moreover, it is easy to see that $\delta_1(x) \leq \delta_2(x)$, for all x, implies $GQ_{\delta_1}(k) \geq GQ_{\delta_2}(k)$. For some functions δ, it is not difficult to obtain upper bounds for $GQ_\delta(k)$. Finding an upper bound for $GQ_{x-1}(k)$, the function discussed in Example 3.18, is left as Exercise 3.5. In the next theorem, we give an upper bound on $GQ_{x-2}(k)$. Our proof makes use of some known values of $GQ_{x-2}(k)$ (these and other values of $GQ_{g(x)}(k)$ are given in Table 3.2 at the end of this section).

Theorem 3.19. *Let $k \geq 2$ and $m_k = 1+2(k-1)^2+2\sum_{j=4}^{k-1} j(j-1)$. Then every 2-coloring of $[1, m_k]$ has a monochromatic k-term GQ_{x-2}-progression with low-difference $d \leq 2(k-1)$. In particular, for $k \geq 5$, $GQ_{x-2}(k) \leq \frac{2}{3}k^3 - \frac{8}{3}k - 13$.*

Proof. The second claim of the theorem follows from the first via a straightforward computation, which we leave as Exercise 3.6.

To prove the first claim, we note the following values are known: $GQ_{x-2}(2) = 3$, $GQ_{x-2}(3) = 9$, and $GQ_{x-2}(4) = 19$. It is obvious from these values that for $k = 2, 3, 4$ it is also true that any k-term GQ_{x-2}-progression has a low-difference not exceeding $2(k-1)$. Hence the theorem is true for $k = 2, 3, 4$.

We shall complete the proof by induction on k. To this end, assume that $k \geq 4$ is an integer for which the statement is true, and let $\chi : [1, m_{k+1}] \to \{0, 1\}$ be any 2-coloring. Therefore, there exists a monochromatic GQ_{x-2}-progression $X = \{x_1 < x_2 < \cdots < x_k\}$ with low-difference $d \leq 2(k-1)$ and $x_k \leq m_k$. Without loss of generality, let $\chi(X) = 1$.

For $0 \leq t \leq 2k-1$, let $A_t = [x_k + d + kt, x_k + d + kt + k - 1]$, and let $A_{2k} = \{x_k + d + 2k^2\}$. Notice that

$$\begin{aligned} x_k + d + 2k^2 &\leq 1 + 2(k-1)^2 + 2\sum_{j=4}^{k-1} j(j-1) + 2k^2 + 2(k-1) \\ &= 1 + 2k^2 + 2\sum_{j=4}^{k} j(j-1) \\ &= m_{k+1}. \end{aligned}$$

Hence, $A_t \subseteq [1, m_{k+1}]$ for $0 \leq t \leq 2k$. Consider

$$A_0 = [x_k + d, x + d + k - 1].$$

If any $y \in A_0$ has color 1, then $X \cup \{y\}$ forms a monochromatic GQ_{x-2}-progression of low-difference d, and we are done (the difference between x_k and y does not exceed $d + (k+1) - 2$). Hence, we may assume that all members of A_0 have color 0. From this we may assume that all members of $A_1 = [x_k + d + k, x_k + d + 2k - 1]$ have color 1 (or else, for some $z \in A_1$, the set $A_0 \cup \{z\}$ is a monochromatic $(k+1)$-term sequence where the gap between the largest two terms lies in the interval $[1, k]$, thus forming a GQ_{x-2}-progression with low-difference 1). Using this same argument, we may assume that all of A_2 has color 0. Continuing in this way, we are left with the situation in which all members of A_t have color 1 for t odd, and color 0 for t even. However, this gives us the monochromatic set $\{x_k + d + 2tk : 0 \leq t \leq k\}$ (it has color 0), which is a $(k+1)$-term GQ_{x-2}-progression (actually an arithmetic progression) with low-difference $2k$, which completes the proof. $\square$

Theorem 3.19 gives a cubic polynomial in the variable k as an upper bound for $GQ_{x-2}(k)$. In Exercise 3.5, you are asked to establish a quadratic upper bound for $GQ_{x-1}(k)$. We might conjecture (although these two results offer quite limited evidence) that for large enough k, $GQ_{x-m}(k)$ is bounded above by a polynomial of degree $m + 1$. Noting that $GQ_{x-k}(k)$ has the same meaning as $w(k)$, the truth of a conjecture of this type could have exciting ramifications for the van der Waerden numbers themselves.

There are other ways in which we could allow the "diameters" to grow. As one example, we could insist that the first few terms of the progression (say the first three terms) form an arithmetic progression, and then allow the diameters to increase as the terms of the progression increase. Such variations on topics covered in this book can be the seeds of interesting and meaningful research projects (and, we may hope, research projects that inspire further exploration by others).

We end this section with a table of known values of $GQ_{f(x)}(k)$ for some particular functions $f(x)$. Notice that the classical 2-color van der Waerden numbers lie along a diagonal in the table.

$f(x) \setminus k$	3	4	5	6	7	8	9	10
x	6	10	14	20	27	34	44	52
$x-1$	9	19	33	52	74	100	?	?
$x-2$	9	22	38	60	?	?	?	?
$x-3$	9	35	59	≥ 88	?	?	?	?
$x-4$	9	35	178	?	?	?	?	?

Table 3.2: Values and lower bounds for $GQ_{f(x)}(k)$

3.3. Descending Waves

In this section we consider another type of sequence that generalizes the notion of an arithmetic progression, called a *descending wave*. We begin with the definition.

Definition 3.20. For $k \geq 3$, an increasing sequence of positive integers $\{x_1, x_2, \ldots, x_k\}$ is a *descending wave* if $x_i - x_{i-1} \leq x_{i-1} - x_{i-2}$ for $i = 3, 4, \ldots, k$.

As examples, the sequence $\{1, 4, 6, 8, 9\}$ is a 5-term descending wave, while $\{1, 5, 10\}$ fails to be a descending wave. We will denote by $DW(k)$ the least positive integer such that for every 2-coloring of $[1, DW(k)]$ there is a monochromatic k-term descending wave. It is obvious that every arithmetic progression is a descending wave, and therefore, by van der Waerden's theorem, $DW(k)$ exists for all k. There does not appear to be any obvious implication between the property of being a descending wave and that of being a quasi-progression. Clearly, not every quasi-progression of diameter $n \geq 1$ is a descending wave; and conversely, for any given $n \geq 1$, it is easy to find a descending wave of length $k \geq 3$ that is not a quasi-progression of diameter n. For example, if $n = 2$, then $\{1, 8, 12\}$ is a descending wave, but is not a quasi-progression of diameter 2. Although here we will confine the discussion of descending waves to the existence and magnitude of the function $DW(k)$, we mention that it is known that any infinite set of positive integers that contains arbitrarily long quasi-progressions of diameter d, for some fixed d, must also contain arbitrarily long descending waves.

We are able to give both upper and lower bounds for $DW(k)$. We see from the bounds given by Theorems 3.21 and 3.24 (below) that $DW(k)$ grows like a polynomial of degree three.

We start with a simple upper bound.

Theorem 3.21. *For all $k \geq 3$, $DW(k) \leq \frac{k^3}{2} - \frac{k^2}{2} + 1$.*

Proof. Let $n = \frac{k^3}{2} - \frac{k^2}{2} + 1$. We will show that for any 2-coloring $\chi : [1, n] \to \{0, 1\}$ there is a monochromatic k-term descending wave. Assume, without loss of generality, that $\chi(1) = 0$. If there exist k consecutive integers of color 1, we are done. So, assume that no k consecutive integers of color 1 exist under χ. Let $x_0 = 0$ and $x_1 = 1$ and define, for $i \geq 1$,

$$x_{i+1} = \min\{y : y - x_i \geq x_i - x_{i-1}, \chi(y) = 0\}.$$

We leave it to the reader in Exercise 3.9 to show that $x_1, x_2, \ldots, x_k$ is a monochromatic k-term descending wave and that $x_k \leq n$. $\square$

The best known upper bound is slightly better than that provided by Theorem 3.21. We state this result, without proof, as Theorem 3.22.

Theorem 3.22. *For all $k \geq 3$, $DW(k) \leq \frac{1}{3}k^3 - \frac{4}{3}k + 3$.*

Turning to lower bounds, we have the following result.

Theorem 3.23. *For all $k \geq 3$, $DW(k) \geq k^2 - k + 1$.*

Proof. Let $k \geq 3$ and let χ be the 2-coloring of $[1, k^2 - k]$ defined by the string

$$\underbrace{00\ldots0}_{k-1}\underbrace{11\ldots1}_{k-1}\underbrace{00\ldots0}_{k-2}\underbrace{11\ldots1}_{k-2}\cdots\underbrace{00}_{2}\underbrace{11}_{2}\underbrace{0}_{1}\underbrace{1}_{1}.$$

We will show that there is no k-term descending wave with color 0. The proof that there is also none with color 1 is similar and is left as Exercise 3.10. Assume, for a contradiction, that

$$X = \{x_1 < x_2 < \cdots < x_k\}$$

is a descending wave with color 0.

Notice that χ includes exactly $k-1$ different blocks of consecutive 0's – call these blocks $B_1, B_2, \ldots, B_{k-1}$, where $|B_i| = k - i$. By the pigeonhole principle, some block contains more than one member of X. Let t be the least integer such that $|B_t \cap X| \geq 2$. Let

$$X' = X \cap (B_1 \cup B_2 \cup \cdots \cup B_t).$$

Since for $i < t$, B_i contains at most one member of X, we have

$$|X'| \leq (t - 1) + k - t = k - 1.$$

Hence, $X' = \{x_1 < x_2 < \cdots < x_r\}$ with $x_{r-1}, x_r \in B_t$ and $r \leq k-1$. Therefore,

$$x_{r+1} - x_r \geq k - t + 1 > x_r - x_{r-1},$$

contradicting our assumption that X is a descending wave. □

We now state, without proof, the best known lower bound, which shows, in part, that the asymptotic rate of growth of $DW(k)$ is that of a cubic polynomial.

Theorem 3.24. *For all $k \geq 3$, there exists a positive constant c such that $DW(k) \geq ck^3$.*

3.4. Semi-progressions

Recall that a quasi-progression of diameter n is a sequence $\{x_i\}_{i=1}^{k}$ such that, for some $d \in \mathbb{Z}^+$, we have $x_i - x_{i-1} \in [d, d+n]$ for $2 \leq i \leq k$. We might think of this as a "loosening" of the property of being an arithmetic progression, by allowing the gaps between consecutive terms "some slack." Another way to generalize the notion of an arithmetic progression is to allow the gaps to vary by *multiples* of some d, rather than by *additions* to d. With this idea in mind, we give the following definition.

Definition 3.25. For $k, m \in \mathbb{Z}^+$, a k-term *semi-progression of scope* m is a sequence of positive integers $\{x_1, x_2, \ldots, x_k\}$ such that, for some $d \in \mathbb{Z}^+$, $x_i - x_{i-1} \in \{d, 2d, \ldots, md\}$ for all i, $2 \leq i \leq k$.

We shall use the notation $SP_m(k)$ for the corresponding Ramsey-type function. That is, we denote by $SP_m(k)$ the least positive integer such that for every 2-coloring of $[1, SP_m(k)]$ there is a monochromatic k-term semi-progression of scope m. Since for each positive integer m, any arithmetic progression is also a semi-progression of scope m, the collection of semi-progressions of scope m is a superset of AP. Hence, $SP_m(k)$ exists for all m and k. In fact, the following observation is an immediate consequence of the definition of a semi-progression.

Proposition 3.26. *For all $k \geq 1$, $w(k) = SP_1(k) \geq SP_2(k) \geq \cdots$.*

One of the motivations for studying the Ramsey properties of supersets of AP is the potential for gaining more information about $w(k)$. In this regard, the functions $SP_m(k)$ may be a more relevant extension of the the van der Waerden numbers than are the functions $Q_m(k)$. Our reason for saying this has to do with the following definition.

Definition 3.27. For $m, k \in \mathbb{Z}^+$, define $\Gamma_m(k)$ to be the least positive integer s such that for every s-element set $S = \{x_1, x_2, \ldots, x_s\}$ with $x_i - x_{i-1} \in \{1, 2, \ldots, m\}$ for $2 \leq i \leq s$, there is a k-term arithmetic progression in S.

To illustrate this definition, we look at some examples.

Example 3.28. Let us find $\Gamma_2(3)$. We want the least s such that whenever $X = \{x_1, x_2, \ldots, x_s\}$ is a sequence with $x_i - x_{i-1} \in \{1, 2\}$ for each i, then X will contain a 3-term arithmetic progression. Does $s = 3$ work? No – the set $\{1, 2, 4\}$ is a 3-term sequence, with the property that the gaps between consecutive terms belong to $\{1, 2\}$, yet there is no 3-term arithmetic progression. Does $s = 4$ work? No – consider $\{1, 2, 4, 5\}$, whose gaps all lie in $\{1, 2\}$, but which also fails to have a 3-term arithmetic progression. As it turns out, $\Gamma_2(3) = 5$. To prove this, by a simple translation, it is sufficient to consider all 5-term increasing sequences whose first element is 1, and where each of the four gaps is either 1 or 2. There are then only $2^4 = 16$ possible sequences to check. We leave it to the reader to check that each of these sixteen sequences contains some 3-term arithmetic progression.

Example 3.29. Consider $\Gamma_1(k)$. Here we want to look at sets of consecutive integers, and ask for the size of such sets that would guarantee a k-term arithmetic progression. It is easy to see that we have $\Gamma_1(k) = k$, since any set of consecutive integers is itself an arithmetic progression.

The above examples prompt an obvious question: does $\Gamma_m(k)$ always exist? This question is answered, in the affirmative, by the following proposition.

Proposition 3.30. $\Gamma_r(k) \leq w(k; r)$ *for all $k, r \in \mathbb{Z}^+$.*

Proof. Let $w = w(k;r)$ and let $X = \{x_i : 1 \le i \le w\}$ be a sequence of positive integers with $1 \le x_i - x_{i-1} \le r$ for $i = 2, 3, \ldots, w$. We wish to show that X contains a k-term arithmetic progression. As noted in Example 3.28, we may assume $x_1 = 1$.

Consider the coloring $\chi : [1, w] \to \{1, 2, \ldots, r\}$ defined by

$$\chi(y) = j \text{ if and only if } j = \min\{x_i - y : x_i \ge y, x_i \in X\}.$$

To see that χ is an r-coloring, note that no consecutive elements of X differ by more than r. Hence, since χ is an r-coloring of $[1, w]$, by the definition of $w(k;r)$ there is a monochromatic arithmetic progression $A = \{a + nd : 0 \le n \le k-1\}$.

Say the color of A is j_0. This tells us that for each $a + nd \in A$, $0 \le n \le k-1$, there is some $x_{i_n} \in X$ such that $x_{i_n} - (a + nd) = j_0$. That is, $\{x_{i_n} : 0 \le n \le k-1\} = \{(j_0 + a) + nd : 0 \le n \le k-1\}$, so that X contains a k-term arithmetic progression, as desired. □

Now that we have established the existence of $\Gamma_m(k)$, we are able to explain the significance of the function $SP_m(k)$ as it relates to the search for an upper bound on $w(k)$. The explanation is simply this:

Proposition 3.31. *For all $k, m \ge 0$, $w(k) \le SP_m(\Gamma_m(k))$.*

This inequality holds because any 2-coloring of $[1, SP_m(\Gamma_m(k))]$ must contain a monochromatic semi-progression of scope m having $\Gamma_m(k)$ terms. By the definition of Γ_m, among these $\Gamma_m(k)$ terms there must be a k-term arithmetic progression.

We next present some results concerning the magnitude of the function $SP_m(k)$. We begin with a simple formula for $SP_m(k)$ when $k \le m$ (we leave the proof as Exercise 3.11).

Theorem 3.32. *If $k \le m$, then $SP_m(k) = 2k - 1$.*

As made evident by Proposition 3.26, the lower the value of m, the more significant (and probably the more difficult to obtain) any upper bounds on $SP_m(k)$ will be. In the following theorem we give a nice upper bound for $SP_m(k)$, but only provided m is more than half the value of k.

Theorem 3.33. *Let $m \ge 2$. Assume $m < k < 2m$. Let $c = \left\lceil \frac{m}{2m-k} \right\rceil$. Then $SP_m(k) \le 2c(k-1) + 1$.*

Proof. Let $\ell = 2c(k-1) + 1$ and let $\chi : [1, \ell] \to \{0, 1\}$ be any 2-coloring. We will show that under χ there is a monochromatic k-term semi-progression of scope m. It is clear that among the $2(k-1)+1$ elements of $[1, \ell]$ that are congruent to 1 modulo c, there is a set A with $|A| \ge k$ such that A is monochromatic. Assume $\chi(A) = 1$, and let $X = \{x_1 < x_2 < \cdots < x_k\}$ consist of the k smallest members of A.

Let $d_i = x_i - x_{i-1}$ for $2 \le i \le k$. If $d_i \le cm$ for $i \in [2, k]$, then since, by the definition of A, $d_i \in \{c, 2c, \ldots, mc\}$ for each i, X is the desired semi-progression of scope m. Thus, we may assume that there exists $j \in \{2, 3, \ldots, k\}$ such that $d_j = cs$ with $s \ge m + 1$.

Now let $S = \{x_{j-1} + ci : 1 \le i \le s - 1\}$. Then $\chi(S) = 0$ and $|S| \ge m$. We consider two cases.

Case 1. $c \ge 3$. Notice that if $x_{j-1} \le m - c$ and $x_j \ge \ell - (m-c) + 1$, then since $x_k - x_1 \le \ell - 1$, this would imply that

$$\sum_{\substack{i=2\\ i\neq j}}^{k} d_i \le (\ell - 1) - (\ell - 2m + 2c + 1) = 2m - 2c - 2,$$

which would contradict the fact that $\sum_{i \ne j} d_i \ge c(k-2)$ (since $c \ge 2$ and $m < k$). Hence, either $x_{j-1} > m - c$ or $x_j < \ell - (m-c) + 1$. We shall cover the case in which $x_{j-1} > m - c$; the case in which $x_j < \ell - (m-c) + 1$ may be done by a symmetric argument.

So, assume that $x_{j-1} > m - c$ and let

$$B = \{b \not\equiv 1 (\text{mod } c) : x_{j-1} - m + c \le b < x_j\}.$$

For every $b \in B$, there is some $t \in S$ so that $|t - b| \le \max\{c, m\} = m$. Hence, if there is a set $B_0 \subseteq B$ such that $|B_0| = k - m$ and $\chi(B_0) = 0$, then $S \cup B_0$ is a monochromatic semi-progression of scope m with length at least k, and we are done (it is a semi-progression of scope m since the gaps between adjacent members all belong to $\{1, 2, \ldots, m\}$). Thus, we may assume that at most $k - m - 1$ members of B have color 0. Let $Y = \{y_1 < y_2 < \cdots < y_n\}$ be those members of B having color 1. To finish the proof of Case 1, we will show that Y provides us with the monochromatic semi-progression we are seeking by showing that the following two statements are true:

$$(3.5) \qquad y_i - y_{i-1} \leq m \text{ for } 2 \leq i \leq n$$

and

$$(3.6) \qquad n \geq k.$$

For convenience, let $r = k - m$. To prove (3.5), first list the elements of B in increasing order: $b_1 < b_2 < \cdots$. Notice that for any two elements, b_h and b_{h+g}, of B,

$$b_{h+g} - b_h \leq g + \left\lceil \frac{g}{c-1} \right\rceil.$$

Hence, because $|B - Y| \leq k - m - 1 = r - 1$, between any two consecutive y_i's there are at most $r - 1$ elements from $B - Y$. Thus, for all i we have that for some h, $y_i - y_{i-1} \leq b_{h+r} - b_h$, and therefore

$$(3.7) \qquad y_i - y_{i-1} \leq r + \left\lceil \frac{r}{c-1} \right\rceil = \left\lceil \frac{rc}{c-1} \right\rceil.$$

Also, we note that $c \geq \frac{m}{2m-k}$ implies that

$$(3.8) \qquad \left\lceil \frac{rc}{c-1} \right\rceil \leq m.$$

By (3.7) and (3.8), we see that (3.5) is true.

To establish (3.6), we first observe that $|B| \geq (c-1)(m + \lceil \frac{m}{c} \rceil)$. Since $|B - Y| \leq r - 1$, (3.6) will follow if we can prove that

$$(3.9) \qquad (c-2)m + (c-1)\left\lceil \frac{m}{c} \right\rceil + 1 \geq 2r.$$

Since $m > r$, (3.9) is obviously true for $c \geq 4$. To finish Case 1, we will establish (3.9) for $c = 3$.

Using (3.8),

$$\begin{aligned} m + 2\left\lceil \frac{m}{3} \right\rceil + 1 &\geq \left\lceil \frac{3r}{2} \right\rceil + 2\left\lceil \frac{\lceil \frac{3r}{2} \rceil}{3} \right\rceil + 1 \\ &\geq \frac{3r}{2} + 2\left\lceil \frac{r}{2} \right\rceil + 1 \\ &\geq 2r, \end{aligned}$$

which completes the proof of Case 1.

Case 2. $c = 2$. We have $x_k - x_1 \leq \ell - 1 = 4(k-1)$ and $3m \geq 2k$. Therefore, if $i \neq j$, then

$$(3.10) \qquad x_i - x_{i-1} \leq m,$$

for otherwise we would have

$$\begin{aligned} x_k - x_1 &= x_j - x_{j-1} + \sum_{i \neq j}(x_i - x_{i-1}) \\ &\geq 2(m+1) + (m+1) + 2(k-3) \\ &= 3m + 2k - 3 \\ &\geq 4k - 3. \end{aligned}$$

Let $B' = \{x_{j-1} + 2i + 1 : 0 \leq i \leq s - 1\}$. Clearly, for every $b \in B'$, there is a $t \in S$ such that $|t - b| \leq 1$, so that, as in Case 1 for the set B, we may assume that at most $k - m - 1 = r - 1$ members of B' have color 0. Let $Y = \{y_1 < y_2 < \cdots < y_u\} = \{b \in B' : \chi(b) = 1\}$. So $u \geq s - (r-1)$. If for some $i \in \{2, 3, \ldots, u\}$ we have $y_i - y_{i-1} > 2r$, then

$$x_j - x_{j-1} > 2r + 2(s - r - 1) + 2 = 2s,$$

a contradiction. Thus, for each $i \in \{2, 3, \ldots, u\}$,

$$(3.11) \qquad y_i - y_{i-1} \leq 2r.$$

By the same reasoning, the following hold:

$$(3.12) \qquad x_j - y_u \leq 2r,$$

$$(3.13) \qquad y_1 - x_{j-1} \leq 2r.$$

By (3.10)–(3.13) and the fact that $m \geq 2r$ (since $3m \geq 2k$), $X \cup Y$ is a semi-progression of scope m. Since $X \cup Y$ is monochromatic and has at least k terms, the proof is complete. □

It is interesting to note that the proof of Theorem 3.33 actually gives a stronger result. Namely, it shows that for every 2-coloring of $[1, 2c(k-1)+1]$ (where c, k, and m are as in Theorem 3.33), there is a monochromatic sequence $\{x_1, x_2, \ldots, x_k\}$ such that either

$$x_i - x_{i-1} \in \{1, 2, \ldots, m\} \text{ for } i = 2, 3, \ldots, k;$$

or

$$x_i - x_{i-1} \in \{c, 2c, \ldots, mc\} \text{ for } i = 2, 3, \ldots, k.$$

That is, the value of d, from the definition of a semi-progression of scope m, belongs to the set $\{1, c\}$. It would be interesting to try to find similar results where $\{1, c\}$ is replaced by some other set.

Example 3.34. For a given m, the "best" (i.e., the largest) value of k for which Theorem 3.33 provides an upper bound is $k = 2m - 1$. For this case, the theorem gives

$$SP_m(2m-1) \leq 2c(2m-2)+1 = 4(m^2-m)+1.$$

Theorem 3.32 gives the somewhat trivial result $SP_m(m) = 2m-1$. Precise formulae for Ramsey-type functions in less trivial cases are desirable, but typically somewhat difficult to come by. In the case of the function $SP_m(k)$, letting $k = m+1$, an exact formula is known, which we now present.

Theorem 3.35. *Let $m \geq 2$. Then*

$$SP_m(m+1) = \begin{cases} 4m+1 & \text{if } m \text{ is even,} \\ 4m-1 & \text{if } m \text{ is odd.} \end{cases}$$

Proof. First note that setting $k = m+1$ in Theorem 3.33 yields $SP_m(m+1) \leq 4m+1$. We next establish that $SP_m(m+1) \leq 4m-1$ if m is odd. To do this, let $m \geq 3$ be odd and let $\chi : [1, 4m-1] \to \{0, 1\}$ be any 2-coloring. From Proposition 2.30, it follows from Theorem 3.32 that in the interval $[m+1, 3m-1]$ there is a monochromatic set $X = \{x_1, x_2, \ldots, x_m\}$ with $x_i - x_{i-1} \in \{1, 2, \ldots, m\}$ for $2 \leq i \leq m$. We may assume that $\chi(X) = 1$. We shall consider two cases.

`Case 1.` $x_i - x_{i-1} > 1$ for some $i \in \{2, 3, \ldots, m\}$. Let j be the least value of i such that $x_i - x_{i-1} > 1$. If $\chi(x_{j-1}+1) = 1$, then $X \cup \{x_{j-1}+1\}$ forms an $(m+1)$-term monochromatic semi-progression of scope m, and we are done. So we shall assume that $\chi(x_{j-1}+1) = 0$. Similarly, we may assume that each of $x_1 - 1, x_1 - 2, \ldots, x_1 - m$ has color 0. Then the set

$$A = [x_1 - m, x_1 - 1] \cup \{x_{j-1}+1\} \subseteq [1, 4m-1]$$

is monochromatic. Also, since

$$(x_{j-1}+1) - (x_1 - 1) = j \leq m,$$

A is a semi-progression of scope m.

`Case 2.` $x_i - x_{i-1} = 1$ for all $i \in \{2, 3, \ldots, m\}$. If any member of $Y = [x_1 - m, x_1 - 1] \cup [x_m + 1, x_m + m]$ has color 1, then clearly we have a monochromatic $(m+1)$-term semi-progression of scope m in $[1, 4m-1]$. So we may assume that $\chi(Y) = 0$. Now let

$$B = \left\{x_1 - (2i-1) : 1 \leq i \leq \tfrac{m+1}{2}\right\} \bigcup \left\{x_m + (2i-1) : 1 \leq i \leq \tfrac{m+1}{2}\right\}.$$

Then B consists of $m+1$ elements, has color 0 (since it is a subset of Y), and each pair of consecutive elements of B has gap 2 or $m+1$. Hence, since $m+1$ is even and $m+1 \leq 2m$, B is a monochromatic $(m+1)$-term semi-progression of scope m.

We have thus far established that $4m-1$ and $4m+1$ serve as upper bounds for their respective cases. To complete the proof of the theorem, we need to show that they also serve as lower bounds. To do so we shall exhibit, for m odd, a 2-coloring of $[1, 4m-2]$ that avoids $(m+1)$-term monochromatic semi-progressions of scope m; and, for m even, a 2-coloring of $[1, 4m]$ that avoids such progressions.

Consider m odd. Color $[1, 4m-2]$ with the coloring α defined by the string

$$\underbrace{11\ldots1}_{m-1}\underbrace{00\ldots0}_{m}\underbrace{11\ldots1}_{m}\underbrace{00\ldots0}_{m-1}.$$

Let $C = \{x_1 < x_2 < \cdots < x_\ell\}$ be a maximum length monochromatic semi-progression of scope m. By the symmetry of α, we may assume $\alpha(C) = 1$. Let $d = \min\{x_i - x_{i-1} : 2 \leq i \leq \ell\}$. If $d = 1$, then $x_i - x_{i-1} \leq m$ for each i. For this case, it is evident from the way α is defined that $\ell \leq m$. If $d \geq 2$, then each of the two blocks of 1's in the representation of α contains at most $\frac{m}{2}$ members of C, so again $\ell \leq m$. Hence, in all cases, α admits no monochromatic $(m+1)$-term semi-progressions of scope m.

Now assume m is even. Essentially the same reasoning as that used in the case of m odd shows that the coloring of $[1, 4m]$ defined by the string

$$\underbrace{11\ldots1}_{m}\underbrace{00\ldots0}_{m}\underbrace{11\ldots1}_{m}\underbrace{00\ldots0}_{m}$$

has no monochromatic $(m+1)$-term semi-progression of scope m. We leave the details to the reader as Exercise 3.12. □

We now derive a lower bound for $SP_m(k)$ that holds for all m and k.

Theorem 3.36. *Let $k \geq 2$ and $m \geq 1$. Let $\lambda(k,m) = \left\lceil \frac{k-1}{\lceil k/m \rceil} \right\rceil$. Then*

$$SP_m(k) \geq 2(k-1)\left(\left\lceil \frac{k}{\lambda(k,m)} \right\rceil - 1\right) + 1.$$

Before giving the proof of Theorem 3.36, we note that for m fixed, the theorem gives us the asymptotic lower bound

$$SP_m(k) \geq \frac{2}{m}k^2(1+o(1)).$$

Proof of Theorem 3.36. Let k and m be given, let $\lambda = \lambda(k,m)$, and let $n = 2(k-1)\left(\lceil \frac{k}{\lambda} \rceil - 1\right)$. To prove the theorem we exhibit a specific coloring of $[1,n]$ that avoids monochromatic k-term semi-progressions of scope m. We define this coloring by a string of 1's and 0's as follows. Let A represent a block of $k-1$ consecutive 1's and B a block of $k-1$ consecutive 0's. Color $[1,n]$ with $ABAB\ldots AB$, where A and B each occur $\lceil \frac{k}{\lambda} \rceil - 1$ times (so there is a total of $2\left(\lceil \frac{k}{\lambda} \rceil - 1\right)$ blocks).

Assume, for a contradiction, that $X = \{x_1, x_2, \ldots, x_k\}$ is a monochromatic semi-progression of scope m that is contained in $[1,n]$. Let $d \in \mathbb{Z}^+$ be such that $x_i - x_{i-1} \in \{d, 2d, \ldots, md\}$ for $i = 2, 3, \ldots, k$. Since each of the blocks A and B has length $k-1$, and since X is monochromatic, there is some i such that $x_i - x_{i-1} \geq k$. This implies that $d \geq \lceil \frac{k}{m} \rceil$, and hence each block contains at most $\lceil \frac{k-1}{d} \rceil \leq \lambda$ members of X. Thus, for each color, there are no more than $\lambda\left(\lceil \frac{k}{\lambda} \rceil - 1\right)$ members of X having that color. Since $\lambda\left(\lceil \frac{k}{\lambda} \rceil - 1\right) \leq k-1$ (this is true for any positive integers k, λ), we have a contradiction, and the proof is complete. □

We conclude this section with a table of values and bounds of $SP_m(k)$ for small m and k, preceded by an exact formula for $SP_m(k)$ under the restriction that $k \leq \frac{3m}{2}$. A proof is outlined in Exercise 3.13.

Theorem 3.37. *Let $m+2 \leq k \leq \frac{3m}{2}$. Then $SP_m(k) = 4k-3$.*

$m \setminus k$	2	3	4	5	6	7	8	9	10	11
1	3	9	35	178	?	?	?	?	?	?
2	3	9	17	33	55	≥ 87	≥ 125	≥ 177	?	?
3	3	5	11	19	31	71	97	≥ 117	?	?
4	3	5	7	17	21	35	44	65	≥ 75	≥ 84
5	3	5	7	9	19	25	33	49	56	≥ 69
6	3	5	7	9	11	25	29	33	55	61
7	3	5	7	9	11	13	27	33	37	47
8	3	5	7	9	11	13	15	33	37	41

Table 3.3: Values and lower bounds for $SP_m(k)$

3.5. Iterated Polynomials

By the n^{th} *iteration* of the function $f(x)$, we mean the composite function $\underbrace{f(f(\ldots(f(x))\ldots))}_{n \text{ times}}$, often denoted $f^{(n)}(x)$. One way of thinking of an arithmetic progression $S = \{a, a+d, a+2d, \ldots, a+(k-1)d\}$ is as a sequence that results from $(k-1)$ iterations of the function $f(x) = x + d$; that is, $S = \{a, f(a), f^{(2)}(a), \ldots, f^{(k-1)}(a)\}$. Looking at arithmetic progressions in this fashion leads us to some natural ways of forming supersets of AP.

We begin with a definition.

Definition 3.38. A p_n-*sequence* is an increasing sequence of positive integers $\{x_1, x_2, \ldots, x_k\}$ such that there exists a polynomial $p(x)$ of degree n, with integer coefficients, so that $p(x_i) = x_{i+1}$ for $i = 1, 2, \ldots, k-1$.

We shall call the polynomial p in Definition 3.38 a p_n-*function* and say that $p(x)$ *generates* $\{x_1, x_2, \ldots, x_k\}$.

It will be convenient to adopt the following notation.

Notation. Denote by $P_{n,k}$ the family of all p_n-sequences of length k, and by P_n the family of all p_n-sequences (regardless of their length), i.e., $P_n = \bigcup_{k=1}^{\infty} P_{n,k}$.

Thus, for example, the sequence $\{1, 3, 7, 15\}$ is a member of $P_{1,4}$ because it is generated by $p(x) = 2x + 1$, a polynomial of degree one. Of course, it is also a member of P_1. Similarly, each arithmetic progression belongs to P_1, since for some d it is generated by $x+d$; that is, $AP \subseteq P_1$.

One easy way to produce p_n-sequences is to begin with any positive integer a, and any polynomial p with integer coefficients, and list the consecutive iterates of a by p; as long as this list is increasing, we have found a p_n-sequence. As one example, let $p(x) = x^2 - 2x + 3$ and $a = 1$. Since p is increasing on the interval $[1, \infty)$, then for each positive integer i, $\{1, 2, 3, 6, 27, \ldots, p^{(i)}(1)\}$ is a p_2-sequence and is a member of $P_{2,i+1}$.

We shall give an explicit upper bound, in terms of k, for the Ramsey-type function $R(AP \cup P_{k-2}, k; 2)$ (i.e., we will want every 2-coloring to produce a monochromatic k-term sequence that is either a p_{k-2}-sequence or is an arithmetic progression). We will also show (in Theorem 3.51, below) that if this bound could be improved to a certain other (somewhat slower growing) function of k, then $w(k)$ would be bounded above by a similar function.

We will sometimes want to consider the family of polynomials having degree not exceeding a specified number n. For this reason, we introduce the following notation:

$$S_n = \bigcup_{i=1}^{n} P_i, \text{ and } S_{n,k} = \bigcup_{i=1}^{n} P_{i,k}.$$

Thus, for example, $S_{3,5}$ consists of all 5-term increasing sequences of positive integers that can be obtained by the iteration of some linear, quadratic, or cubic polynomial having integer coefficients.

Before getting to the main results on p_n-sequences, we need some lemmas. The first is a rather interesting fact in its own right: if X is an increasing k-term sequence that is generated by a polynomial with integer coefficients, then there exists a polynomial with integer coefficients and degree not exceeding $k-2$ that also generates X; i.e., every k-term sequence that is a member of P_n for some n is also a member of S_{k-2}.

Lemma 3.39. *Let $k \geq 3$. Then $\bigcup_{i=1}^{\infty} P_{i,k} = S_{k-2,k}$.*

Proof. By the definition of $S_{k-2,k}$, we need only show that for all $k \geq 3$ and $n \geq 1$, every k-term p_n-sequence is a member of $S_{k-2,k}$.

Let $X = \{x_1, x_2, \ldots, x_k\}$ be generated by the polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$, with $a_i \in \mathbb{Z}$ for each i and $a_n \neq 0$. To complete the proof we must show that there exist integers $b_0, b_1, \ldots, b_{k-2} \in \mathbb{Z}$ satisfying the system

$$\begin{aligned} b_0 + b_1 x_1 + \cdots + b_{k-2} x_1^{k-2} &= x_2 \\ b_0 + b_1 x_2 + \cdots + b_{k-2} x_2^{k-2} &= x_3 \\ \vdots \qquad\qquad & \quad \vdots \\ b_0 + b_1 x_{k-1} + \cdots + b_{k-2} x_{k-1}^{k-2} &= x_k. \end{aligned} \tag{3.14}$$

We see that (3.14) is a system of $k-1$ equations in the $k-1$ variables $b_0, \ldots, b_{k-2}$. The determinant of the coefficient matrix is the well-known Vandermonde determinant, known to equal

$$\prod_{1 \leq \ell < m \leq k-1} (x_m - x_\ell).$$

Since the x_i's are distinct, this determinant is nonzero. By Cramer's rule, system (3.14) has the unique solution $(b_0', b_1', \ldots, b_{k-2}')$, where for each j, $0 \leq j \leq k-2$,

$$b_j' = \frac{D_j}{\prod_{1 \leq \ell < m \leq k-1} (x_m - x_\ell)},$$

where D_j is the determinant of the matrix

$$\begin{pmatrix} 1 & x_1 & \ldots & x_1^{j-1} & x_2 & x_1^{j+1} & \ldots & x_1^{k-2} \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ \cdot & & & & & & & \cdot \\ 1 & x_{k-1} & \ldots & x_{k-1}^{j-1} & x_k & x_{k-1}^{j+1} & \ldots & x_{k-1}^{k-2} \end{pmatrix}.$$

The proof is completed by showing that each b_j' is an integer. The proof of this fact requires a bit of abstract algebra, and is left to the exercises (see Exercise 3.16). $\square$

Remark 3.40. It is worth noting from the above proof that for every sequence of positive integers generated by a polynomial with integer coefficients, there exists a *unique* polynomial of degree at most $k-2$ that generates the sequence.

The next lemma provides a useful characterization of the p_1-sequences.

Lemma 3.41. *A sequence is a p_1-sequence of length $k \geq 3$ if and only if it has the form*

$$\left\{t, t+d, t+d+ad, t+d+ad+a^2d, \ldots, t+d\sum_{i=0}^{k-2} a^i\right\}$$

for some positive integers t, d, a.

Proof. Let S be a set of the described form. It is easy to see that the function $f(x) = ax + t(1-a)d$ generates S. Hence S is a p_1-sequence.

Conversely, assume that $X = \{x_1, x_2, \ldots, x_k\}$ is an increasing sequence of positive integers for which there exist $a, b \in \mathbb{Z}$ such that the function $f(x) = ax + b$ generates X. Let $t = x_1$ and $d = x_2 - x_1$. Then, since $ax_1 + b = x_2$ and $ax_2 + b = x_3$,

$$a = \frac{x_3 - x_2}{x_2 - x_1} = \frac{x_3 - t - d}{d}. \tag{3.15}$$

Therefore $x_2 = t + d$ and $x_3 = t + d(1 + a)$. Now assume that $n \geq 3$ and that for $j \in \{n-1, n\}$, $x_j = t + d\sum_{i=0}^{j-2} a^i$. We shall complete the proof via induction on n by showing that $x_{n+1} = t + d\sum_{i=0}^{n-1} a^i$.

Note that

$$\begin{aligned} a &= \frac{x_{n+1} - x_n}{x_n - x_{n-1}} \\ &= \frac{x_{n+1} - \left(t + d\sum_{i=0}^{n-2} a^i\right)}{\left(t + d\sum_{i=0}^{n-2} a^i\right) - \left(t + d\sum_{i=0}^{n-3} a^i\right)}. \end{aligned}$$

Therefore,

$$\begin{aligned} x_{n+1} &= a\left(d\sum_{i=0}^{n-2} a^i - d\sum_{i=0}^{n-3} a^i\right) + t + d\sum_{i=0}^{n-2} a^i \\ &= ada^{n-2} + t + d\sum_{i=0}^{n-2} a^i \\ &= t + d\sum_{i=0}^{n-1} a^i, \end{aligned}$$

and the proof is complete. □

According to Lemma 3.41, every p_1-sequence is completely determined by its first three terms. Extending this idea to an arbitrary 3-term sequence, we can pose the following "riddle": if x, y, and z are the first three numbers of a sequence, what is the the fourth number? Mathematically, the idea is very elementary. Using (3.15), letting $a = \frac{z-y}{y-x}$ and $b = y - ax$, we have a method for getting from one number in the sequence to the next: multiply by a and then add b. For example, given the sequence $-\frac{1}{9}, -\frac{1}{3}, \frac{5}{9}$, then by taking $a = -4$ and $b = -\frac{7}{9}$, we may answer the riddle by saying that the fourth number is $(-4)\frac{5}{9} - \frac{7}{9} = -3$. This solution to the riddle works for any three numbers x, y, z (even complex numbers) provided $x \neq y$.

The following two technical lemmas concerning the growth of iterated polynomials are stated without proof.

Lemma 3.42. *Let $\{x_1 < x_2 < \cdots < x_k\}$ be a p_{k-2}-sequence, generated by the p_{k-2}-function $g(x)$. Then*

$$g(x_k) \leq x_k + \prod_{i=1}^{k-1}(x_k - x_i).$$

Example 3.43. The sequence $\{1, 2, 3, 6\}$ is generated by the p_2-function $f(x) = x^2 - 2x + 3$. Hence, $f(6) \leq 6 + 3 \cdot 4 \cdot 5 = 126$ (in fact, $f(6) = 27$).

Lemma 3.44. *Let $k \geq 3$ and let $\{x_1, x_2, \ldots, x_k\} \in S_{k-2}$ be generated by the p_n-function $g(x)$ of degree $n \leq k-2$. Then for each nonnegative* *integer j, the sequence*

$$\{x_1, x_2, \dots, x_k, g(x_k) + j \prod_{i=1}^{k-1}(x_k - x_i)\}$$

belongs to S_{k-1} and is generated by the function $h(x) = \sum_{i=1}^{k-1} b_i x^i$, where $b_{k-1} = j$. Furthermore, all members of S_{k-1} may be obtained from members of S_{k-2} in this way.

Remark 3.45. Note that, using the notation of Lemma 3.44, the lemma tells us that for $j \geq 1$, the sequence

$$\{x_1, x_2, \dots, x_k, g(x_k) + j \prod_{i=1}^{k-1}(x_k - x_{i-1})\}$$

belongs to P_{k-1}.

To help clarify what Lemma 3.44 is saying, we consider two examples.

Example 3.46. Using the notation of Lemma 3.44, if $j = 0$, then $\{x_1, x_2, \dots, x_k, g(x_k)\}$ is generated by $g(x)$, so it is, of course, a member of S_{k-2} (and hence of S_{k-1}, where we may take $h(x) = g(x)$, with $b_{k-1} = 0$).

Example 3.47. Consider $\{1, 2, 3\}$, a p_1-sequence generated by the polynomial $g(x) = x + 1$. Lemma 3.44 tells us that each sequence $\{1, 2, 3, 4 + 2j\}$, where $j \geq 0$, is a member of S_2, and, for $j \geq 1$, is generated by a p_2-function whose leading coefficient is j. For instance, according to Example 3.43, $\{1, 2, 3, 6\}$ is generated by the p_2-function $x^2 - 2x + 3$. In turn, we can build 5-term p_3-sequences from $\{1, 2, 3, 6\}$: each sequence $\{1, 2, 3, 6, 27 + j(5 \cdot 4 \cdot 3)\}$, $j \geq 1$, is a member of P_3, generated by a cubic polynomial with leading coefficient j. Of more significance, perhaps, is what the last sentence of the lemma says: that *every* p_3-sequence of length 5 may be found in this way – by starting with the 4-term members of S_2.

Remark 3.48. Extending the previous example, notice that if $X = \{x_1, x_2, \dots, x_k\}$ is any member of $S_{k-2} - AP$ whose first $k - 1$ terms form an arithmetic progression $\{x_1 + id : 0 \leq i \leq k - 2\}$, then X is generated by a member of P_{k-2}, i.e., $j \neq 0$. Furthermore, $x_k = x_1 + (k-1)d + j(d)(2d) \cdots ((k-2)d) = x_1 + (k-1)d + jd^{k-2}(k-2)!$.

We now are able to give an upper bound for $R(AP \cup P_{k-2})$.

Theorem 3.49. *For all $k \geq 5$, $R(AP \cup P_{k-2}, k) \leq k^{\frac{(k-2)!}{2}}$.*

Proof. The inequality holds for $k = 5$, since it is known that $R(AP \cup P_3, 5) = 85$ (see Table 3.4 at the end of this section). We proceed by induction on k, letting $k \geq 6$ and assuming the theorem is true for $k - 1$. Let χ be any 2-coloring of $[1, k^{(k-2)!/2}]$, using the colors red and blue. By the induction hypothesis there exists a monochromatic $(k-1)$-term sequence $X \in AP \cup P_{k-3}$ contained in $[1, (k-1)^{(k-3)!/2}]$. Say $X = \{x_1 < x_2 < \cdots < x_{k-1}\}$ has color red. By Remark 3.40, there is a unique polynomial $f \in S_{k-3}$ that generates X. Hence, by Remark 3.45, for each $j = 1, 2, \dots, k$,

$$\left\{x_1, x_2, \dots, x_{k-1}, f(x_{k-1}) + j \prod_{i=1}^{k-2}(x_{k-1} - x_i)\right\} \in P_{k-2}. \tag{3.16}$$

For ease of notation, let us denote $\prod_{i=1}^{k-2}(x_{k-1} - x_i)$ by Π. Now let $A = \{f(x_{k-1}) + j\Pi : 1 \leq j \leq k\} = \{y_1 < y_2 < \cdots < y_k\}$. We wish to show that each member of A belongs to $[1, k^{(k-2)!/2}]$. To show this, note that by Lemma 3.42,

$$\begin{aligned} y_k &\leq x_{k-1} + (k+1)\Pi \leq x_{k-1} + (k+1)x_{k-1}^{k-2} \\ &\leq (k-1)^{(k-3)!/2} + (k+1)(k-1)^{(k-2)!/2}. \end{aligned} \tag{3.17}$$

That the expression (3.17) does not exceed $k^{(k-2)!/2}$ will follow if

$$\frac{(k-1)^{(k-3)!/2}}{(k-1)^{(k-2)!/2}} + (k+1) \leq \left(\frac{k}{k-1}\right)^{\frac{(k-2)!}{2}} = \left(1 + \frac{1}{(k-1)}\right)^{\frac{(k-2)!}{2}}.$$

This last inequality does indeed hold, since

$$k + 2 \leq \left(1 + \frac{1}{k-1}\right)^{(k-2)!/2}. \tag{3.18}$$

We leave the justification for (3.18) to the reader (hint: consider the binomial expansion of the right-hand side of the inequality). Thus, we have established that $A \subseteq [1, k^{(k-2)!/2}]$.

We consider two possibilities. If every member of A has color blue, then A is a monochromatic arithmetic progression. Otherwise, some member y_m of A has color red, in which case, by (3.16), $X \cup \{y_m\}$ is a red p_{k-2}-sequence. In either case we have a monochromatic k-term member of $AP \cup P_{k-2}$ that lies in $[1, k^{(k-2)!/2}]$. This completes the proof. $\square$

The upper bound given by Theorem 3.49, although it may be an interesting result, seems to involve a much larger collection of sequences than does AP; therefore, there is no obvious reason to think that having an upper bound on $AP \cup P_{k-2}$ could help us to find a reasonable upper bound on $w(k) = R(AP, k)$. However, in the next theorem we show that if we were able to establish a certain improved upper bound on $R(AP \cup P_{k-2}, k)$, then we would also have a similar bound for $w(k)$.

We first prove the following lemma.

Lemma 3.50. *Let $k \geq 7$. If $\{x_1, x_2, \ldots, x_k\} \in S_{k-2} - AP$, then $x_k \geq 2^{k-1}$.*

Proof. Let $\{x_1, x_2, \ldots, x_k\} \in S_{k-2} - AP$, and let j be the largest integer such that $\{x_1, x_2, \ldots, x_j\}$ is an arithmetic progression (clearly, $2 \leq j \leq k-1$).

If $j = k-1$, then by Remark 3.48, $x_k \geq k + (k-2)! \geq 2^{k-1}$. If $j < k-1$, then by Lemma 3.44, $x_{j+1} \geq x_j + (x_j - x_{j-1}) + \prod_{i=1}^{j-1}(x_j - x_i)$. Now, x_{j-1}, x_j, x_{j+1} is a p_1-sequence and hence is generated by the polynomial $f(x) = ax + b$, where

$$\begin{aligned} a &= \frac{x_{j+1} - x_j}{x_j - x_{j-1}} \\ &\geq \frac{x_j - x_{j-1} + \prod_{i=1}^{j-1}(x_j - x_i)}{x_j - x_{j-1}} \\ &= 1 + \prod_{i=1}^{j-2}(x_j - x_i) \\ &\geq 1 + (j-1)!. \end{aligned}$$

Since $\{x_{j-1}, x_j, x_{j+1}, x_{j+2}\} \in S_2$, by Lemma 3.44 $x_{j+2} \geq f(x_{j+1})$ and hence, by Lemma 3.41,

$$x_{j+2} \geq x_j + 1 + (j-1)! + (1 + (j-1)!)^2.$$

Applying this same argument to each of the triples $\{x_\ell, x_{\ell+1}, x_{\ell+2}\}$, $j-1 \leq \ell \leq k-3$, we obtain

$$x_k \geq j + \sum_{i=1}^{k-j}[1 + (j-1)!]^i.$$

That this last expression is no less than 2^{k-1} is left as Exercise 3.18, which completes the proof. $\square$

Theorem 3.51. *Let $\tau_k = 2^{k-2} + \prod_{i=1}^{k-3}(2^{k-3} - 2^{i-1})$. If $k \geq 7$ and $R(AP \cup P_{k-2}, k) \leq \tau_k - 1$, then $w(k-1) \leq \tau_k - 2 = 2^{k^2-4k}(1 + o(1))$.*

Proof. To prove the result, we will show that, for all $k \geq 7$, if $X = \{x_1 < x_2 < \cdots < x_k\}$ is any k-term p_{k-2}-sequence such that $\{x_1, x_2, \ldots, x_{k-1}\}$ is not an arithmetic progression, then $x_k \geq \tau_k$. From this it follows by Lemma 3.44 that all k-term members of $AP \cup P_{k-2}$ that are contained in $[1, \tau_k - 1]$ are formed by adding $j \prod_{i=1}^{k-2}(x_{k-1} - x_i)$ to the k^{th} term of some arithmetic progression, for some nonnegative integer j. Hence, by the hypothesis, it follows that $w(k-1) \leq \tau_k - 2$, which is the desired conclusion.

Thus, we shall show that whenever X is a k-term p_{k-2}-sequence whose first $k-1$ terms do not form an arithmetic progression, then $x_k \geq \tau_k$. For the case of $k = 7$, we ask the reader to prove this by inspection of all 7-term members of $AP \cup P_5$, with the help of Lemma 3.44. Now let $k \geq 8$. If $\{x_1, x_2, \ldots, x_{k-1}\}$ is generated by the function $g(x) = 2x$, then $x_{k-1} \geq 2^{k-2}$, so that $x_k \geq \tau_k$ by Lemma 3.44.

If $\{x_1, x_2, \ldots, x_{k-1}\}$ is generated by a function other than $2x$, then by Lemma 3.50, there is a least integer $j \leq k-1$ such that $x_j > 2^{j-1}$. Also, by Lemmas 3.44 and 3.50, if $i < j$, then

$$x_{k-1} - x_i \geq 2^{k-2} - 2^{i-1}.$$

If $j \leq i \leq k-2$, then since $x_j - x_{j-1} > 2^{j-1} - 2^{j-2} = 2^{j-2}$, and since $\{x_{i-1}, x_i, x_{i+1}\}$ is a p_1-sequence, but not an arithmetic progression,

$x_{i+1} - x_i \geq 2(x_i - x_{i-1})$, and again we have

$$x_{k-1} - x_i \geq \sum_{\ell=i-1}^{k-3} 2^\ell = 2^{k-2} - 2^{i-1}.$$

Now let h be the p_{k-3}-function that generates $\{x_1, x_2, \ldots, x_{k-1}\}$. Then by Lemma 3.50, $h(x_{k-1}) \geq 2^{k-1}$. Since X is a p_{k-2}-sequence (but not a p_{k-3}-sequence), we know by Lemma 3.44 that

$$x_k \geq h(x_{k-1}) + \prod_{i=1}^{k-2}(x_{k-1} - x_i) > \tau_k,$$

completing the proof. □

Other results that are similar to Theorem 3.51 have been found, where obtaining a seemingly mild improvement over the known bound on the Ramsey function corresponding to the superset of AP will lead to a like bound on $w(k)$. References to such work are mentioned in Section 3.9.

We may also consider Ramsey functions involving iterated polynomials where the number of colors, r, is greater than two. As one example, the next theorem gives an upper bound for $R(P_1, 3; r)$ (hence an upper bound for $R(S_1, 3; r)$).

Theorem 3.52. *$R(P_1, 3; r) \leq r!r^2 + r$ for $r \geq 2$.*

Proof. Direct calculation gives $R(P_1, 3; 2) = 7$, so that the theorem holds for $r = 2$. Now assume $r \geq 3$, and let χ be any r-coloring of $[1, r!r^2 + r + 1]$. For $i = 0, 1, \ldots, r^2$, define $M_i = [r!i + 1, r!i + r + 1]$. Clearly, each M_i contains a pair $a_i < b_i$, such that $\chi(a_i) = \chi(b_i)$. For each i, $0 \leq i \leq r^2$, let $h(i)$ be the ordered pair $(a_i - r!i, \chi(i))$. Note that there are a total of only r^2 possible ordered pairs (the set of possible ordered pairs is the same for all i). Thus, by the pigeonhole principle, there exist $i < j$ such that $h(i) = h(j)$. Since $a_j = a_i + r!(j - i)$ and $b_i - a_i \leq r$,

$$a_j = a_i + \lambda(b_i - a_i),$$

where $\lambda \geq 2$. Thus, by Lemma 3.41, $\{a_i, b_i, a_j\}$ is a p_1-sequence. Since $\{a_i, b_i, a_j\}$ is monochromatic, and since $a_j < b_j \leq r!j + r + 1$, the theorem is proved. □

We end this section with a table. Note that by Lemma 3.39, there is no need to include values of $R(S_n, k; r)$ for which $n \geq k - 1$.

n	k	r	$R(S_n, k; r)$	$R(AP \cup P_n, k; r)$
1	3	2	7	7
1	4	2	23	?
2	4	2	20	21
1	5	2	76	76 – 177
2	5	2	72	?
3	5	2	67	85
4	6	2	68 – 6^{12}	192 – 6^{12}
1	3	3	14	14
1	4	3	≥ 71	?
2	4	3	≥ 71	≥ 71
1	3	4	24	24
1	3	5	38	38

Table 3.4: Values and bounds of $R(S_n, k; r)$ and $R(AP \cup P_n, k; r)$

3.6. Arithmetic Progressions as Recurrence Solutions

By a *linear recurrence relation* of order n, we mean an equation of the form

$$x_k = c_1 x_{k-1} + c_2 x_{k-2} + \cdots + c_n x_{k-n},$$

where the c_i's are given constants. Solving such a system means finding a closed formula for x_k that holds for all k. As a simple example, consider the following recurrence of order one: $x_k = 3x_{k-1}$, $k \geq 1$. A "general" solution is $x_k = 3^k x_0$. To get a "particular" solution, it suffices to know the initial value x_0.

An arithmetic progression may be thought of as the solution to the recurrence

$$x_k = 2x_{k-1} - x_{k-2}. \tag{3.19}$$

To get a particular solution, it is sufficient to be given two initial values. For example, if $x_1 = 1$ and $x_2 = 3$, then $x_3 = 2x_2 - x_1 = 5$,

$x_4 = 7$, etc. This is simply another way of looking at the fact that arithmetic progressions $\{x_1, x_2, \dots\}$ are completely determined by knowing x_1 and x_2, since $x_k - x_{k-1} = x_{k-1} - x_{k-2}$ for all k.

Thus we see that the family AP may be considered a special subfamily of the family of those sets that occur as solutions to some linear recurrence. There are many ways that we can generalize (3.19). One way is the following. For $k \geq 3$, consider the family of those sets $\{x_1, x_2, \dots, x_k\}$ with $x_1 < x_2$, having the property that there exists a set $\{a_i \geq 2 : 3 \leq i \leq k\}$ such that for $i = 3, 4, \dots, k$, $x_i = a_i x_{i-1} + (1 - a_i)x_{i-2}$. Let us call this collection of sets R_1. We see that every arithmetic progression belongs to R_1 (let $a_i = 2$ for every i). As another example, let us consider the 6-term sequence obtained by letting $a_3 = 2$, $a_4 = 3$, $a_5 = 2$, $a_6 = 7$, $x_1 = 1$, and $x_2 = 3$. Then we have $x_3 = 2x_2 - x_1$, $x_4 = 3x_3 - 2x_2$, $x_5 = 2x_4 - x_3$, and $x_6 = 7x_5 - 6x_4$, yielding the sequence $\{1, 3, 5, 9, 13, 37\}$, a 6-term member of R_1.

It is not hard to obtain an upper bound on the Ramsey-type function $R(R_1, k)$. We do so in the following theorem.

Theorem 3.53. *Let $k \geq 3$. Then $R(R_1, k) \leq \frac{7}{24}(k+1)!$.*

Proof. We may calculate directly that $R(R_1, 3) = 7$, so that the result is true for $k = 3$. Proceeding by induction, assume that $k \geq 4$, and that $R(R_1, k-1) \leq \frac{7}{24}k!$. Let $m_k = \frac{7}{24}(k+1)!$ and consider an arbitrary 2-coloring $\chi : [1, m_k] \to \{0, 1\}$. By the induction hypothesis, in $[1, m_{k-1}]$ there is a monochromatic $(k-1)$-term member of R_1. Say $X = \{x_1, x_2, \dots, x_{k-1}\}$ is such a sequence and that $\chi(X) = 0$. For $1 \leq i \leq k$, let $y_i = x_{k-1} + i(x_{k-1} - x_{k-2})$. Notice that, for each i, the set $\{x_1, x_2, \dots, x_{k-1}, y_i\}$ is a k-term member of R_1 (where, according to the notation used in the definition of R_1, $a_i = i + 1$). Also notice that every y_i is no greater than m_k, since

$$m_{k-1} + k(m_{k-1} - x_{k-2}) \leq (k+1)m_{k-1} = m_k.$$

To complete the proof, we consider two possibilities. If some y_j, $j \in \{1, 2, \dots, k\}$, has color 0, then $X \cup \{y_j\}$ is a monochromatic member of R_1 that is contained in $[1, m_k]$. Otherwise $\{y_1, y_2, \dots, y_k\}$

is a monochromatic arithmetic progression (hence a member of R_1) contained in $[1, m_k]$, and the proof is complete. □

There are many other generalizations of (3.19) that may be considered. For example, instead of having $b_i = 1 - a_i$, we can simply require that $b_i = -1$ for all i. Some examples are given in the exercises. We encourage the reader to experiment with other generalizations – it seems likely that there are some interesting ones that have yet to be explored.

3.7. Exercises

3.1 a) Find all 3-term quasi-progressions with diameter 1 that are contained in [1,6]. How many are contained in [1,10]? How many are contained in $[1, m]$?
b) How many 3-term quasi-progressions with diameter k are contained in $[1, m]$?

3.2 Prove Corollary 3.11(i).

3.3 Complete the details of Case 3 in the proof of Theorem 3.12.

3.4 Prove Corollary 3.15.

3.5 Show that $\frac{k(k+1)}{2}$ is an upper bound for $GQ_{x-1}(k)$ (see Example 3.18).

3.6 Show that the last sentence in the statement of Theorem 3.19 follows from the previous sentence.

3.7 Calculate $DW(3)$ and $DW(4)$.

3.8 How many descending waves of length three are contained in $[1, 10]$? in $[1, m]$?

3.9 Finish the proof of Theorem 3.21

3.10 Finish the proof of Theorem 3.23.

3.11 Prove Theorem 3.32.

3.12 Complete the proof of Theorem 3.35 by showing that for m even, the string $\underbrace{11\dots1}_{m}\underbrace{00\dots0}_{m}\underbrace{11\dots1}_{m}\underbrace{00\dots0}_{m}$ avoids monochromatic $(m+1)$-term semi-progressions of scope m.

3.13 This exercise provides the outline for a proof of Theorem 3.37.

a) Assume $m+2 \leq k \leq \frac{3m}{2}$. Show that if k is even, then the 2-coloring $11\underbrace{00\ldots0}_{k-2}\underbrace{11\ldots1}_{k-2}\underbrace{00\ldots0}_{k-2}\underbrace{11\ldots1}_{k-2}$ avoids monochromatic k-term semi-progressions of scope m. Hence, for k even, $SP_m(k) \geq 4k-3$.
b) Use Theorem 3.36 to show that the lower bound of (a) also holds for k odd.
c) Use the results of (a) and (b), along with Theorem 3.33, to prove Theorem 3.37, i.e., that $SP_m(k) = 4k-3$.

3.14 For i an integer, $0 \leq i \leq k-2$, let $\lambda(k,m,i) = \left\lceil \frac{k-i-1}{\lceil (k-i)/m \rceil} \right\rceil$. Prove that

$$SP_m(k) \geq 2(k-i-1)\left(\left\lceil \frac{k}{\lambda(k,m,i)} \right\rceil - 1\right) + 1.$$

3.15 a) Find specific values of m, k, and i for which the lower bound on $SP_m(k)$ given by Exercise 3.14 is better (greater) than that given by Theorem 3.36.
b) Find specific values of m and k for which the lower bound provided by Theorem 3.37 is better than any obtainable by Exercise 3.14.

3.16 Complete the proof of Lemma 3.39 by showing that each b_j' is an integer. (*May require abstract algebra.*)

3.17 Let $d \geq 1$. Use Lemma 3.41 to show that $X = \{x_1, x_2, \ldots, x_k\}$ is a member of P_1 if and only if $dX = \{dx_1, dx_2, \ldots, dx_k\}$ is a member of P_1.

3.18 Complete the proof of Lemma 3.50 by proving that, for $2 \leq j \leq k-2$ and $k \geq 7$,

$$j + \sum_{i=1}^{k-j} [1 + (j-1)!]^i \geq 2^{k-1}.$$

(Hint: verify that the inequality holds whenever $j=2$ and use $n! > (\frac{n}{e})^n$ to show that it is true for $j \geq 7$; then, show that it holds for each pair (j,k) in the set $\{(3,7),(4,7),(5,7),(6,8)\}$, and then use induction on k (where j is fixed).)

3.19 Define a C-sequence to be a sequence of distinct (but not necessarily increasing) positive integers $\{x_1, x_2, \ldots, x_k\}$ such that there exists a function $f(x) = ax+b$, with $a \in \mathbb{Z}^+$, $b \in \mathbb{Z}$, where for each $i = 2, 3, \ldots, k$, $f(x_{i-1}) = x_i$. Obviously, every p_1-sequence is a C-sequence. Prove that a sequence S is a C-sequence if and only if

$$S = \left\{ t+d, t+d+ad, t+d+ad+a^2d, \ldots, t+d\sum_{i=0}^{k-2} a^i \right\}$$

for some $t, a, d \in \mathbb{Z}$, where $t, a \geq 1$ and $d \neq 0$.

3.20 Define an E-sequence of length k to be an increasing sequence of positive integers generated by a function $p(x) = ax+b$, where $a \geq 1$ and $b \geq 0$ are integers. It is clear that every arithmetic progression is an E-sequence, and that every E-sequence is a p_1-sequence. Prove that (x_1, x_2, x_3) is an E-sequence if and only if it has the form $\{x, x+d, x+d+ad\}$, where $a \leq \frac{d}{x}+1$.

3.21 Denote by $N(AP,k,m)$ the number of different k-term arithmetic progressions contained in $[1,m]$. Define $N(P_1,k,m)$ and $N(C,k,m)$ analogously, replacing the family of arithmetic progressions with p_1-sequences and C-sequences, respectively (see Exercise 3.19). Prove that

$$N(C,k,m) = 2N(P_1,k,m) - N(AP,k,m).$$

3.22 Define a D-sequence to be any sequence of distinct positive integers of the form $\{t, f(t), f^{(2)}(t), \ldots, f^{(k-1)}(t)\}$, where $f(x) = ax+b$, $a, b \in \mathbb{Z}$. It is clear that every C-sequence (see Exercise 3.19) is a D-sequence.
a) Prove that every D-sequence of length three is also a C-sequence.
b) Prove that a set of size $k \geq 4$ is a D-sequence but not a C-sequence if and only if it has the form

$$\left\{ t, t+d, t+d-ad, t+d-ad+a^2d, \ldots, t+d\sum_{i=0}^{k-2}(-1)^i a^i \right\},$$

where $a, d \in \mathbb{Z}$, $a \geq 2$, and $d \neq 0$.

c) Using Exercise 3.21's notation, prove that for all $k \geq 4$,
$$N(D,k,m) = N(C,k,m) + 2\sum_{a=2}^{j}\sum_{d=1}^{s}(m - a^{k-2}d),$$
where j is the greatest integer such that $a^{k-2} \leq m-1$ and $s = \left\lfloor \frac{m-1}{a^{k-2}} \right\rfloor$.

3.23 Define R_2 to be the same as R_1 (as in Theorem 3.53), except instead of using the recurrence $x_i = a_i x_{i-1} - b_i x_{i-2}$, where $b_i = 1 - a_i$, we now use the recurrence $x_i = a_i x_{i-1} - x_{i-2}$ for all i. Notice that R_2 is a superset of AP. Prove that $R(R_2, k) \leq \frac{1}{3}(k+1)!$ for $k \geq 3$.

3.24 As in Exercise 3.23, define R_3 to differ from R_1 only in that we require that $a_i = 2$ and $b_i \leq -1$ for all i. Find an upper bound on $R(R_3, k)$.

3.8. Research Problems

3.1 Let $B_2(k)$ be the least positive integer with the following property: for every 2-coloring of $[1, B_2(k)]$, there is a monochromatic sequence $\{x_1, x_2, \ldots, x_k\}$ such that $x_i - x_{i-1} \in \{d_1, d_2\}$, $2 \leq i \leq k$, for some $d_1, d_2 \in \mathbb{Z}^+$. Write a computer program to calculate $B_2(k)$. Try to get bounds for $B_2(k)$.
References: **[58]**, **[163]**, **[167]**

3.2 Repeat Research Problem 3.1 above, except consider $B_m(k)$, the Ramsey-type function for those sequences such that for some m-element set $D = \{d_1, d_2, \ldots, d_m\}$, the gaps must belong to D. Try to determine relationships that exist between $B_m(k)$ and the Ramsey-type functions discussed in this chapter. Write a computer program to calculate specific values of $B_m(k)$. Try to obtain bounds for $B_m(k)$.
References: **[58]**, **[163]**, **[167]**

3.3 It has been conjectured that $Q_{k-i}(k) = 2ik - 4i + 3$ if $k \not\equiv 1 \pmod{i}$ and $k \geq 2i \geq 2$. Prove or disprove this conjecture. As a special case, determine if $Q_{k/2}(k) = k^2 - 2k + 3$, for k even. Table 3.1 provides some known values of this function. One possible approach to solving this problem may come from the following conjecture (based on some known computer output): if $k \equiv 0 \pmod{i}$ and $k > i$, then the only valid coloring of maximal length (i.e., of length $2ik - 4k + 2$) is the coloring $A_1 \underbrace{B \ldots B}_{i-1} A_0$, where A_1 is a string of 1's having length $k - i$, A_0 is a string of 0's having length $k - i$, and $B = \underbrace{00\ldots0}_{k-1}\underbrace{11\ldots1}_{k-1}$.
References: **[58]**, **[167]**

3.4 It has been conjectured that $Q_{k-i}(k) = 2ik - 2i + 1$ if $k \equiv 1 \pmod{i}$ and $k \geq 2i \geq 1$. Prove or disprove this conjecture. Some specific values are given in Table 3.1. As in Research Problem 3.3, computer output suggests a rather regular pattern in the valid colorings of maximal length.
References: **[58]**, **[167]**

3.5 Denote by r^j a string of length rj consisting of j monochromatic blocks, each of length r, where the colors of the blocks alternate. For example, 3^4 represents the string 000111000111 or the string 111000111000, and 2^5 represents a string such as 1100110011. Prove or disprove the following conjecture: Whenever $k \geq 2i$ and $k \equiv 2 \pmod{i}$, the coloring represented by $(k-2)^{i-1}(k-1)^2(k-2)^{i-1}$ is a maximal valid coloring. A proof of this would support the conjecture of Research Problem 3.3.
References: **[58]**, **[167]**

$*$ **3.6** Determine a relationship between $w(k)$ and $Q_1(k)$ asymptotically (i.e., what happens to the ratio as $k \to \infty$?).
References: **[58]**, **[167]**

3.7 It was mentioned after the proof of Theorem 3.33 that its proof establishes an upper bound for the Ramsey function associated with semi-progressions of scope m, where the value of d (as in the definition of semi-progression) is restricted to the set $D = \{1, c\}$. Try replacing the set D with various other sets, and attempt to find upper bounds. Perhaps a more general result can be obtained, where the case of $D = \{1, c\}$ would be one special case.
Reference: **[163]**

3.8 Obtain an upper bound on $GQ_{x-3}(k)$. In particular, can we bound it above by a polynomial of degree four? Attempt to generalize this to $GQ_{x-t}(k)$.
References: [**82**], [**170**]

3.9 Denote by $SP_m(k;r)$ the least positive integer M such that for every r-coloring of $[1,M]$ there will be a monochromatic k-term semi-progression of scope m. As in the case of two colors, it is clear that $w(k;r) \leq SP_m(\Gamma_m(k);r)$. Study the function $SP_m(k;r)$ for values of r greater than two.
Reference: [**163**]

3.10 In Table 3.3, we observe that for $m \in \{3,4,5,6\}$,

$$SP_m(2m) = 6(2m-1) + \epsilon,$$

where $\epsilon = 1$ or 2. Determine if $SP_m(2m) = 12m(1+o(1))$.
Reference: [**163**]

3.11 We see from Table 3.3 that, for the pairs $(m,k) = (2,5)$ and $(m,k) = (4,9)$, $SP_m(k) = 8(k-1)+1$. Determine if this is true for the pair $(m,k) = (6,13)$ or, more generally, for $(m,k) = (2j, 4j+1)$.
Reference: [**163**]

3.12 Let us say a set of positive integers S has property QP if, for some fixed n, S contains a k-term quasi-progression of diameter n for every $k \geq 1$ (that, is S contains arbitrarily long quasi-progressions of diameter n).
a) Determine if the set of squares $\{i^2 : i = 1,2,\dots\}$ has property QP.
b) It is known that there are infinitely many 4-term quasi-progressions of diameter 1 among the set of squares. Determine if there exist any such progressions of length five. Is so, how many?
References: [**58**], [**61**], [**167**]

3.13 Improve on the lower bound given by Theorem 3.9.
Reference: [**163**]

3.14 As in Theorem 3.12, obtain an exact formula for $Q_{k-3}(k)$.
Reference: [**163**]

3.15 Improve on Theorem 3.14 by obtaining an upper bound on $Q_n(n+t)$ when t is a number that is greater than $\frac{n}{2}$.
Reference: [**163**]

$*$ **3.16** Attempt to find the precise rate of growth of $DW(k)$; i.e., find a such that $DW(k) = ak^3(1+o(1))$.
References: [**15**], [**58**], [**179**]

3.17 Find bounds on $DW(k;r)$ for $r \geq 3$.
References: [**15**], [**58**], [**179**]

3.18 Improve upon Table 3.4 (either by adding new entries, or by improving any of the bounds).
References: [**131**], [**171**], [**172**]

3.19 Find an upper bound for $R(C,k;r)$ (see Exercise 3.19). In particular, find an upper bound for $R(C,3;r)$ that is less than that of Theorem 3.52. Also, the following are known: $R(C,3;2) = 5$, $R(C,4;2) = 20$, $R(C,3;3) = 13$, $R(C,3;4) = 17$, $R(C,4;3) \geq 28$ and $R(C,5;2) \geq 53$. Find more of these values; in particular, find $R(C,5;2)$.
References: [**131**], [**161**], [**162**], [**169**], [**171**], [**172**], [**177**]

3.20 Referring to Exercise 3.20, find upper and/or lower bounds for the Ramsey-type function $R(E,k;2)$ associated with k-term E-sequences. Find a lower bound for $R(E,3;r)$ (it is obvious that $w(k;r) \geq R(E,k;r) \geq R(P_1,k;r)$). It is known that $R(E,3;2) = 9$, $R(E,4;2) = 35$, $R(E,3;3) = 26$, and $62 \leq R(E,5;2) \leq 177$. Find more exact values; in particular, find $R(E,5;2)$.
References: [**131**], [**161**], [**162**], [**169**], [**171**], [**172**], [**177**]

3.21 Repeat Research Problem 3.19, using D in place of C (see Exercise 3.22). Also, the following values and bounds are known: $R(D,3;2) = 5$, $R(D,4;2) = 19$, $R(D,3;3) = 13$, $R(D,3;4) = 17$, $R(D,4;3) \geq 28$, and $52 \leq R(D,5;2) \leq 177$. Find more exact values; in particular, find $R(D,5;2)$.
References: [**131**], [**161**], [**162**], [**169**], [**171**], [**172**], [**177**]

3.22 Look for other ways to define supersets of AP, and attempt to obtain non-trivial upper and/or lower bounds on the corresponding Ramsey-type function.

3.23 Calculate values and attempt to get bounds on any function analogous to the Erdős and Turán function, but where the family AP is replaced by one of the families discussed in this chapter. If any bound is obtained, use it to try to get a bound on the corresponding Ramsey-type function.

3.9. References

§**3.1**. The bounds on quasi-progressions are from [**167**], which also includes a proof of Lemma 3.13, as well as a table of computer-generated data. Brown, Erdős, and Freedman [**58**] study the question of which sets contain arbitrarily long quasi-progressions, and the relationship between this property and several other properties, including that of containing arbitrarily long descending waves. They also prove the equivalence of two properties, one a conjecture of Erdős' (a Ramsey-type statement involving infinite reciprocal sums; see Research Problem 2.3) and the other involving quasi-progressions. Other results on infinite reciprocal sums can be found in [**59**] and [**73**]. Brown, Freedman, and Shiue [**61**] consider the problem of finding quasi-progressions that are contained in the set of squares.

§**3.2**. The notion of generalized quasi-progressions is introduced in [**170**], which contains a proof of Theorem 3.19, some related results, and Table 3.2. Functions similar to $GQ_\delta(k)$ are considered in [**82**].

§**3.3**. Proofs of Theorems 3.21 and 3.24 may be found in [**15**]. A proof of Theorem 3.22 can be found in [**58**]. Some results concerning the notion of an ascending wave (the gaps increase rather than decrease) are also discussed in [**15**] and [**58**]. We can generalize descending waves to $DW(k,\ell)$, the least positive integer d such that $[1,d]$ contains either an ascending wave of length k or a descending wave of length ℓ. See work by Erdős and Szekeres [**101**] and Lefmann [**179**] for more information. The fact that any infinite set of positive integers that contains arbitrary long quasi-progressions must also contain arbitrarily long descending waves is given in [**58**], where it is also shown that the reverse implication is false.

§**3.4**. The proof of Proposition 3.30 is from [**195**], which also has other related results. Rabung provides another proof of the existence

of $\Gamma_m(k)$ [**209**]. Further results on $\Gamma_m(k)$ are given in [**63**]. Theorems 3.32, 3.33, and 3.35–3.37 are from [**163**].

§**3.5**. Lemmas 3.42 and 3.44 are from [**162**]. A slightly weaker result than that of Theorem 3.49 appears in [**131**], which also contains proofs of Lemma 3.50 and Theorem 3.51. In [**168**], results like Theorem 3.51 are given for certain families of sequences that are properly contained in $AP \cup P_{k-2}$. Theorem 3.52 is from [**161**]. Other results involving Ramsey functions based on polynomial iteration appear in [**172**]. The values and bounds in Table 3.4 are from [**131**], [**171**], and [**172**]. Some properties on the growth of iterated polynomials, irrespective of Ramsey theory, are explored in [**177**]. Work on p_n-sequences involving more than two colors is considered in [**169**] and [**171**].

§**3.6**. Theorem 3.53 and related work may be found in [**166**].

§**3.7**. Exercise 3.5 is from [**170**]. Exercise 3.14 is proven in [**163**]. Exercises 3.19–3.22 are taken from [**162**]. Exercises 3.23 and 3.24 are from [**166**].

Chapter 4

Subsets of AP

In Chapter 3 we considered functions analogous to $w(k;r)$ by replacing AP, the collection of arithmetic progressions, with certain collections $\mathcal{F}$ such that $AP \subseteq \mathcal{F}$. The existence of the corresponding van der Waerden-type functions $R(\mathcal{F},k;r)$ was guaranteed by van der Waerden's theorem, since each arithmetic progression is a member of $\mathcal{F}$. The purpose of this chapter is to consider the reverse situation, $\mathcal{F} \subseteq AP$. Thus, we wish to restrict, in some way, the allowable arithmetic progressions.

We note that if $\mathcal{F}$ is a proper subset of AP, then van der Waerden's theorem does not guarantee the existence of $R(\mathcal{F},k;r)$ for all k and r. In fact, any positive result we can uncover about such a function can be considered a strengthening of van der Waerden's theorem. For example, if $R(\mathcal{F},k;r) < m$ for some $\mathcal{F} \subseteq AP$, then $w(k;r) < m$.

As a start, we show that for certain choices of $\mathcal{F}$, k, and r it is relatively easy to conclude that $R(\mathcal{F},k;r)$ does not exist. We begin with some examples.

Example 4.1. Let $\mathcal{F}$ be the collection of all arithmetic progressions $\{a, a+d, a+2d, \ldots\}$ such that $a \geq 1$ and $d \in \{1,2\}$. To show that $R(\mathcal{F},k)$ does not exist, it is enough for us to find a 2-coloring of the positive integers that does not yield any k-term monochromatic members of $\mathcal{F}$. Consider the coloring $\chi : \mathbb{Z}^+ \to \{0,1\}$ represented by the sequence $11001100\ldots$. It is easy to see that there do not exist more than two consecutive numbers with the same color, nor do there exist $x, x+2$ of the same color. Hence, $R(\mathcal{F}, 3)$ does not exist and therefore $R(\mathcal{F}, k)$ does not exist if $k \geq 3$.

Example 4.2. Let $\mathcal{F}$ be the family of all arithmetic progressions $\{a, a+d, a+2d, \dots\}$ such that $a = 2^i$ for some $i \geq 0$. Consider the 2-coloring of $\mathbb{Z}^+$ defined by coloring all the powers of 2 red, and all other positive integers blue. Then obviously there is no blue member of $\mathcal{F}$. Also, if $X = \{x, y, z\}$ is red, where $x < y < z$, then X cannot be an arithmetic progression. This is due to the fact that for all $k \geq 0$, $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$, so that $z - y > y - x$. Hence, $R(\mathcal{F}, k)$ does not exist for $k \geq 3$.

Examples 4.1 and 4.2 may not be very surprising, since the collection $\mathcal{F}$ consists of a rather small part of AP. The next example uses a much larger collection $\mathcal{F}$.

Example 4.3. Let $\mathcal{F}$ be the collection of arithmetic progressions whose gaps are odd. Then the coloring of the positive integers defined by coloring the even numbers red and the odd numbers blue yields no monochromatic 2-term member of $\mathcal{F}$. Thus, $R(\mathcal{F}, 2)$ does not exist.

There are many ways that one may choose a subcollection of AP. Examples 4.1 and 4.3 illustrate one natural way, which is to require that the gap d of the arithmetic progressions belong to some prescribed set of positive integers. This type of subcollection of AP has been the subject of several recent research articles, and many interesting problems remain unanswered. For this reason, we devote most of this chapter to the idea of placing certain restrictions on the gaps that the desired arithmetic progressions may have.

Before proceeding, we introduce the following notation and language.

Notation. For D a set of positive integers, denote by A_D the collection of all arithmetic progressions whose gaps belong to D. We will refer to D as the *gap set*. We refer to an element of A_D as a D-a.p.. Also, if d is a positive integer, by a *d-a.p.* we shall mean an arithmetic progression whose gap is d.

We begin the study of the Ramsey properties of sets of type A_D by considering the case in which D is a finite set.

4.1. Finite Gap Sets

Example 4.1 hints that, for D finite, A_D may not be regular (see Definition 1.28). Indeed, the following theorem shows that this is true, even when only two colors are used.

Theorem 4.4. *For D a finite set of positive integers, there is a k large enough so that $R(A_D, k; 2)$ does not exist.*

Proof. Let $n = \max\{d : d \in D\}$. Define the 2-coloring of $\mathbb{Z}^+$ by the string

$$\underbrace{11\dots1}_{n}\underbrace{00\dots0}_{n}\underbrace{11\dots1}_{n}\underbrace{00\dots0}_{n}\dots.$$

To prove the theorem, we will show that under this coloring there is no monochromatic $(n+1)$-term arithmetic progression whose gap belongs to D. Obviously, any arithmetic progression with gap n alternates color, so there cannot exist even a 2-term monochromatic arithmetic progression with gap n.

Now assume X is an arithmetic progression with gap less than n. Then X cannot have more than n consecutive elements of the same color. Hence $R(A_D, n+1; 2)$ does not exist. □

The proof of Theorem 4.4 shows that $R(A_D, k; 2)$ does not exist if $n = \max\{d : d \in D\}$. However, it does not say that $n+1$ is the minimum value of k such that $R(A_D, k; 2)$ does not exist. So what is the minimum value? The answer seems to depend not only on the size of D, but also on the specific elements of D.

For example, the third van der Waerden number, $w(3)$, is known to equal 9. Obviously, any 3-term arithmetic progression that is contained in $[1, 9]$ will have as its gap an element of $D = \{1, 2, 3, 4\}$. Hence, for this choice of D, $R(A_D, 3) = w(3) = 9$. On the other hand, if $E = \{1, 3, 5, 7\}$, then (as discussed in Example 4.3) the coloring $101010\dots$ avoids 2-term arithmetic progressions with gaps in E; hence $R(A_E, 2) = \infty$.

For D such that $|D| \leq 3$, much is known about $R(A_D, k; 2)$. In particular, a complete answer is known if $|D| = 1$ or 2. We summarize these results in the next three theorems.

Theorem 4.5. *If* $|D| = 1$, *then* $R(A_D, 2; 2) = \infty$.

Proof. Let $D = \{d\}$. Define a 2-coloring χ of $\mathbb{Z}^+$ as follows. First, color $[1, d]$ in any way. Then, for all $x > d$, define $\chi(x)$ such that $\chi(x) \neq \chi(x - d)$. Then χ is a 2-coloring of $\mathbb{Z}^+$ that does not have any 2-term monochromatic arithmetic progressions with gap d. This proves the theorem. □

The next theorem makes use of Lemma 4.6, below, for which we remind the reader of the following notation.

Notation. For D a set of positive integers and t a real number, we write tD to denote $\{td : d \in D\}$.

Lemma 4.6. *Let D be a set of positive integers, and let* $k, t \geq 1$. *Then* $R(A_{tD}, k; r) = t[R(A_D, k; r) - 1] + 1$ *(if* $R(A_D, k; r) = \infty$, *then* $R(A_{tD}, k; r) = \infty$*).*

Proof. Let $m = R(A_D, k; r)$, and assume $m < \infty$. To show that $R(A_{tD}, k; r) \leq t(m-1)+1$, let χ be any r-coloring of $[1, t(m-1)+1]$. Define χ' on $[1, m]$ as follows:

$$\chi'(x) = \chi(t(x-1)+1).$$

By the definition of m, for some $d \in D$, within $[1, m]$ there is a d-a.p., $\{x_i : 1 \leq i \leq k\}$, that is monochromatic under χ'. Then $\{t(x_i - 1) + 1 : 1 \leq i \leq k\}$ is monochromatic under χ and is a k-term td-a.p. Thus, $R(A_{tD}, k; r) \leq t(m-1)+1$.

To obtain the reverse inequality, note that by the definition of m, there exists an r-coloring ϕ of $[1, m-1]$ that avoids monochromatic k-term arithmetic progressions whose gaps belong to D. Define ϕ' on $[1, t(m-1)]$ as follows:

$$\phi'[t(j-1)+1, tj] = \phi(j) \text{ for each } j = 1, 2, \ldots, m-1.$$

Then ϕ' avoids monochromatic k-term arithmetic progressions whose gaps belong to tD. Hence, $R(A_{tD}, k; r) \geq t(m-1)+1$.

It is easy to check that this same line of reasoning takes care of the cases when $R(A_D, k; r) = \infty$. □

We now consider 2-element gap sets.

Theorem 4.7. *Let* $D = \{a, b\}$, *let* $g = \gcd(a, b)$, *and let* $k \geq 2$. *If* $\frac{a}{g}$ *and* $\frac{b}{g}$ *are not both odd and if* $k = 2$, *then* $R(A_D, k) = a + b - g + 1$. *Otherwise,* $R(A_D, k) = \infty$.

Proof. We first observe that it is sufficient to prove the theorem for the situation in which $g = 1$. To see this, assume that the statement of the theorem is true for all 2-element sets whose greatest common divisor equals one. Now let $D = \{a, b\}$ and let $a' = \frac{a}{g}$ and $b' = \frac{b}{g}$ not both be odd. Then $\gcd(a', b') = 1$ and therefore by Lemma 4.6 we have

$$R(A_{\{a,b\}}, 2) = g[R(A_{\{a',b'\}}, 2)-1]+1 = g(a'+b'-1)+1 = a+b-g+1.$$

If both a' and b' are odd, then by our assumption, $R(A_{\{a',b'\}}, 2) = \infty$, so that by Lemma 4.6, $R(A_{\{a,b\}}, 2) = \infty$.

We begin with the case in which $R(A_D, k)$ is finite. We prove $R(A_D, 2) \leq a + b$ by contradiction. Thus, we assume that a and b are not both odd, that $k = 2$, and that there exists a 2-coloring χ of $[1, a+b]$ that yields no monochromatic 2-term arithmetic progression whose gap belongs to $\{a, b\}$.

Let $\oplus$ represent addition modulo $a + b$. Then for all $i \in [1, a+b]$ we have $|i \oplus a - i| \in \{a, b\}$. Hence, by our assumption,

$$\chi(i) \neq \chi(i \oplus a) \text{ for all } i \in [1, a+b]. \tag{4.1}$$

Now,

$$1 \oplus (a+b-1)a \equiv (1-a) \pmod{(a+b)}$$

so that

$$1 \oplus (a+b-1)a \equiv (b+1) \pmod{(a+b)}. \tag{4.2}$$

Also, since $a + b - 1$ is even, by repeated use of (4.1) we have

$$\chi(1 \oplus (a+b-1)a) = \chi(1). \tag{4.3}$$

Combining (4.2) and (4.3), we have $\chi(b+1) = \chi(1)$, contradicting our assumption about χ.

To prove $R(A_D, 2) \geq a + b$, we give a 2-coloring of $[1, a + b - 1]$ that yields no 2-term monochromatic D-a.p. Since $\gcd(a, a + b) = 1$, for each $i \in [1, a + b - 1]$ there exists a unique $j \in [1, a + b - 1]$ such that $i \equiv ja\,(\mathrm{mod}\,(a+b))$. Hence the coloring χ of $[1, a+b-1]$ defined by

$$\chi(i) = \begin{cases} 1 & \text{if } i \equiv ea\,(\mathrm{mod}\ (a+b)) \text{ with } e \text{ even}, \\ 0 & \text{if } i \equiv ua\,(\mathrm{mod}\ (a+b)) \text{ with } u \text{ odd} \end{cases}$$

is a well-defined 2-coloring of $[1, a + b - 1]$.

Now, assume that y, z are both in $[1, a + b - 1]$ with $z = y + a$. If $y \equiv ja\,(\mathrm{mod}\ (a + b))$, then $z \equiv (j + 1)a\,(\mathrm{mod}\ (a + b))$, so that, by the way χ is defined, $\chi(y) \neq \chi(z)$. Also, if $y, z \in [1, a + b - 1]$ with $z = y + b$, then

$$y \equiv z \oplus a\,(\mathrm{mod}\,(a + b)), \tag{4.4}$$

so that $y \equiv ta(\mathrm{mod}\ (a+b))$ where $t \in [2, a+b-1]$ (check that $t \neq 1$). So, from (4.4) and the definition of χ, we have $\chi(y) \neq \chi(z)$. Hence χ is $R(A_D, 2)$-valid on $[1, a + b - 1]$, which completes the proof that $R(A_D, 2) = a + b$.

We now do the cases in which $R(A_D, k)$ is infinite. The case in which a and b are both odd is covered by Example 4.3. Now assume one of a and b is even (they cannot both be even). It remains only to show that $R(A_{\{a,b\}}, 3) = \infty$. We may assume that a is even.

To complete the proof, we shall exhibit a 2-coloring γ of $\mathbb{Z}^+$ that has period $2a$ and that has no monochromatic 3-term arithmetic progression with gap in D.

If $b \equiv i\,(\mathrm{mod}\,2a)$ with $a \leq i < 2a$, then, by the periodicity of γ, if $X = \{x, x+b, x+2b\}$ were monochromatic under γ, then for any $t > 0$ with $2ta > b$, the set $\{x+4ta, x+2ta+b, x+2b\}$ would be a monochromatic arithmetic progression with gap $2ta - b \equiv -i\,(\mathrm{mod}\,2a)$. Hence, it is sufficient for us to consider only those b such that $b \equiv i\,(\mathrm{mod}\,2a)$, where $1 \leq i \leq a$. Also, since γ has period $2a$, and since b is odd (why?), we may assume $1 \leq b < a$.

We consider the following two cases, and define the coloring γ (with period $2a$) according to the case. In each case we let B_j denote the interval $[(j - 1)a + 1, ja]$ for $j \geq 1$.

`Case 1.` $1 \leq b < \frac{a}{2}$. For each $i \leq a$, let $\gamma(i) = 1$ if i is odd, and $\gamma(i) = 0$ if i is even; and for each $i > a$, let $\gamma(i) \neq \gamma(i-a)$. Then there is no 2-term monochromatic a-a.p., and γ has period $2a$. If $\{x, x+b\}$ is monochromatic with $x \in B_j$, then by the way γ is defined, since b is odd, $x+b \in B_{j+1}$. Since $x+b \leq ja+\frac{a}{2}$, we have $x+2b \in B_{j+1}$, so that $\gamma(x + 2b) \neq \gamma(x + b)$. Thus there is no 3-term monochromatic b-a.p. in $\mathbb{Z}^+$, and γ has no monochromatic 3-term arithmetic progressions with gap in D.

`Case 2.` $\frac{a}{2} \leq b < a$. Define γ as follows. For every j, let $\gamma(B_\ell) = 1$ for ℓ odd, and $\gamma(B_\ell) = 0$ for ℓ even. Clearly, there is no monochromatic 2-term a-a.p. Also, if $\{x, x + b\}$ is monochromatic, with $x \in B_j$, then $x+b$ must belong to B_j. Thus, $x+2b \in B_{j+1}$ (or else $2b < a$), so that $\gamma(x + 2b) \neq \gamma(x)$. So γ yields no monochromatic 3-term arithmetic progressions with gap in D. □

If we look over the proof of Theorem 4.7, we see that we have proved something stronger than $R(A_{\{a,b\}}, 3) = \infty$: in certain cases in which $|D| = \{a, b\}$, it is possible to 2-color the positive integers so that there are neither monochromatic 2-term a-a.p.'s nor monochromatic 3-term b-a.p.'s. In fact, it is possible to prove several results about this type of "mixed" van der Waerden-type function. You will find some interesting results on this in the exercises of this chapter.

The last proof was rather long for a situation in which the structure seems quite uncomplicated and small. If we up the ante to $|D| = 3$, then, as you may have guessed, we do not have a theorem akin to Theorem 4.7 – but we can say something.

Theorem 4.8. *Let $|D| = 3$. Then $R(A_D, 4) = \infty$.*

Proof. Let $D = \{a, b, c\}$ and assume $a < b < c$. The proof splits naturally into two cases.

`Case 1.` $2b \leq c$. In this case let χ be the 2-coloring of $\mathbb{Z}^+$ defined recursively as follows:

(i) $\chi(i) = 1$ for $1 \leq i \leq a$,

(ii) $\chi(i) \neq \chi(i - a)$ for $a < i \leq b$,

(iii) $\chi(i) \neq \chi(i - b)$ for $b < i \leq c$,

(iv) $\chi(i) \neq \chi(i - c)$ for $i > c$.

Note that χ has period $2c$. We will show that under χ there is no monochromatic 4-term arithmetic progression whose gap belongs to D.

Let $j = \lceil \frac{c}{b} \rceil$. For $1 \le i < j$, let $B_i = [(i-1)b+1, ib]$, and let $B_j = [(j-1)b+1, c]$. Clearly, in each B_i, $1 \le i \le j$, there is no monochromatic 2-term a-a.p. Therefore there is no 3-term a-a.p. in $[1, c]$. Notice that there are no monochromatic 2-term a-a.p.'s in $[1, 2a]$, and therefore, for each nonnegative integer k, there also cannot be any in $[kc+1, kc+2a]$. Thus, in $\mathbb{Z}^+$ there is no monochromatic 4-term a-a.p. By the same reasoning, there is no monochromatic 3-term b-a.p. in $\mathbb{Z}^+$ (check this). Finally, it is clear from (iv) that there is no monochromatic 2-term c-a.p.

Case 2. $2b > c$. Let χ' be the following 2-coloring of $\mathbb{Z}^+$ (note that χ' has period $4c$):

(i) $\chi'(i) = 1$ for $1 \le i \le a$,

(ii) $\chi'(i) \ne \chi'(i-a)$ for $a < i \le b$,

(iii) $\chi'(i) \ne \chi'(i-b)$ for $b < i \le 2c$,

(iv) $\chi'(i) \ne \chi'(i-2c)$ for $i > 2c$.

Let $d = 2c$. Then in the same way that we used $2b \le c$ in Case 1, we may use $2b \le d$ to show that under χ' there is no monochromatic 4-term a-a.p., no monochromatic 3-term b-a.p., and no monochromatic 2-term d-a.p. (we leave the details as Exercise 4.3). Since there is no monochromatic 2-term d-a.p., there cannot be a monochromatic 3-term c-a.p.

In each case, we have given a coloring of $\mathbb{Z}^+$ that yields no monochromatic 4-term arithmetic progression whose gap belongs to D, proving the theorem. □

Based on the results obtained above, let us denote by $m(n)$ the least positive integer m such that whenever D is a set of positive integers with $|D| = n$ we have $R(A_D, m) = \infty$. The above theorems show that $m(1) = 2$, $m(2) = 3$, and $3 \le m(3) \le 4$. It would be nice to know more about the function $m(k)$ (see Section 4.4 for some open problems).

4.2. Infinite Gap Sets

Van der Waerden's theorem says that for all k and r, $R(A_{\mathbb{Z}^+}, k; r)$ exists (is finite). If we can find a proper subset D of $\mathbb{Z}^+$ such that $R(A_D, k; r)$ exists for all k and r, then we have strengthened van der Waerden's theorem. In the previous section we learned that no finite D will work. For infinite D, the answer is not so simple.

It is not hard to find certain sets D that strengthen van der Waerden's theorem. For example, by the next theorem we see that we may take D to be the set of all multiples of a fixed positive integer m. The theorem follows immediately from Lemma 4.6 by taking D (of Lemma 4.6) to be $\mathbb{Z}^+$.

Theorem 4.9. *Let m be a fixed integer with $m \ge 2$. Then for all positive integers k and r,*

$$R(m\mathbb{Z}^+, k; r) = m(w(k; r) - 1) + 1.$$

The following corollary is an easy consequence of Theorem 4.9. We leave its proof as Exercise 4.7.

Corollary 4.10. *Let F be a finite set of positive integers. Then $R(\mathbb{Z}^+ - F, k; r) < \infty$.*

We next consider a special case of Corollary 4.10 in which the results are quite interesting.

For c a positive integer, denote by $w'(c, k; r)$ the least positive integer w' such that for every r-coloring of $[1, w']$ there is a monochromatic k-term arithmetic progression whose gap is at least c. Notice that if $c = 1$, then $w'(c, k; r)$ coincides with the classical van der Waerden number $w(k; r)$. We also see that $w'(c, k; r) = R(A_D, k; r)$, where $D = \{c, c+1, c+2, \dots\}$, and by Corollary 4.10 we know that this number always exists. Let us take a closer look at the numbers $w'(c, k; r)$.

For arithmetic progressions of length three, we have the following theorem, which is a generalization of the fact that $w(3) = 9$.

Theorem 4.11. *Let c be a positive integer. Then $w'(c, 3) = 8c + 1$.*

Proof. Since $w(3) = 9$, by Proposition 2.29, any 2-coloring of
$$\{1, 1+c, 1+2c, \ldots, 1+8c\}$$
must have a monochromatic arithmetic progression of length three. Obviously, its gap is at least c. This shows that $w'(c,3) \leq 8c+1$.

To show that $8c+1$ is also a lower bound, consider the coloring of $[1, 8c]$ represented by the string $A_1B_1A_2B_2$, where $A_i = 1^{2c}$ and $B_i = 0^{2c}$ for $i = 1, 2$.

We shall show, by contradiction, that under this coloring there is no monochromatic 3-term arithmetic progression whose gap is at least c. Assume there is such an arithmetic progression, $P = \{x, y, z\}$ with $x < y < z$. By the symmetry of the coloring, we may assume that P has color 1. Since $z - x \geq 2c$, one of the A_i's must contain two members of P, and the other A_i must contain one member of P. Without any loss of generality, say $x, y \in A_1$ and $z \in A_2$. Then $z - y > 2c$, contradicting the fact that $y - x < 2c$. This shows that $w'(c,3) \geq 8c+1$, completing the proof. □

We can use the same idea by which we obtained the lower bound in Theorem 4.11 to prove the following more general fact that has the lower bound of Theorem 4.11 as a special case.

Before stating the theorem, we mention some convenient notation.

Notation. For positive integers c, k, and r with $k, r \geq 2$, let $\lambda(c,k,r)$ denote the r-coloring $\lambda : [1, cr(k-1)^2] \to \{0, 1, \ldots, r-1\}$ defined by the string $(B_0B_1 \ldots B_{r-1})^{k-1}$, where for each $i = 0, 1, \ldots, r-1$, B_i is a string of i's having length $c(k-1)$, and where there are $k-1$ copies of the block $(B_0B_1 \ldots B_{r-1})$.

Example 4.12. The coloring $\lambda(3,5,2)$ is the coloring of $[1, 72]$ represented by the string $(0^{12}1^{12})^4 = 0^{12}1^{12}0^{12}1^{12}0^{12}1^{12}0^{12}1^{12}$.

Theorem 4.13. *For all $c \geq 1$ and $k, r \geq 2$,*
$$w'(c,k;r) \geq cr(k-1)^2 + 1.$$

Proof. To prove this, it suffices to show that, under $\lambda(c,k,r)$, there is no k-term monochromatic arithmetic progression whose gap is at least c. We leave the details as Exercise 4.8. □

When dealing with the functions $w'(c,k;r)$, the following modification of the terminology used for valid colorings will be useful.

Definition 4.14. An r-coloring that admits no monochromatic k-term arithmetic progressions with gap at least c is called a $(c,k;r)$-*valid* coloring. For $r = 2$, we call the coloring (c,k)-*valid.*

Theorem 4.13 gives a lower bound for $w'(c,k;r)$. Of course, finding an upper bound would be much more significant, as it would provide an upper bound for the classical van der Waerden numbers $w(k;r)$ (letting $c = 1$). This is one of the main reasons for investigating families $\mathcal{F}$ that are subsets of AP. One possible approach to finding an upper bound on $w'(c,k;r)$ would be to first find a fairly simple description of all maximal length (c,k)-valid colorings. Some conjectures along these lines have been formulated (see Sections 4.4 and 4.5). For the case of $k = 3$ and $r = 2$, it turns out that for each $c \geq 2$, the maximal length valid colorings are quite simple to describe – in fact, as we shall see in the next theorem, there is only one!

Notice that if χ is any valid r-coloring of the interval $[a, b]$, then the r-coloring χ' obtained from χ by merely renaming the colors is still valid. Hence, when counting valid colorings we will not consider such pairs χ and χ' to be distinct colorings, unless otherwise stated.

It is well known that there are three distinct 2-colorings of $[1, 8]$ for which there is no 3-term monochromatic arithmetic progression, i.e., which are $(1,3)$-valid. These are represented by 11001100, 10011001, and 10100101 (the reader should verify this).

Note that the first of these colorings is the coloring $\lambda(1,3,2)$.

Theorem 4.17 below shows that for $c \geq 2$, $\lambda(c,3,2)$ is the *only* maximal length $(c,3)$-valid 2-coloring of $[1, 8c]$.

We shall need the following two lemmas.

Lemma 4.15. *Let $c, k, m \in \mathbb{Z}^+$, and let χ be a (c,k)-valid 2-coloring of $[1, mc]$. Let $i \in \{1, 2, \ldots, c\}$ and let χ^* be the 2-coloring of $[1, m]$ defined by $\chi^*(j) = \chi((j-1)c + i)$ for each $j = 1, 2, \ldots, m$. Then χ^* is $(1,k)$-valid on $[1, m]$.*

Proof. Assume χ^* is not $(1,k)$-valid on $[1,m]$. Then under χ^* there is a monochromatic arithmetic progression

$$\{x, x+d, \ldots, x+(k-1)d\} \subseteq [1,m].$$

Now consider the set $S = \{(x-1+jd)c+i : j = 0,1,\ldots,k-1\}$, an arithmetic progression contained in $[1,mc]$. It follows from the definition of χ^* that S is monochromatic under χ. Also, the gap of S is $cd \geq c$, contradicting the assumed validity of χ. □

In the next two proofs we will be using the $(AP,3)$-valid 2-colorings (i.e., the $(1,3)$-valid 2-colorings) of $[1,8]$, including ones that can be obtained from another coloring by interchanging the names of the two colors. Hence, there are six such colorings, which we denote as follows:

$$\sigma = 11001100, \qquad \tau = 10011001, \qquad \mu = 10100101,$$
$$\sigma' = 00110011, \qquad \tau' = 01100110, \qquad \mu' = 01011010.$$

Lemma 4.16. *Let $c \geq 3$, and assume that $\chi : [1,8c] \to \{0,1\}$ is a $(c,3)$-valid 2-coloring with $\chi(c) = 1$. Then $A = \{c, 2c, \ldots, 8c\}$ must have the color pattern $\sigma = 11001100$.*

Proof. Define χ^* on $[1,8]$ by $\chi^*(j) = \chi(jc)$. By Lemma 4.15 (taking $i = c$ in its statement), χ^* is (1,3)-valid on [1,8]. Hence, because $\chi(c) = 1$, as noted before, χ^* has one of the color patterns σ, τ, or μ. Thus A has one of these three color patterns. To complete the proof, we shall show that it is impossible for A to have color pattern τ or μ.

We consider two cases.

`Case 1.` c is odd. Let $B = \{1, c+1, 2c+1, \ldots, 7c+1\}$. By Lemma 4.15, the function χ' defined on [1,8] by $\chi'(j) = \chi((j-1)c+1)$ has one of the six color patterns σ, σ', τ, τ', μ, μ'. Hence, under χ, B has one of these six color patterns.

Let us first assume that A has color pattern τ; we will reach a contradiction.

If B has either coloring σ or μ', then we have $\chi(c+1) = \chi(8c) = 1$. Hence, $\chi\left(\frac{9}{2}c + \frac{1}{2}\right) = 0$, for otherwise $\left\{c+1, \frac{9}{2}c+\frac{1}{2}, 8c\right\}$ would be a monochromatic arithmetic progression with gap at least c, which is not possible. This implies that $\left\{2c+1, \frac{9}{2}c+\frac{1}{2}, 7c\right\}$ is monochromatic under χ, a contradiction.

If B has one of the colorings σ' or τ, then $\chi(4c) = \chi(7c+1) = 1$. Hence, $\chi(c-1) = 0$. It follows that $\{c-1, 3c, 5c+1\}$ has color 0, a contradiction.

The remaining possibilities for coloring B are τ' and μ. For each of these cases, $\chi(2c+1) = \chi(5c) = 1$, so that $\chi(8c-1) = 0$. This implies that $\{4c+1, 6c, 8c-1\}$ is monochromatic under χ and its gap is at least c; again a contradiction.

To complete Case 1, now assume that A has color pattern μ.

If B has any of the color patterns σ, τ', or μ, then $\chi(3c) = \chi(5c+1) = 1$. Therefore, we must have $\chi(c-1) = 0$ (by the validity of χ). This implies that $\{c-1, 2c, 3c+1\}$ is monochromatic, which is not possible. If B has either of the color patterns σ' or μ', then $\chi(c) = \chi(6c+1) = 1$, which implies that $\chi\left(\frac{7}{2}c + \frac{1}{2}\right) = 0$; but then $\left\{2c, \frac{7}{2}c+\frac{1}{2}, 5c+1\right\}$ has color 0, a contradiction. Finally, if B has color pattern τ, then since $\chi(3c) = \chi(4c+1) = 1$, we must have $\chi(2c-1) = 0$; but then $\{2c-1, 4c, 6c+1\}$ is monochromatic, again a contradiction.

`Case 2.` c is even. This case has a very similar proof to that of Case 1, the main difference being that instead of using the set B, we use $C = \{2, c+2, 2c+2, \ldots, 7c+2\}$. We shall work out two of the subcases and leave the other subcases as Exercise 4.9.

`Subcase i.` A has color pattern τ and C has color pattern σ. In this case, $\chi(c+2) = \chi(8c) = 1$, so that $\chi(\frac{9}{2}c+1) = 0$. This then implies that $\left\{2c+2, \frac{9}{2}c+1, 7c\right\}$ must be monochromatic, which is not possible.

`Subcase ii.` A has color pattern τ and C has one of the color patterns σ' or τ. In this case, $\chi(4c) = \chi(7c+2) = 1$, and therefore $\chi(c-2) = 0$ (by assumption, $c-2 \geq 1$). Thus, all members of $\{c-2, 3c, 5c+2\}$ have color 0, a contradiction. □

We are now ready to prove the theorem which shows that for all $c \geq 2$, $\lambda(c,3,2)$ is the only $(c,3)$-valid coloring of $[1,8c]$.

Theorem 4.17. *Let $c \geq 2$, and assume that χ is a $(c,3)$-valid 2-coloring of $[1,8c]$ with $\chi(1)=1$. Then $\chi = \lambda(c,3,2)$.*

Proof. We can show that the theorem is true for $c=2$ directly by checking that $\lambda(2,3,2) = 1111000011110000$ is the only $(2,3)$-valid 2-coloring of $[1,16]$ (we leave this to the reader as Exercise 4.10).

Now let $c \geq 3$ and let χ be any $(c,3)$-valid 2-coloring of $[1,8c]$ such that $\chi(c)=1$. To complete the proof, it is sufficient to show that for each $i = 1,2,\ldots,c$,

(4.5) $A_i = \{(j-1)c+i : 1 \leq j \leq 8\}$ has color scheme 11001100.

We know from Lemma 4.16 that (4.5) is true for $i=c$, so let us assume that $i \in \{1,2,\ldots,c-1\}$. Let χ_i be the coloring of $[1,8]$ defined by $\chi_i(j) = \chi((j-1)c+i)$. By Lemma 4.15, χ_i is $(1,3)$-valid on [1,8]. Therefore χ_i has one of the color patterns σ, σ', τ, τ', μ, μ'. Hence, we will complete the proof if we can show that A_i does not have any of the color patterns σ', τ, τ', μ, or μ'. We shall do this by contradiction.

First, assume A_i has one of the color patterns σ', τ, or μ. Then $\chi(c+i) = \chi(3c) = 0$, so $\chi(5c-i) = 1$. Hence, $\{5c-i, 6c, 7c+i\}$ is monochromatic, contradicting the $(c,3)$-validity of χ.

Next, assume A_i has the color pattern τ'. Then $\chi(2c+i) = \chi(5c) = 1$, implying that $\chi(8c-i) = 0$. This gives the monochromatic arithmetic progression $\{i, 4c, 8c-i\}$, a contradiction.

Finally, assume A_i has the pattern μ'. In this case, $\chi(i) = \chi(4c) = 0$, which implies that $\{3c+i, 5c, 7c-i\}$ is monochromatic, again impossible. □

In the above discussion, for some constant c, the gaps in the arithmetic progressions are required to be no less than c. We can generalize this type of restriction if, instead of choosing c to be the same constant for all arithmetic progressions, the value of this minimum gap size is dependent on the particular arithmetic progression. Specifically, given a function $f : \mathbb{Z}^+ \to \mathbb{R}^+$, define $w'(f(x),k;r)$ to be the least positive integer w', if it exists, such that for every r-coloring of $[1,w']$ there is a monochromatic k-term arithmetic progression $\{a, a+d, \ldots, a+(k-1)d\}$ with $d \geq f(a)$. Hence, the function

$w'(c,k;r)$ represents the special case of $w'(f(x),k;r)$ in which f is the constant function c.

Example 4.18. Let $f(x) = x^2$. Here we are interested in the collection of arithmetic progressions that consists of all those whose first term is 1, those whose first term is 2 with gap at least 4, those whose first term is 3 with gap at least 9, etc. Hence, for $w'(f(x),3;r)$ we would not consider progressions such as $\{2,3,4\}$, $\{2,4,6\}$, $\{2,5,8\}$, and $\{3,11,19\}$, but would consider those such as $\{2,6,10\}$, $\{2,7,12\}$, $\{3,12,21\}$, $\{3,13,23\}$, and $\{4,20,36\}$.

Definition 4.19. Let f be a function defined on the positive integers. If $X = \{a, a+d, \ldots, a+(k-1)d\}$ is an arithmetic progression such that $d \geq f(a)$, we shall call X an *f-a.p.* Further, if an r-coloring avoids monochromatic k-term f-a.p.'s, we will say that it is *$(f,k;r)$-valid*, or just *(f,k)-valid* if $r=2$.

The existence of $w'(c,k;r)$ is a direct consequence of van der Waerden's theorem. As we can see, for many choices of $f(x)$, there is a much greater restriction placed on the arithmetic progressions than there is if we are only requiring that the gap be at least c. Hence, as you might guess, the existence of $w'(f(x),k;r)$ is not as easy to show. In fact, as we shall see, for many functions $f(x)$ it does not exist.

For 2-term arithmetic progressions, the question of the existence of $w'(f(x),k;r)$ is relatively easy to answer. According to the next theorem, not only does it always exist, but we are able to give a precise formula for its value. For the purpose of keeping the statement of the theorem as simple as possible, we will assume that f is integer-valued; however, it is easy to see that $w'(f(x),2;r)$ will still exist in the more general case.

Before stating the theorem we remind the reader of the following notation.

Notation. If g is a function, we denote by $g^{(n)}(x)$ the n^{th} iterate of g (or the composition of g with itself n times). That is,

$$g^{(2)}(x) = g(g(x)),\ g^{(3)}(x) = g(g(g(x))), \text{ etc.}$$

Also, we define $g^{(0)}(a)$ to equal a (that is, $g^{(0)}$ is the identity function).

Theorem 4.20. *Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a nondecreasing function. Let $g(x) = f(x) + x$. Then $w'(f(x), 2; r) = g^{(r)}(1)$.*

Before delving into the proof, we look at why the existence of $w'(f(x), 2; r)$ is not surprising. Obviously, for any r-coloring of $\mathbb{Z}^+$, some color must occur an infinite number of times. Hence, if integer x is colored with this color, then clearly there is another integer y of the same color such that $y - x \geq f(x)$ for any function f.

Proof of Theorem 4.20. To show that $g^{(r)}(1)$ is an upper bound, let χ be any r-coloring of $[1, g^{(r)}(1)]$. Applying the pigeonhole principle, we see that there exist two members of the $(r+1)$-element set $\{1, g(1), g^{(2)}(1), \ldots, g^{(r)}(1)\}$ that have the same color. Say

$$\chi(g^{(i)}(1)) = \chi(g^{(j)}(1)),$$

where $0 \leq i < j \leq r$. Since

$$g^{(j)}(1) - g^{(i)}(1) \geq g^{(i+1)}(1) - g^{(i)}(1) = f(g^{(i)}(1)), \tag{4.6}$$

$\{g^{(i)}(1), g^{(j)}(1)\}$ is a 2-term monochromatic f-a.p. This proves the upper bound.

To show that $g^{(r)}(1)$ is also a lower bound for $w'(f(x), 2; r)$, we give a 2-coloring of $[1, g^{(r)}(1) - 1]$ that has no monochromatic pair $\{a, a+d\}$ with $d \geq f(a)$. Note that by (4.6), for each $i = 1, 2, \ldots, r$, the interval $A_i = [g^{(i-1)}(1), g^{(i)}(1) - 1]$ is not empty. Now define $\chi(A_i) = i$ for each $i = 1, 2, \ldots, r$. By (4.6), no two members of A_i differ by more than $f(g^{(i-1)}(1)) - 1$. Since f is nondecreasing, for each i there does not exist $\{a, a+d\} \in A_i$ with $d \geq f(a)$. This completes the proof. □

We now consider the existence of $w'(f(x), 3; 2)$ when f is a function from $\mathbb{Z}^+$ to $\mathbb{R}^+$. The next theorem shows that $w'(f(x), 3; 2)$ always exists.

Theorem 4.21. *For any function $f : \mathbb{Z}^+ \to \mathbb{R}^+$, $w'(f(x), 3; 2) < \infty$.*

Proof. Obviously, if the theorem is true for a function f_1 and if $f_1(x) \geq f_2(x)$ for all $x \in \mathbb{Z}^+$, then the theorem is also true for f_2. Hence, there is no loss of generality if we assume that f is a nondecreasing function.

We will show that for every 2-coloring of $\mathbb{Z}^+$ there is a monochromatic 3-term arithmetic progression of the desired type. The theorem then follows by the Compactness Principle (see Section 2.1).

Let χ be a 2-coloring of $\mathbb{Z}^+$. We can think of χ as a sequence of 0's and 1's: $\chi = \chi(1)\chi(2)\chi(3)\ldots$. We consider two cases.

Case 1. χ does not include infinitely many copies of the pattern 001 or the pattern 110 (i.e., there do not exist infinitely many t such that $\chi(t) = \chi(t+1)$ and $\chi(t) \neq \chi(t+2)$). In this case, there is some n so that $\chi(n)\chi(n+1)\chi(n+2)\ldots$ is one of the infinite binary sequences $000\ldots$, $111\ldots$, or $101010\ldots$. For each of these three possibilities, there is obviously a 3-term arithmetic progression whose first term is n and whose gap is at least $f(n)$.

Case 2. One of the patterns 001 or 110 occurs infinitely often. Without loss of generality, say there are infinitely many occurrences of 001. Let $\chi(a) = 0$, $\chi(a+1) = 0$, and $\chi(a+2) = 1$ be one of these occurrences. Then there exists another occurrence, $\chi(b) = 0$, $\chi(b+1) = 0$, and $\chi(b+2) = 1$, where $b - a \geq f(a+2)$ (otherwise there would not be infinitely many occurrences of 001). Let $d = b - a$. So $\{a, a+1, a+d, a+d+1\}$ has color 0, and $\{a+2, a+d+2\}$ has color 1.

If $\chi(a+2d+2) = 1$, then $\{a+2, a+d+2, a+2d+2\}$ is a monochromatic arithmetic progression with gap $d \geq f(a+2)$. If $\chi(a+2d+2) = 0$, then $\{a, a+d+1, a+2d+2\}$ is a monochromatic arithmetic progression with gap $d+1 \geq f(a+2) \geq f(a)$.

In both cases we have produced the desired type of arithmetic progression. □

The following theorem, which also tells us that $w'(f(x), 3; 2)$ exists, is more useful than Theorem 4.21 because it provides more information about the magnitude of $w'(f(x), 3; 2)$. We do not include the proof.

Theorem 4.22. *Let $f : \mathbb{Z}^+ \to \mathbb{R}^+$ be a nondecreasing function. Let $\beta = 1 + 4\left\lceil \frac{f(1)}{2} \right\rceil$. Then*

$$w'(f(x), 3; 2) \leq \left\lceil 4f\left(\beta + 4\left\lceil \frac{f(\beta)}{2} \right\rceil\right) + 14\left\lceil \frac{f(\beta)}{2} \right\rceil + \frac{7\beta - 13}{2} \right\rceil.$$

Theorem 4.22 gives an upper bound for $w'(f(x),3;2)$. The next theorem provides a lower bound. To keep the notation simpler, we assume $f : \mathbb{Z}^+ \to \mathbb{Z}^+$, although a more general result can be obtained by the same method of proof.

Theorem 4.23. *Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a nondecreasing function with $f(n) \geq n$ for all $n \in \mathbb{Z}^+$. Let $h = 2f(1)+1$. Then*

$$w'(f(x),3;2) \geq 8f(h) + 2h + 2 - t,$$

where t is the largest integer such that $f(t) + t \leq 4f(h) + h + 1$.

Proof. Let $m = 8f(h) + 2h + 1 - t$. To prove the theorem we shall give a 2-coloring of $[1,m]$ under which there is no monochromatic 3-term f-a.p.

We begin by partitioning $[1,m]$ into the following four intervals:

$$\begin{aligned} I_1 &= [1, h-1], \\ I_2 &= [h, 2f(h)+h-1], \\ I_3 &= [2f(h)+h, 4f(h)+h], \\ I_4 &= [4f(h)+h+1, m]. \end{aligned}$$

Now define the 2-coloring χ on $[1,m]$ as follows: let $\chi(I_1 \cup I_3) = 1$ and $\chi(I_2 \cup I_4) = 0$.

Assume that there is a monochromatic 3-term arithmetic progression $P = \{a,b,c\}$ with $d = b-a = c-b \geq f(a)$. Since $\chi(a) = \chi(b)$, we have the following six cases, each of which leads to a contradiction.

Case 1. $a, b \in I_1$. Then $d \leq h-2$, and therefore $c = b+d \leq 2h-3$. Also, since $d \geq f(1)$, we have $c \geq 1 + 2f(1) = h$. Thus, $c \in I_2$, which contradicts the fact that P is monochromatic.

Case 2. $a \in I_1$ and $b \in I_3$. In this case $d \geq 2f(h)+1$, and therefore $c \geq 4f(h)+h+1$. This implies $\chi(c) = 0$, a contradiction.

Case 3. $a, b \in I_3$. Then $c \geq a + 2f(a) \geq 3a \geq 6f(h) + 3h$, and therefore $\chi(c) = 0$, so P is not monochromatic.

Case 4. $a, b \in I_2$. In this case we have $d \leq 2f(h) - 1$, and hence $c \leq b + 2f(h) - 1 \leq 4f(h) + h - 2$. Also, $c \geq a + 2f(a) \geq h + 2f(h)$. So $c \in I_3$, a contradiction.

Case 5. $a \in I_2$, $b \in I_4$. Then $d \geq 4f(h) + h + 1 - a$, which implies $c \geq 8f(h) + 2h + 2 - a$. Thus, from the definition of m, it must be the case that $a \geq t+1$. Hence, by the meaning of t,

$$\begin{aligned} c &\geq a + 2f(a) \\ &\geq t + 1 + 2f(t+1) \\ &> 4f(h) + h + 1 + f(t+1) \\ &> 8f(h) + 2h + 2 - (t+1) = m, \end{aligned}$$

which is not possible.

Case 6. $a, b \in I_4$. Then $c \geq a + 2f(a) \geq 3a > m$, which is not possible. □

Example 4.24. It is not hard to compute bounds based on Theorems 4.22 and 4.23 for the function $f(x) = x + c$, where c is a nonnegative integer. If c is odd, we obtain

$$\tfrac{1}{2}(43c + 49) \leq w'(x+c,3;2) \leq 64c + 61.$$

Similar bounds may be obtained if c is even; we leave this as Exercise 4.11.

By the next theorem, we can improve upon the upper bound that is supplied by Theorem 4.22 for $w'(x+c,3;2)$. The proof makes use of the fact that $w'(x,3;2) = 24$; this and some other computer-generated values are presented in the following table.

$f(x)$	$w'(f(x),3;2))$
x	24
$x+1$	46
$x+2$	67
$x+3$	89
$x+4$	110
$x+5$	132
$2x$	77
$2x+1$	114

Table 4.1: Values of $w'(f(x),3;2)$

Interestingly, every one of these values agrees exactly with the lower bound given by Theorem 4.23.

Theorem 4.25. *Let c be a nonnegative integer. Then*

$$\frac{43}{2}c + 24 + \delta \leq w'(x+c, 3; 2) \leq 23c + 24,$$

where $\delta = 0$ if c is even and $\delta = \frac{1}{2}$ if c is odd.

Proof. The lower bound follows immediately from Example 4.24. For the upper bound, let χ be any 2-coloring of $[1, 23c+24]$. Let χ' be the 2-coloring of [1,24] defined by $\chi'(i) = \chi((c+1)(i-1)+1)$. Since, as noted above, $w'(x,3;2) = 24$, under χ' there exists a monochromatic arithmetic progression $\{a, a+d, a+2d\}$ with $d \geq a$. Therefore

$$\{(c+1)(a-1)+1, (c+1)(a+d-1)+1, (c+1)(a+2d-1)+1\}$$

is an arithmetic progression that is monochromatic under χ with a gap that is no less than $(c+1)a = (c+1)(a-1)+1+c$. Hence, it is a monochromatic f-a.p. where $f(x) = x + c$, which establishes the upper bound. □

We know from Theorem 4.21 that $w'(f(x), 3; 2)$ always exists. In contrast, $w'(f(x), 4; 2)$ does not always exist; in fact, it only exists if f is a rather slowly growing function. The situation is similar when more than two colors are used, even for 3-term arithmetic progressions. The details are found in the following theorem.

Theorem 4.26. *For $k \geq 3$ and $r \geq 2$, let*

$$c = \frac{\sqrt[r-1]{2} - 1}{k-1}.$$

If $k \geq 4$ or $r \geq 3$, then $w'(cx, k; r)$ does not exist.

Proof. Assume $k \geq 4$ or $r \geq 3$ and let $b = \sqrt[r-1]{2}$. To prove the theorem, we give an r-coloring of $\mathbb{Z}^+$ for which there does not exist a monochromatic arithmetic progression $\{a + jd : 0 \leq j \leq k-1\}$ with $d \geq ca$.

Let $\chi : \mathbb{Z}^+ \to \{0, 1, \ldots r-1\}$ be defined as follows. Whenever $x \in B_i = [b^i, b^{i+1} - 1]$, where i is a nonnegative integer, let $\chi(x) = \bar{\imath}$ where $i \equiv \bar{\imath} \pmod{r}$ and $0 \leq \bar{\imath} \leq r-1$.

Let us assume that $\{a, a+d, \ldots, a+(k-1)d\}$ is monochromatic under χ. We will complete the proof by showing that d must be less than ca. Let n be such that $b^n \leq a + d < b^{n+1}$. Then since $d < b^{n+1}$, we know that $a + 2d < 2b^{n+1} = b^{n+r}$. Since $\chi(a+2d) = \chi(a+d)$, the only B_i that $a+2d$ can belong to is B_n.

Now let us look at $a + 3d$. Since $a + 2d \in B_n$, we have

$$a + 3d = (a+2d) + d < 2b^{n+1} = b^{n+r}.$$

Since $\chi(a+3d) = \chi(a+d)$, this implies $b^n \leq a + d < a + 3d < b^{n+1}$, so that $a + 3d \in B_n$.

We see that by applying this same line of reasoning, we will obtain $a+4d, a+5d, \ldots, a+(k-1)d \in B_n$. Since

$$b^n \leq a + d < a + (k-1)d < b^{n+1},$$

we have

$$d < \frac{b^{n+1} - b^n}{k-2}.$$

Thus,

$$a \geq b^n - d > b^n\left(1 - \frac{b-1}{k-2}\right) \geq \frac{b^n}{2} = b^{n-(r-1)}.$$

Since $\chi(a) = \chi(a+d)$, it follows that $a \in B_n$.

We have shown that $b^n \leq a < a + (k-1)d < b^{n+1}$, so that

$$d < \frac{b^{n+1} - b^n}{k-1} = cb^n \leq ca.$$

Thus, we have shown that there is no monochromatic k-term arithmetic progression $\{a + jd : 0 \leq j \leq k-1\}$ whose gap is at least ca, thereby proving the theorem. □

For the case in which $r = 2$, the following theorem, which we offer without proof, improves upon Theorem 4.26.

Theorem 4.27. *Let $k \geq 4$. Then*

$$w'\left(\frac{x}{k^2 - 4k + 3}, k; 2\right)$$

does not exist.

Theorems 4.11, 4.13, 4.17, 4.20–4.23, 4.26 and 4.27 deal with van der Waerden-type problems in which the set of allowable gaps is restricted by insisting that the gaps exceed a certain value. In particular, in Theorems 4.11, 4.13, and 4.17 we restricted the allowable gaps to belong to sets of the form $D = \{c, c+1, c+2, \dots\}$ for a fixed positive integer c, and found, among other things, that in this case the associated van der Waerden-type numbers always exist. There are, of course, other ways to restrict the set of allowable gaps and ask if the corresponding van der Waerden-type function exists. It would be very nice if we could characterize which sets D have the property that $R(A_D, k; r)$ exists for all k and r. Such a characterization is not known. However, we will give a partial answer by addressing the following questions:

1. What general properties must a set D have in order for $R(A_D, k; r)$ to exist for all k and r?;
2. Are there any properties that preclude A_D from having this property?

It will be convenient to introduce some terminology.

Definition 4.28. Let $D \subseteq \mathbb{Z}^+$. For a fixed positive integer r, we say that D is *r-large* if $R(A_D, k; r)$ exists for all k. We call D *large* if it is r-large for all $r \geq 1$.

Note that if $s \geq r$, then any set that is s-large must also be r-large.

Let us consider Question 1 above. That is, what properties are necessary for D to be large? One such property is fairly easy to prove. We state it in the following theorem. Note that the theorem gives a condition that is necessary in order for a set to be 2-large. By the previous paragraph, it is also a necessary condition in order for the set to be large.

Theorem 4.29. *If D is 2-large, then for each positive integer m, D contains a multiple of m.*

Proof. Assume that D is a set not containing a multiple of every positive integer. Let $n \in \mathbb{Z}^+$ be such that D contains no multiple of n. The proof is completed by showing that D is not 2-large, i.e., that there is a 2-coloring of $\mathbb{Z}^+$ under which there do not exist arbitrarily long monochromatic arithmetic progressions with gap belonging to D. This may be done by an argument that is essentially the same as the proof of Theorem 4.4; we leave the details to the reader as Exercise 4.16. □

Note that the condition of Theorem 4.29 – that D must contain a multiple of every positive integer – is equivalent to the stronger-sounding condition that D must contain an infinite number of multiples of every positive integer.

As it turns out, a sequence of positive integers that grows too fast cannot be large. Here is one result along these lines. For other interesting results of this type, we refer the reader to the references given in Section 4.5.

Theorem 4.30. *Let $D = \{d_i\}_{i=1}^{\infty}$ be an increasing sequence of positive integers such that d_i divides d_{i+1} for all i. Then D is not 2-large.*

Proof. Define a 2-coloring $\chi : \mathbb{Z}^+ \to \{0, 1\}$ recursively as follows. First, for all $x \in [1, d_1]$ let $\chi(x) = 1$. Once χ has been defined on $[1, d_i]$, we extend χ to $[1, d_{i+1}]$ by having $\chi(x) \neq \chi(x - d_i)$ for each $x \in [d_i + 1, d_{i+1}]$.

To complete the proof we will show that for each $i \geq 1$ and each $j \geq 1$, there is no 3-term monochromatic d_i-a.p. contained in $[1, d_j]$.

First note that, by the way χ is defined on each of the intervals $[d_i + 1, d_{i+1}]$, for every $i \geq 1$ there can be no 2-term monochromatic d_i-a.p. contained in $[1, d_{i+1}]$.

Now assume that $j \geq i+2$ and that $x_1 < x_2 < x_3$ is a monochromatic d_i-a.p. that is contained in $[1, d_j]$. Since d_{i+1} divides d_j, by the way χ is defined we see that every subinterval of $[1, d_j]$ of the form $[kd_{i+1} + 1, (k+1)d_{i+1}]$, $k \geq 1$, either has the same color pattern as $[1, d_{i+1}]$, or has the pattern obtainable from that of $[1, d_{i+1}]$ by replacing all 1's by 0's and vice versa. Hence, since there is no 2-term monochromatic d_i-a.p. in $[1, d_{i+1}]$, neither of the pairs $\{x_1, x_2\}$ or $\{x_2, x_3\}$ can be contained in any one interval $[kd_{i+1} + 1, (k+1)d_{i+1}]$. This implies that $x_3 - x_1 > d_{i+1}$, and hence $x_3 - x_1 \geq 2d_i$, which contradicts the fact that $\{x_1, x_2, x_3\}$ is a d_i-a.p.. □

Example 4.31. Let $G = \{a, ar, ar^2, ar^3, \dots\}$ be an infinite geometric sequence. Then $R(A_G, k; 2)$ does not exist for large enough k. In other words, there exists a 2-coloring of $\mathbb{Z}^+$ that does not have, for some k, a monochromatic k-term arithmetic progression with difference from G.

We now shift our attention to finding sufficient conditions for a set to be large. We begin with a very useful theorem that tells us that if a union of a finite number of sets is large, then at least one of the sets itself must be large. This theorem is itself a Ramsey theorem in the sense in which we described Ramsey theory in Chapter 1: *Ramsey theory is the study of the preservation of properties under set partition.* That is, this theorem states that if we have a set that is large, and if we partition this set into n subsets, then one of the subsets must be a large set, i.e., if we n-color a large set, then there must exist a monochromatic large subset.

Theorem 4.32. *Let D be a large set, and let $n \geq 1$. If*

$$D = D_1 \cup D_2 \cup \cdots \cup D_n,$$

then some D_i is large.

Proof. We will prove the theorem for $n = 2$. It is then a simple induction argument to extend it to general n. Let $D = D_1 \cup D_2$, and assume that neither D_1 nor D_2 is large. We will prove the theorem by showing that D is not large.

Since D_1 is not large, there exist positive integers k_1 and r, and some r-coloring χ of $\mathbb{Z}^+$, under which there is no monochromatic k_1-term D_1-a.p. Similarly, there exist positive integers k_2 and s, and an s-coloring ϕ of $\mathbb{Z}^+$, under which there is no monochromatic k_2-term D_2-a.p.

Now, let σ to be the rs-coloring of $\mathbb{Z}^+$ given by $\sigma(i) = (\chi(i), \phi(i))$. Thus, the "colors" of σ consist of all ordered pairs whose first coordinate is one of the r colors of χ and whose second coordinate is one of the s colors of ϕ. Let $k = \max\{k_1, k_2\}$. To show that D is not large it will suffice to show that if P is any k-term arithmetic progression that is monochromatic under σ, then the gap of P does not belong to D.

Let $P = \{x_1, x_2, \ldots, x_k\}$ be an arithmetic progression that is monochromatic under σ. Then $\chi(x_1) = \chi(x_2) = \cdots = \chi(x_k)$. Hence, by our assumptions about χ and the fact that $k \geq k_1$, the gap of P cannot belong to D_1. Similarly, $\phi(x_1) = \phi(x_2) = \cdots = \phi(x_k)$, which implies that the gap of P cannot belong to D_2. Hence the gap of P does not belong to $D_1 \cup D_2$. Since $D = D_1 \cup D_2$, the proof is complete. □

The following fact is an immediate consequence of Theorems 4.4 and 4.32.

Corollary 4.33. *Let D be large, and F be finite. Then $D - F$ is large.*

Note that the proof of Theorem 4.32 actually tells us more than the statement of the theorem. It says that if D_1 is not r-large, and if D_2 is not s-large, then $D_1 \cup D_2$ is not rs-large. Extending this statement to n sets, it tells us that if, for each $i = 1, 2, \ldots, n$, D_i is not r_i-large, then $D_1 \cup D_2 \cup \cdots \cup D_n$ is not $r_1 r_2 \cdots r_n$-large. Looking at the contrapositive of this, and speaking in somewhat more general terms, it describes a way of going from a set that has a certain degree of largeness, to a subset of the set that has another (smaller) degree of largeness (that is, going from $r_1 r_2 \cdots r_n$-largeness to the r_i-largeness of D_i for some i). On the other hand, it is generally more difficult to prove results that involve showing that the r-largeness of a certain set implies the r-largeness of a proper subset of that set. One such result that is relatively easy to prove, however, is the following refinement of the above corollary.

Theorem 4.34. *Let r be a positive integer, and assume D is r-large. If F is finite, then $D - F$ is r-large.*

Proof. It is sufficient to show that for every $d \in D$, the set $D - \{d\}$ is r-large (why?). We shall do this by contradiction. So assume that $d_0 \in D$, and that $D - \{d_0\}$ is not r-large. Thus, there exist an r-coloring χ of $\mathbb{Z}^+$, and a positive integer k, such that under χ there is no monochromatic k-term arithmetic progression whose gap belongs to $D - \{d_0\}$. Since D is r-large, this implies that under χ there are arbitrarily long monochromatic d_0-a.p.'s. By Theorem 4.29, $md_0 \in D$ for some $m \geq 2$. It is clear that under χ there are arbitrarily long monochromatic md_0-a.p.'s. Since $md_0 \in D - \{d_0\}$, we have a contradiction. □

In this text we are limiting our methods to what are considered elementary in the sense that we are relying on the basic techniques of combinatorics and number theory on the set of integers. Some very useful and powerful theorems in Ramsey theory have also been proven by entirely different techniques. For example, several important improvements over van der Waerden's original theorem have been found by the use of ergodic theory and the methods of measure-preserving systems. These are very deep results, and we will not be including any of their proofs here, but we would like to mention at least one important special case of one of these results, which we shall phrase in the language of large sets.

Theorem 4.35. *Let $p(x)$ be a polynomial with integer coefficients, with positive leading coefficient, and whose constant term is 0. Then $\{p(i) > 0 : i \in \mathbb{Z}^+\}$ is large.*

Theorem 4.35 tells us that the range of any polynomial with the stated conditions gives us a large set. Denoting, for a polynomial f, the set $\{f(x) : x \in \mathbb{Z}^+\}$ by range(f), we see, for example, that range(p) is large when $p(x) = x^2$. That is, $\{1^2, 2^2, 3^2, \ldots\}$ is a large set. Even a set as sparse as $\{x^{1000} : x \in \mathbb{Z}^+\}$ is large.

Theorem 4.35 is a generalization of van der Waerden's theorem, because van der Waerden's theorem tells us that $\mathbb{Z}^+$ itself is large, which may be considered as the range of the polynomial $f(x) = x$, $x \in \mathbb{Z}^+$.

One hypothesis of Theorem 4.35 is that the constant term of $p(x)$ equals 0, i.e., that $x|p(x)$. We may extend Theorem 4.35 to polynomials with any linear factor $x + a$, where a is an integer, as given by the following corollary.

Corollary 4.36. *Let $p(x)$ be a polynomial with integer coefficients, with leading coefficient positive, and such that $x + a$ divides $p(x)$ for some integer a. Then $\{p(i) : i \in \mathbb{Z}^+\}$ is large.*

Proof. Let $p(x) = (x + a)q(x)$, and let $\hat{p}(x) = p(x - a)$. Then $\hat{p}(x) = xq(x - a)$.

By Theorem 4.35, range$(\hat{p}) \cap \mathbb{Z}^+$ is large. If $a \leq 0$, then we have range$(\hat{p}) \subseteq$ range(p), and therefore range$(p) \cap \mathbb{Z}^+$ is large. If $a > 0$, then range(p) = range$(\hat{p}) - F$, where F is a finite set, and hence by Corollary 4.33, range$(p) \cap \mathbb{Z}^+$ is large. □

There are some other interesting results, both positive and negative, concerning monochromatic arithmetic progressions that satisfy certain properties. We mention these in Section 4.5.

4.3. Exercises

4.1 Show that if $|D| = 1$, then $R(A_D, 2; 2) = \infty$, by exhibiting a specific 2-coloring of the positive integers.

4.2 Let s be a positive integer. Prove that if $R(A_D, k; 2)$ is finite, then
$$R(A_{sD}, k; 2) = s[R(A_D, k; 2) - 1] + 1;$$
and that if $R(A_D, k; 2) = \infty$, then $R(A_{sD}, k; 2) = \infty$.

4.3 Fill in the details of the proof of Case 2 of Theorem 4.8 (i.e., explain why there is no 4-term monochromatic a-a.p., no 3-term monochromatic b-a.p., and no 3-term monochromatic c-a.p.).

4.4 Define $w((d_1, d_2), k_1, k_2)$ to be the least positive integer n such that for every 2-coloring of $[1, n]$ there is either a monochromatic k_1-term arithmetic progression whose gap is d_1, or a monochromatic k_2-term arithmetic progression whose gap is d_2. Prove that for all $d_2, k_2 \geq 2$,
$$w((1, d_2), 2, k_2) = d_2(k_2 - 1) + 1.$$

4.5 In this exercise, we extend the notation of Exercise 4.4 above by denoting by $w((d_1, d_2, d_3), k_1, k_2, k_3)$ the least positive integer n such that for every 2-coloring of $[1, n]$ there is some $i \in \{1, 2, 3\}$ such that there is a k_i-term monochromatic arithmetic progression with gap d_i. Let $g = \gcd(d_1, d_2)$, and assume that exactly one of the numbers $\frac{d_1}{g}$ and $\frac{d_2}{g}$ is even.

a) Use Theorem 4.7 to prove that
$$w((d_1, d_2, d_3), 2, 2, 3) \leq d_1 + d_2 - g + 1.$$

b) Assuming further that $d_3 = \max\{d_1, d_2, d_3\}$, prove that
$$w((d_1, d_2, d_3), 2, 2, 3) = d_1 + d_2 - g + 1.$$

4.6 Using the notation of Exercise 4.5, prove that

$$w((1, d_2, d_3), 2, k_2, k_3) = (k_2 - 1)d_2 + 1$$

if exactly one of d_2 and d_3 is odd.

4.7 Prove Corollary 4.10.

4.8 Complete the proof of Theorem 4.13.

4.9 Complete the proof of Case 2 of Lemma 4.16.

4.10 Let χ be a $(2,3)$-valid 2-coloring of [1,16] with $\chi(1) = 1$. Show that $\chi = 1111000011110000$. Do this by direct computation; do not use Theorem 4.17.

4.11 Finish Example 4.24 for the case when c is even.

4.12 Use Theorems 4.22 and 4.23 to find upper and lower bounds for $w'(mx, 3; 2)$ if m is an odd integer and $m \geq 3$.

4.13 Use the method employed in the proof of Theorem 4.25 to prove that for any positive integer b,

$$w'(bx + bc, 3; 2) \leq (w'(bx, 3; 2) - 1)c + w'(bx, 3; 2).$$

4.14 It is known that $w'(2x, 3; 2) = 77$. Use this to find an upper bound for $w'(2(x + c), 3; 2)$ in terms of c.

4.15 Prove that for all k and r,

$$w'\left(\frac{x}{w(k;r) - k + 1}, k; r\right) = w(k;r),$$

so that there is at least some constant $c = c(k;r)$ such that $w'(cx, k; r)$ exists ($w(k;r)$ here represents the ordinary van der Waerden function).

4.16 Complete the proof of Theorem 4.29.

4.17 Prove that if D is large, and m is a positive integer, then mD is large.

4.18 Prove that if D is large and m is a positive integer, then $D - \{x : m \nmid x\}$ is large.

4.19 Let us call a set of positive integers "small" if it is not large. Must the complement (with respect to $\mathbb{Z}^+$) of a small set be large? Must the complement of a large set be small?

4.20 Prove: if D is r-large, and if all elements of D are multiples of the positive integer m, then $\frac{1}{m}D$ is r-large. Hence if D is large, then $\frac{1}{m}D$ is large.

4.4. Research Problems

4.1 Do a more complete study of $R(A_D, k; 2)$. In particular, define $m = m(n)$ to be the least positive integer such that for all D with $|D| = n$, $R(A_D, m; 2) = \infty$. By the work discussed in this chapter, we know that $m(1) = 2$, $m(2) = 3$, and $m(3) = 3$ or 4. Determine the exact value of $m(3)$ (it has been conjectured that $m(3) = 3$).
Reference: [**164**]

4.2 Using the notation of Research Problem 4.1, find an upper bound on $m(n)$ as a function of n. In particular, is it true that $m(n) \leq n$ for all n?
Reference: [**164**]

4.3 Using the notation of Exercises 4.4–4.6, characterize those triples (d_1, d_2, d_3) for which $w((d_1, d_2, d_3), 2, 3, 3) < \infty$.
Reference: [**164**]

4.4 Find a formula for $w((d_1, d_2, d_3), 2, 2, 3)$ (see Exercise 4.5).
Reference: [**164**]

4.5 Prove or disprove: the lower bound of Theorem 4.25 is the precise value of $w'(x+c, k; 2)$ (computer output suggests that this may well be the precise value; see Table 4.1).
Reference: [**64**]

4.6 It is known that there is a function $f(x)$ that tends to infinity as x goes to infinity, such that for each r, $w'(f(x), k; r)$ exists for all k. This function grows very slowly (something like an inverse of the classical van der Waerden function $w(k;r)$). Try to find a faster growing function that still has this property.
References: [**53**], [**64**]

4.7 Find results analogous to Theorem 4.11 or 4.17 for $w'(c, 4; 2)$ or $w'(c, 3; 3)$.
Reference: [**64**]

4.8 Let $A(k;r)$ be the set of all positive real numbers a such that $w'(ax,k;r)<\infty$. It has been discovered that $w'(\frac{x}{4},4;2)$ exists (it happens to equal 134), so that $\frac{1}{4}\in A(4;2)$. As noted in this chapter, $w'(\frac{x}{3},4;2)$ does not exist. Thus, letting $\beta(k;r)$ denote $\sup\{a : a\in A(k;r)\}$ we have $\frac{1}{4}\le\beta(4;2)\le\frac{1}{3}$. Find the exact value of $\beta(4;2)$, or improve its bounds. Do the same for $\beta(3;3)$ (it is known that $\frac{1}{25}\le\beta(3;3)\le\frac{\sqrt{2}-1}{2}$) and for $\beta(3;4)$ (here it is known that $\frac{1}{74}\le\beta(3;4)\le\frac{\sqrt[3]{2}-1}{2}$).
Reference: [**64**]

4.9 The following is known: if $w'(c,k;2)=2c(k-1)^2+1$ and j is a positive integer, then $w'(jc,k;2)=2jc(k-1)^2$. It would be desirable to know if the following stronger statement holds: if $w'(c,k;2)=2c(k-1)^2+1$ and $m\in\mathbb{Z}^+$, then

$$w'(c+m,k;2)=2(c+m)(k-1)^2+1.$$

References: [**64**], [**165**]

4.10 Theorem 4.27 prompts us to ask whether the fastest growing function f, such that $w'(f(x),k;2)$ exists for large enough k, must grow like the function $\frac{x}{k^2}$. If we cannot answer this, or if the answer is no, then one might try to answer the following question: does $w'(\frac{x}{2^k},k;2)$ exist for k large enough? Writing a computer program to calculate various values of $w'(f(x),k;2)$ is likely to be a big help with these types of questions.
Reference: [**64**]

4.11 Theorem 4.32 says that if a finite union of sets is large, then at least one of the sets must be large. Is the same true if we replace the word "large" with "r-large?" In particular, is it true that if $D\cup E$ is 2-large, then either D or E must be 2-large? The proof of Theorem 4.32 shows that if D is not 2-large and E is not 2-large, then $D\cup E$ is not 4-large, but it does not tell us about the 2-largeness or 3-largeness of $D\cup E$. It follows from Theorems 4.29 and 4.30 that neither $\{2n-1 : n\in\mathbb{Z}^+\}$ nor $\{n! : n\in\mathbb{Z}^+\}$ is 2-large. Hence, the union of these two sets is not 4-large. Is the union 2-large? 3-large?
Reference: [**62**]

4.12 Which sets B have the property that some translation of B is large? In other words, for which sets B does there exist an integer t such that $D=B+t=\{b+t : b\in B\}$ is large? By Theorem 4.35, the range of a polynomial with integer coefficients and positive leading coefficient has this property, because if $f(x)$ is the polynomial, then $f(x)-f(0)$ is a polynomial with a zero constant term.
References: [**62**], [**65**]

4.13 Let $p(x)$ be any polynomial with integer coefficients, positive leading coefficient, and $p(0)=0$, and let D be a large set. Determine if $\{p(d) : d\in D\}$ must be large? In particular, must $\{d^2 : d\in D\}$ be large?
References: [**62**], [**65**]

4.14 For m a positive integer and $0\le a<m$, denote by $S_{a(m)}$ the set of all arithmetic progressions whose gaps are not congruent to $a\ (\mathrm{mod}\ m)$. Do a study of $R(S_{a(m)},k;r)$.
References: [**64**], [**65**], [**160**]

4.5. References

§**4.1**. Theorems 4.7 and 4.8 are found in [**164**]. More generally, [**164**] considers the function $w(\vec{d},\vec{k})$, where $\vec{d}=(d_1,d_2,\ldots,d_n)$ and $\vec{k}=(k_1,k_2,\ldots,k_n)$, defined as the least positive integer m such that for every 2-coloring of $[1,m]$ there will be, for some i, a monochromatic k_i-term d_i-a.p.

§**4.2**. The proof of Theorem 4.11 is from [**165**]. Theorems 4.13, 4.17, 4.20–4.23, and 4.25–4.27 come from [**64**], which also gives a result that is slightly more general than Theorem 4.9. Several conjectures, and some evidence for them, are also found in [**64**]. The proofs of Theorems 4.29, 4.30, 4.32, 4.34, and Corollary 4.36 are found in [**62**], along with some other conditions that are necessary, or are sufficient, for a set to be large. For instance, if there exists $\epsilon>0$ such that the ratio $\frac{d_{i+1}}{d_i}$ exceeds $1+\epsilon$ (asymptotically), then $\{d_i : i\ge 1\}$ is not large. Some examples and questions are also presented in [**62**]. Some additional results and examples on large sets are given in [**151**]. Theorem 4.35 is a special case of a result due to Bergelson and Leibman

[**33**]. They employ methods of ergodic theory and measure theory. Extensions of this, and related results, may be found in [**112**], [**113**], [**114**], [**115**], [**185**], and [**233**]. See [**28**] and [**184**] for excellent extensive surveys of work in ergodic Ramsey theory. Walters [**272**] provides a combinatorial proof of the polynomial extensions of van der Waerden's theorem due to Bergelson and Leibman.

§**4.3**. Exercises 4.4–4.6 come from [**164**]. Exercise 4.14 is from [**64**]. Exercises 4.17–4.20 are from [**62**].

§**4.4**. Research Problems 4.1–4.4 come from [**164**]. Research Problems 4.5–4.10 are discussed in [**64**], and 4.11–4.13 are from [**62**].

Additional References: Brown [**53**] finds a 2-coloring of $\mathbb{Z}^+$ and a function h such that if $P = \{a, a+d, \ldots a+(k-1)d\}$ is monochromatic then $k \leq \min\{h(a), h(d)\}$. That is, in order to have the van der Waerden property hold, we cannot require d or a to be too small as a function of k (one corollary of this is Theorem 4.4). Other work along these lines is done in [**21**] and [**152**]. This general problem is mentioned in the excellent monograph of Erdős and Graham [**92**, p. 17]; several other intriguing problems concerning monochromatic arithmetic progressions are discussed in Chapter 2 of [**92**].

Chapter 5

Other Generalizations of $w(k;r)$

In Chapter 3 we looked at ways of generalizing the van der Waerden function $w(k;r)$ by using supersets of the family of arithmetic progressions. Thus, we ended up with functions, such as $Q_n(k)$, which, under special circumstances (in this case when $n = 0$), coincide with the classical van der Waerden function itself. In this chapter we will consider the Ramsey-type functions for some other generalizations of arithmetic progressions, constructed by introducing other parameters, so that the number $w(k;r)$ is a special case of a more general function.

5.1. Sequences of Type $x, ax + d, bx + 2d$

In this section we consider the following generalization of a 3-term arithmetic progression.

Definition 5.1. Let $a \leq b$ be fixed positive integers. An *(a,b)-triple* is any set $\{x, ax + d, bx + 2d\}$, where x and d are positive integers.

It is clear from the definition that we have a generalization of 3-term arithmetic progressions: (x, y, z) is a 3-term arithmetic progression if and only if it is a (1,1)-triple. As further examples, $\{1, 3, 8\}$

is a (1,4)-triple (with $x = 1$ and $d = 2$), $\{2, 7, 10\}$ is a (3,4)-triple (with $d = 1$), and $\{1, 3, 4\}$ is a (2,2)-triple (with $d = 1$).

Notation. Denote by $T_{a,b}$ the set of all (a, b)-triples. Since in this discussion we shall be dealing only with sets of size three, for ease of notation, we shall denote the Ramsey-type function $R(T_{a,b}, k; r)$ more simply as $T(a, b; r)$.

We see that $T(1, 1; r)$ has the same meaning as $w(3; r)$. Thus, from the known exact values of $w(k; r)$, we have $T(1, 1; 2) = 9$, $T(1, 1; 3) = 27$, and $T(1, 1; 4) = 76$. However, the existence of $T(a, b; r)$ for the general pair (a, b) would not seem to follow from van der Waerden's theorem. In fact, as we shall see, $T(a, b; r)$ does *not* always exist. We seek to answer two natural questions:

1. For which values of a, b, and r does $T(a, b; r)$ exist?
2. When it does exist, what can we say about the value of $T(a, b; r)$?

One case for which we are able to give a rather complete answer to these questions is that in which $b = 2a - 1$ and $r = 2$. The result is given by Proposition 5.2 below. We first investigate the relationship between the problem at hand and some families of arithmetic progressions that were discussed in Chapter 4.

Note that $\{1, 3, 5\}$ is not only a (1,1)-triple, but also a $(2, 3)$-triple (taking $d = 1$). Thus, $(a_1, b_1) \neq (a_2, b_2)$ does not imply that T_{a_1,b_1} and T_{a_2,b_2} are disjoint. Is there more we can say about the relationship between such a pair of families?

To help answer this question, let's look more closely at the particular case of (1,1)-triples versus (2,3)-triples. In the example of $\{1, 3, 5\}$ mentioned above, it is not a coincidence that this triple belongs to both $T_{2,3}$ and AP. Actually, if (x, y, z) is any (2,3)-triple, then

$$z - y = (3x + 2d) - (2x + d) = x + d = y - x, \tag{5.1}$$

so that (x, y, z) is an arithmetic progression. Hence $T_{2,3} \subseteq T_{1,1}$. On the other hand, the reverse inclusion does not hold since, by (5.1), for any (2,3)-triple (x, y, z), we must have $z - y = x + d \geq 2$.

More generally, we have the following.

Proposition 5.2. *Let $a \geq 1$. Then $(x, y, z) \in T_{a,2a-1}$ if and only if (x, y, z) is an arithmetic progression with gap $y - x \geq (a - 1)x + 1$.*

Proof. $X = \{x, y, z\} \in T_{a,2a-1}$ if and only if

$$z - y = (2a - 1)x + 2d - (ax + d) = (a - 1)x + d = y - x$$

for some positive integer d, i.e., if and only if X is an arithmetic progression with gap at least $(a - 1)x + 1$. □

As a result of Proposition 5.2, we see that $T(a, 2a - 1; 2)$ is a special case of the Ramsey-type function $w(g(x), 3; 2)$ we studied in Chapter 4, which, you may recall, corresponds to arithmetic progressions $\{y, y+d, y+2d\}$ such that $d \geq g(y)$. Making use of bounds given for this function in Chapter 4, we are able to give upper and lower bounds for $T(a, 2a - 1; 2)$ (for the particular case in which $a = 1$, we know that $T(1, 1; 2) = w(3; 2) = 9$; see Table 5.1 at the end of this section for other values of $T(a, b; 2)$).

Theorem 5.3. *For all $a \geq 2$,*

$$16a^2 - 12a + 6 \leq T(a, 2a - 1; 2) \leq \begin{cases} 16a^3 - 2a^2 + 4a - 3 & \textit{for } a \textit{ even}, \\ 16a^3 + 14a^2 + 2a - 3 & \textit{for } a \textit{ odd}. \end{cases}$$

Proof. By Proposition 5.2, we have $T(a, 2a - 1; 2) = w'(g(x), 3; 2)$ where $g(x) = (a - 1)x + 1$. Applying Theorem 4.23, we obtain

$$T(a, 2a - 1; 2) \geq 8g(2a + 1) + 2(2a + 1) + 2 - c,$$

where c is the greatest integer such that

$$(a - 1)c + 1 + c \leq 4g(2a + 1) + 2a + 2,$$

i.e., such that $ac + 1 \leq 4[(a - 1)(2a + 1) + 1] + 2a + 2$. This implies $c = \lfloor \frac{8a^2 - 2a + 1}{a} \rfloor = 8a - 2$. Therefore,

$$T(a, 2a - 1; 2) \geq 8\left((a - 1)(2a + 1) + 1\right) + 4a + 4 - 8a + 2,$$

which gives us the desired lower bound.

For the upper bound, we give the proof for the case of a even, and leave the case in which a is odd as Exercise 5.2. Assuming a is even and $g(x) = (a - 1)x + 1$, then, using the notation of Theorem

4.22, we have $\beta = 1 + 2a$ and $g(\beta) = 2a^2 - a$. Hence, by Proposition 5.2, Theorem 4.22 yields

$$\begin{aligned} T(a, 2a-1; 2) &\le 4g\left(1 + 2a + 4\tfrac{2a^2-a}{2}\right) + 7(2a^2 - a) + \tfrac{7}{2}(1+2a) - \tfrac{13}{2} \\ &= 4g(4a^2+1) + 14a^2 - 3 \\ &= 4[(a-1)(4a^2+1) + 1] + 14a^2 - 3, \end{aligned}$$

which equals the desired upper bound. □

By virtue of Proposition 5.2, we may also use the results of Chapter 4 to tell us about the function $T(a, 2a-1; r)$ when $r \ge 3$; the following theorem gives us a simple answer.

Theorem 5.4. *Let $a \ge 2$ and $r \ge 3$. Then $T(a, 2a-1; r)$ does not exist.*

Proof. By Proposition 5.2, we have $T(a, 2a-1; r) = w(g(x), 3; r)$, where $g(x) = (a-1)x + 1$. From Theorem 4.26, we know that $w(\frac{r-1/2}{2}x, 3; r)$ does not exist. Since $g(x) \ge \frac{r-1/2}{2}x$ for all x, we see that $w(g(x), 3; r)$ does not exist. □

As we see from the last two theorems, the existence of $T(a, b; r)$ is dependent on the value of r. This prompts the next definition. For convenience we will stray a bit from the way that "regular" was presented in Chapter 1 (where it is associated with a collection of sets).

Definition 5.5. Let a and b be positive integers with $a \le b$. If $T(a, b; r)$ does not exist for some positive integer r, the *degree of regularity* of (a, b), denoted $dor(a, b)$, is the largest value of r such that $T(a, b; r)$ exists. If $T(a, b; r)$ exists for all positive integers r, we say that $dor(a, b) = \infty$, and also say that (a, b) is *regular.*

Obviously, for every pair (a, b), $dor(a, b) \ge 1$. By van der Waerden's theorem (1,1) is regular, while Theorems 5.3 and 5.4 tell us that $dor(a, 2a-1) = 2$ for each $a \ge 2$.

As a brief aside, it is worth noting that the notion of regularity may be applied to any type of sequence. Thus, for example, any of the types of supersets of AP discussed in Chapter 3 may be considered

to be regular. Meanwhile, as we saw in Chapter 4, certain types of sets may yield a finite Ramsey-type value only when the number of colors is not too great (see, for example, Theorem 4.26), so that they would have a finite degree of regularity. The general question of which sets are regular and, if not, *how* regular, is rather intriguing. In later chapters we shall see some other interesting cases of families of sequences that are r-regular up to some specific finite value of r, but which fail to be regular.

Getting back to the degree of regularity for (a, b)-triples, we would like to know whether there are pairs besides those of the form $(a, 2a-1)$ whose degree of regularity is greater than one. The following theorem provides a complete answer to this question. It also gives an upper bound for $T(a, b; 2)$ whenever $dor(a, b) \ge 2$.

Theorem 5.6. *Let $a, b \in \mathbb{Z}^+$ with $a \le b$. Then $dor(a, b) = 1$ if and only if $b = 2a$. Furthermore,*

$$T(a, b; 2) \le \begin{cases} 4a(b^3 - 3b - 3) + 2b^3 + 4b^2 + 6b & \text{if } b > 2a, \\ 4a(b^3 + 2b^2 + 2b) - 4b^2 & \text{if } b < 2a. \end{cases}$$

Proof. First, assume $b = 2a$. To show $dor(a, b) = 1$, we need only exhibit a 2-coloring of $\mathbb{Z}^+$ that admits no monochromatic $(a, 2a)$-triple. Note that for any $(a, 2a)$-triple (x, y, z), we have $z = 2y$. Thus any 2-coloring γ such that for each even number $2n$, $\gamma(2n) \ne \gamma(n)$, avoids monochromatic $(a, 2a)$ triples (color the odd numbers first, arbitrarily, and then color the even numbers appropriately).

Now assume $b > 2a$. Let

$$m = 4a(b^3 + b^2 - 3b - 3) + 2b^3 + 4b^2 + 6b,$$

and let $\chi : [1, m] \to \{0, 1\}$ be an arbitrary 2-coloring. We will show that under χ there is a monochromatic (a, b)-triple. Assume, for a contradiction, that there is no such (a, b)-triple. In particular, $\{2, 2a+2, 2b+4\}$ is not monochromatic. Thus, there exist $x, x+2 \in \{2, 4, 6, \ldots, 2b+4\}$ such that $\chi(x) \ne \chi(x+2)$. Without loss of generality, let $\chi(x) = 0$ and $\chi(x+2) = 1$.

Let z be the least integer greater than $a(x+2)$ such that z is a multiple of $b-2a$. Hence

$$z \le a(x+2)+b-2a-1 \le a(2b+4)+b-2a-1 = 2a(b+1)+b-1.$$

Thus

$$bz+2(b-2a) \le 2a(b^2+b-2)+b^2+b \le m. \tag{5.2}$$

Let S be the (a,b)-triple $(z, az+(b-2a), bz+2(b-2a))$. By (5.2), $S \subseteq [1,m]$, so by our assumption some member of S must have color 1. Let $s \in S$ with $\chi(s)=1$, and let

$$T = \left\{ s+i(b-2a) : 0 \le i \le \frac{s(b-1)}{b-2a}+2 \right\}.$$

Note that since $(b-2a)$ divides s and since $a \le b$, we have that

$$s + \left[\frac{s(a-1)}{(b-2a)}+1\right](b-2a) = as+(b-2a) \in T.$$

Note also that the largest member of T is $bs+2(b-2a)$, and that by (5.2) we have

$$\begin{aligned} bs+2(b-2a) &\le b(bz+2(b-2a))+2(b-2a) \\ &\le 2a(b^3+b^2-2b-2)+b^3+b^2+2b \\ &\le m. \end{aligned}$$

Thus, $T \subseteq [1,m]$, and since $\{s, as+(b-2a), bs+2(b-2a)\}$ is an (a,b)-triple contained in T (hence cannot be monochromatic), some member of T must have color 0.

Let t be the least member of T with color 0. Thus, since $t > s$ and $\chi(s)=1$, we have $\chi(t-(b-2a))=1$. Since $x \le 2b+2$ and $t \le bs+2(b-2a)$, from (5.2) we have

$$\begin{aligned} b(x+2)+2(t-ax-b) &= 2t+x(b-2a) \\ &\le 4a(b^3+b^2-2b-2) \\ &\quad +2(b^3+b^2+2b)+(2b+2)(b-2a) \\ &= m. \end{aligned}$$

Let $(\alpha,\beta,\gamma) = (x+2, t-(b-2a), b(x+2)+2(t-ax-b))$. We claim that (α,β,γ) is an (a,b)-triple. To see this, first note that by the definition of t, $\beta = t-(b-2a) > a(x+2)$. Now let $d = \beta - a(x+2)$. Then

$\beta = a\alpha+d$ and $\gamma = b\alpha+2d$, establishing the claim. Since we assumed that $\chi(\alpha)=1$, and since $\chi(\beta)=1$, we know that $\chi(\gamma)=0$. However, this gives the monochromatic (a,b)-triple (x,t,γ), a contradiction (we leave it to the reader to show that (x,t,γ) is, in fact, an (a,b)-triple). This finishes the case in which $b>2a$.

Now assume $b<2a$, and let $m = 4a(b^3+2b^2+2b)-4b^2$. Assume, by way of contradiction, that there is a 2-coloring χ of $[1,m]$ under which there is no monochromatic (a,b)-triple. As in the previous case, there exist $y-2, y \in \{2,4,\dots,2b+4\}$ that are not of the same color; say $\chi(y-2)=1$ and $\chi(y)=0$. Let z be the least integer greater than $ay-(2a-b)$ that is a multiple of $2a-b$, and let S be the (a,b)-triple $\{z, az+(2a-b), bz+2(2a-b)\}$. As in the previous case, there is an $s \in S$ having color 1, and if

$$T = \{s, s+(2a-b), s+2(2a-b), \dots, bs+2(2a-b)\},$$

then $as+(2a-b) \in T$ and $T \subseteq [1,m]$. It follows that T contains a least member, t, with color 0 (why?). Then $\chi(t-(2a-b))=1$ and, since $(\alpha,\beta,\gamma) = (y-2, t-(a-2b), b(y-2)+2(t-ay+b))$ is an (a,b)-triple that is contained in $[1,m]$ (the verification of this is left to the reader as Exercise 5.3), we must have $\chi(\gamma)=\chi(by+2(t-ay))=0$. This gives the monochromatic (a,b)-triple (y,t,γ), a contradiction. □

Theorem 5.6 provides an upper bound for $T(a,b;2)$. The next theorem gives a lower bound.

Theorem 5.7. *Let $a,b \in \mathbb{Z}^+$ with $a \le b$. Then*

$$T(a,b;2) \ge \begin{cases} 2b^2+5b-2a+4 & \text{if } b>2a, \\ 3b^2+5b-4a+4 & \text{if } b<2a. \end{cases}$$

Proof. For each of the two cases we shall exhibit a 2-coloring (of the appropriate interval) that avoids monochromatic (a,b)-triples.

For the case of $b>2a$, let $\chi : [1, 2b^2+5b-2a+3] \to \{0,1\}$ be the coloring defined by $\chi([1,b+1])=0$, $\chi([b+2, b^2+2b+1])=1$, and $\chi([b^2+2b+2, 2b^2+5b-2a+3])=0$. There cannot be a monochromatic (a,b)-triple of color 1, because the largest term of any (a,b)-triple whose least term lies in $[b+2, b^2+2b+1]$ would lie outside of this interval.

Now assume that (x,y,z) is an (a,b)-triple of color 0. Obviously, $z \notin [1,b+1]$. Also, if $x,y \in [1,b+1]$, then $d \leq b+1-ax$ (where $ax+d=y$), so that $z \leq bx+2(b+1-ax) \leq b^2+2b$. Hence, we must have $y \geq b^2+2b+2$. It follows that if $x \leq b+1$, then $d \geq b^2+2b+2-ax$, so that

$$\begin{aligned} bx+2d &\geq 2b^2+4b+4-2ax+bx \\ &= 2b^2+5b+4-2a+(x-1)(b-2a) \\ &> 2b^2+5b-2a+3, \end{aligned}$$

which is not possible. Thus, we assume that $x \geq b^2+2b+2$. Then, since $d \geq 1$ and $b \geq 3$,

$$z \geq b^3+2b^2+2b+2 \geq 2b^2+5b-2a+4,$$

which is also impossible. This completes the case in which $b > 2a$.

To establish the lower bound for $b < 2a$, consider the 2-coloring of $[1, 3b^2+5b-4a+3] = [1,m]$ defined by coloring $[b+2, b^2+2b+1]$ with color 1, and its complement in $[1,m]$ with color 0. We leave the proof that this coloring avoids monochromatic (a,b)-triples as Exercise 5.4. □

Now that we have established upper and lower bounds for the function $T(a,b;2)$, we look at a few examples.

Example 5.8. Consider $T(1,b;2)$ where b is some fixed positive integer. Thus we are concerned with the family of triples that have the form $(x, x+d, bx+2d)$ for some $d \in \mathbb{Z}^+$, a natural generalization of AP. We know that $T(1,1;2) = w(3;2) = 9$, and (by Theorem 5.6) $T(1,2;2)$ does not exist. For $b \geq 3$, Theorem 5.6 gives $4b^3(1+o(1))$ as an asymptotic upper bound for $T(1,b;2)$, while Theorem 5.7 gives $2b^2(1+o(1))$ as an asymptotic lower bound (Exercise 5.5 gives a lower bound for $T(1,b;2)$ that is slightly better than that of Theorem 5.7, but which is asymptotically the same).

Example 5.9. Let $b=a$. Theorems 5.6 and 5.7 give the asymptotic bounds $3a^2(1+o(1)) \leq T(a,a;2) \leq 4a^4(1+o(1))$. It is known, in fact, that $T(a,a;2) = O(a^2)$, so that in this case the actual value of $T(a,a;2)$ is very close to the bound of Theorem 5.7. Moreover, it has been shown that $T(a,a;2) \leq 3a^2+a$ for all even $a \geq 4$, and that $T(a,a;2) \leq 8a^2+a$ for all odd a.

Example 5.10. Theorems 5.6 and 5.7 yield the asymptotic bounds $12a^2(1+o(1)) \leq T(a,2a-1;2) \leq 32a^4(1+o(1))$. However, these bounds are not as good as the bounds provided by Theorem 5.3: $16a^2(1+o(1)) \leq T(a,2a-1;2) \leq 16a^3(1+o(1))$.

We now return to the problem of determining the degree of regularity of (a,b)-triples. So far we know, by Theorem 5.6 that

$$dor(a,b) \geq 2 \text{ unless } b = 2a;$$

by Theorems 5.3 and 5.4 that

$$dor(a, 2a-1) = 2 \text{ for } a > 1;$$

and by van der Waerden's theorem that $dor(1,1) = \infty$.

It has been conjectured that $(1,1)$-triples (the arithmetic progressions) are the only (a,b)-triples that do not have a finite degree of regularity. In the next theorem, we show that, whenever b is large enough in comparison to a, the degree of regularity is indeed finite.

Theorem 5.11. *Let $1 \leq a < b$, with $b \geq (2^{3/2}-1)a - (2^{3/2}-2)$. Then $dor(a,b) \leq \lceil 2\log_2 c \rceil$, where $c = \left\lceil \frac{b}{a} \right\rceil$.*

Proof. Let a and b be as stated and let $r = \lceil 2\log_2 c\rceil + 1$. To prove the theorem, we will provide an r-coloring of $\mathbb{Z}^+$ that yields no monochromatic (a,b)-triple. For convenience of notation, let $\beta = \sqrt{2}$.

We define $\chi : \mathbb{Z}^+ \to \{0,1,\ldots,r-1\}$ as follows. For each $i \geq 1$, whenever $\beta^i \leq x < \beta^{i+1}$, let $\chi(x) = \bar{\imath}$ where $i \equiv \bar{\imath} \pmod{r}$ and $0 \leq \bar{\imath} \leq r-1$. For example, (assuming $r \geq 5$), $\chi(1) = 1$, $\chi(2) = 2$, $\chi(3) = 3$, $\chi([4,5]) = 4$, and $\chi([\lceil \beta^r \rceil, \lfloor \beta^{r+1} \rfloor]) = 0$.

Assume, for a contradiction, that there exists an (a,b)-triple $(x,y,z) = (x, ax+d, bx+2d)$ that is monochromatic under χ. Let $j \geq 1$ be the integer such that $\beta^j \leq y < \beta^{j+1}$. Since $c \geq 2$, we have $z \leq cy$, so that from the meaning of r it follows that $z < \beta^{r-1}\beta^{j+1} = \beta^{j+r}$. Hence, by the way χ is defined and the fact that $\chi(y) = \chi(z)$, we must have

$$\beta^j \leq y < z < \beta^{j+1}. \tag{5.3}$$

We consider two cases.

Case 1. $b \le 2a$. By the hypothesis of the theorem,

$$b - a \ge 2a(\beta - 1) - 2(\beta - 1).$$

Therefore,

$$\frac{(a-1)x}{(b-a)x} \le \frac{1}{2(\beta-1)}. \tag{5.4}$$

Since $b \le 2a - 1$, we have that $(b-a)x \le (a-1)x$. Hence, by (5.4),

$$\frac{(a-1)x+d}{(b-a)x+d} \le \frac{1}{2(\beta-1)}.$$

Thus, using (5.3),

$$y - x \le \frac{z-y}{2(\beta-1)} < \frac{\beta^{j+1}-\beta^j}{2(\beta-1)} = \frac{\beta^j}{2} = \beta^{j-2}.$$

Because $y \ge \beta^j$, this implies $x \ge \beta^j - \beta^{j-2} = \beta^{j-2}$. Since, in this case, $r = 3$, and since $\chi(x) = \chi(y)$, by the way χ is defined we must have $\beta^j \le x < \beta^{j+1}$. Therefore, $x, y, z \in [\beta^j, \beta^{j+1})$. Hence,

$$z - x = (b-1)x + 2d < \beta^j(\beta - 1) \le x(\beta - 1),$$

contradicting the fact that $b - 1 > \beta - 1$.

Case 2. $b > 2a$. In this case,

$$y - x = (a-1)x + d < (b-a)x + d = z - y.$$

Hence, $\beta^j(\beta-1) \le \beta^j\left(1 - \frac{1}{\beta^{r-1}}\right)$. Since $y > \beta^j$, this implies

$$x \ge \beta^j - \beta^j\left(1 - \frac{1}{\beta^{r-1}}\right) = \beta^{j-r+1}.$$

Since $\chi(x) = \chi(y)$, by the definition of χ we must have

$$\beta^j \le x < y < \beta^{j+1}.$$

Thus, all three numbers x, y, z belong to the interval $[\beta^j, \beta^{j+1})$. As in Case 1, this yields a contradiction. □

We may extend the idea of (a,b)-triples to k-tuples. That is, for fixed positive integers $a_1 \le a_2 \le \cdots \le a_{k-1}$, we may ask about the Ramsey-type functions corresponding to k-tuples of the form $\{x, a_1x + d, a_2x + 2d, \ldots, a_{k-1}x + (k-1)d\}$. (Analogously to the notation $T(a,b)$, these more general Ramsey-type functions are denoted by $T(a_1, a_2, \ldots, a_{k-1})$; see, for example, Research Problem 5.6.)

For this discussion, we shall limit ourselves to the special case in which $a_1 = a_2 = \cdots = a_{k-1}$. Therefore, for convenience, we use the notation $dor_k(a)$ to denote the largest integer r (or possibly ∞) such that for every r-coloring of the positive integers there is a monochromatic k-term progression of the form

$$\{x, ax+d, ax+2d, \ldots, ax+(k-1)d\}. \tag{5.5}$$

Noting that for $a = 1$ we are simply dealing with arithmetic progressions, by van der Waerden's theorem $dor_k(1) = \infty$ for each k. It is also known that $2 \le dor(2,2) \le 5$ and $2 \le dor(3,3) \le 5$ (see Table 5.2 at the end of this section), i.e, that $2 \le dor_3(a) \le 5$ if $a = 2$ or $a = 3$. In contrast to van der Waerden's result, the following theorem shows that for all $a \ne 1$, and large enough k, $dor_k(a) < \infty$ (in fact, the degree of regularity is not greater than three).

Theorem 5.12. *For all $a \ge 2$ and all $k \ge \frac{a^2}{a+1} + 2$, $dor_k(a) \le 3$*

Proof. To prove the theorem, it suffices to show that if a and k satisfy the given hypotheses, then there exists a 4-coloring of $\mathbb{Z}^+$ that avoids monochromatic k-term sequences having the form of (5.5). Clearly, we may assume $k = \lceil \frac{a^2}{a+1} \rceil + 2$.

Define χ to be the 4-coloring of the positive integers defined by coloring each of the intervals $[1, a-1]$, $[a, a^2-1]$, $[a^2, a^3-1]$, ... as follows: $\chi([a^j, a^{j+1}-1]) = \bar{\jmath}$, where $j \equiv \bar{\jmath} \pmod 4$ and $0 \le \bar{\jmath} \le 3$. We will complete the proof by showing that if $\chi(x) = \chi(ax+d)$, then $\chi(ax + (k-1)d) \ne \chi(x)$.

Assume that x and $ax+d$ have the same color under χ, and let i be the integer such that $x \in [a^i, a^{i+1})$. Obviously, $ax + d \notin [a^i, a^{i+1})$. Hence, by the way χ is defined, there is some $m \in \mathbb{Z}^+$ such that $ax + d \in [a^{i+4m}, a^{i+4m+1})$. From this it follows that

$$a^i(a^{4m} - a^2) \le d \le a^{i+1}(a^{4m} - 1). \tag{5.6}$$

Note that, by the way m and χ are defined, if we can show that

$$a^{i+4m+1} \le ax + (k-1)d < a^{i+4(m+1)}, \tag{5.7}$$

then $ax + (k-1)d$ must be colored differently from x and $ax + d$, thereby completing the proof. Hence, we proceed to prove (5.7).

To prove $ax + (k-1)d < a^{i+4(m+1)}$, first note that $k < a^3 + 1$ for all $a \geq 2$. Hence, $1 + (k-2)(1 - a^{-4m}) < a^3$, and therefore

$$a^{i+4m+1} + (k-2)a^{i+1}(a^{4m} - 1) < a^{i+4(m+1)}. \tag{5.8}$$

By (5.6), we have

$$\begin{aligned} ax + (k-1)d &= ax + d + (k-2)d \\ &\leq a^{i+4m+1} + (k-2)a^{i+1}(a^{4m} - 1). \end{aligned}$$

This, together with (5.8), implies $ax+(k-1)d < a^{i+4(m+1)}$, as desired.

To complete the proof, we show that $a^{i+4m+1} \leq ax + (k-1)d$. Since $k \geq \frac{a^2}{a+1} + 2$, we have $(k-2)(a^2-1) \geq a^3 - a^2$, and therefore $(k-2)(a^2 - a^{-4(m-1)}) \geq a^3 - a^2$. From this we know that

$$a^{i+4m} + (k-2)a^i(a^{4m} - a^2) \geq a^{i+4m+1}. \tag{5.9}$$

Also, from (5.6), we have

$$ax + (k-1)d \geq a^{i+4m} + (k-2)a^i(a^{4m} - a^2). \tag{5.10}$$

By (5.9) and (5.10) we have $a^{i+4m+1} \leq ax + (k-1)d$, and the proof is complete. □

The following two tables give the known values and lower bounds for $T(a,b;2)$, and the known degrees of regularity for some small a and b, respectively.

$a \setminus b$	1	2	3	4	5	6	7
1	9	∞	39	58	81	≥ 108	≥ 139
2		16	46	∞	139	≥ 106	≥ 133
3			39	60	114	∞	≥ 135
4				40	87	≥ 124	≥ 214
5					70	100	≥ 150
6						78	≥ 105
7							95

Table 5.1: Values and lower bounds for $T(a,b;2)$

(a,b)	$dor(a,b)$	(a,b)	$dor(a,b)$
(1,1)	∞	(2,5)	$2-3$
(1,2)	1	(2,6)	$2-3$
(1,3)	$2-3$	(2,7)	$2-4$
(1,4)	$2-4$	(2,8)	$2-4$
(1,5)	$2-5$	(2,9)	$2-5$
(1,6)	$2-6$	(3,3)	$2-5$
(1,7)	$2-6$	(3,4)	$2-5$
(1,8)	$2-6$	(3,5)	2
(1,9)	$2-7$	(3,6)	1
(2,2)	$2-5$	(3,7)	$2-4$
(2,3)	2	(3,8)	$2-3$
(2,4)	1	(3,9)	$2-3$

Table 5.2: Degree of regularity of (a,b)-triples

5.2. Homothetic Copies of Sequences

Definition 5.13. Let $s_1, s_2, \ldots, s_{k-1}$ be positive integers. A *homothetic copy* of the k-tuple $(1, 1+s_1, 1+s_1+s_2, \ldots, 1+\sum_{i=1}^{k-1} s_i)$ is a k-tuple $(a, a+bs_1, a+b(s_1+s_2), \ldots, a+b\sum_{i=1}^{k-1} s_i)$, where a and b are any positive integers.

How are homothetic copies related to arithmetic progressions? Well, consider the collection of all homothetic copies of $(1,2,\ldots,k)$, i.e., where $s_i = 1$ for all i in Definition 5.13. Then this is the collection of all sequences of the form $\{a, a+b, a+2b, \ldots, a+(k-1)b\}$; in other words, the family of all k-term arithmetic progressions.

For fixed $s_1, s_2, \ldots, s_{k-1}$, we shall denote by $H(s_1, s_2, \ldots, s_{k-1})$ the 2-color Ramsey-type function associated with the family of all homothetic copies of $(1, 1+s_1, \ldots, 1+\sum_{i=1}^{k-1} s_i)$. Hence, the classical van der Waerden function $w(k)$ has the same meaning as $H(1,1,\ldots,1)$. It follows easily from van der Waerden's theorem that for all k and all $(k-1)$-tuples $(s_1, s_2, \ldots, s_{k-1})$, the number $H(s_1, s_2, \ldots, s_{k-1})$ exists (this is left to the reader as Exercise 5.6).

We know that $H(1,1) = w(3) = 9$. We now examine the function $H(s,t)$ for general pairs (s,t). That is, for a fixed pair (s,t), we want the least positive integer $h = H(s,t)$ such that for every 2-coloring of

$[1,h]$ there is a monochromatic triple of the form $(a, a+bs, a+bs+bt)$ for some positive integers a and b. As we will see, this is a Ramsey-type function for which we are able to come rather close to providing a precise formula.

We begin with an upper bound for $H(s,t)$. We shall make use of the following lemmas. We leave the proof of Lemma 5.14 as Exercise 5.7.

Lemma 5.14. *For all $s,t \geq 1$, $H(s,t) = H(t,s)$.*

For convenience, from now on we will refer to a homothetic copy of $\{1, 1+s, 1+s+t\}$ as an (s,t)-progression.

Lemma 5.15. *Let s,t,c be positive integers. Then*

$$H(cs, ct) = c(H(s,t) - 1) + 1.$$

Proof. Let $m = H(s,t)$. By the definition of $H(s,t)$, every 2-coloring of $[0, m-1]$ yields a monochromatic (s,t)-progression. Hence, every 2-coloring of $\{0, c, 2c, \ldots, (m-1)c\}$ yields a monochromatic (cs, ct)-progression, giving $c(m-1)+1$ as an upper bound for $H(cs, ct)$.

To prove the reverse inequality, first note that there exists a 2-coloring χ of $[1, m-1]$ that yields no monochromatic (s,t)-progression. Now define χ' on $[1, c(m-1)]$ by

$$\chi'([c(i-1)+1, ci]) = \chi(i),$$

for $1 \leq i \leq m-1$. To complete the proof, we will show that χ' avoids monochromatic (cs, ct)-progressions.

Assume, for a contradiction, that $\{x_1 < x_2 < x_3\} \subseteq [1, c(m-1)]$ is a (cs, ct)-progression that is monochromatic under χ'. Then there exists $u > 0$ such that $x_3 - x_2 = uct$ and $x_2 - x_1 = ucs$. For $j = 1,2,3$, let $y_j = \lceil \frac{x_j}{c} \rceil$. Then

$$y_3 - y_2 = \left\lceil \frac{x_3}{c} \right\rceil - \left\lceil \frac{x_2}{c} \right\rceil = ut.$$

Similarly, $y_2 - y_1 = us$.

Hence $\{y_1 < y_2 < y_3\}$ is an (s,t)-progression. Furthermore, $\chi(y_j) = \chi\left(\lceil \frac{x_j}{c} \rceil\right) = \chi'(x_j)$ for each j. This contradicts our assumption that there is no monochromatic (s,t)-progression under χ. □

Theorem 5.16. *For all $s,t \geq 1$, $H(s,t) \leq 4(s+t)+1$.*

Proof. By Lemma 5.14, we may assume that $s \leq t$. We may also assume that $\gcd(s,t) = 1$. To justify this last statement, assume that the theorem is true for all pairs (s_1, t_1) such that $\gcd(s_1, t_1) = 1$. Let $d = \gcd(s,t)$. Then by Lemma 5.15, we have

$$\begin{aligned} H(s,t) &= d\left(H\left(\tfrac{s}{d}, \tfrac{t}{d}\right) - 1\right) + 1 \\ &\leq d\left(4\left(\tfrac{s}{d} + \tfrac{t}{d}\right) + 1 - 1\right) + 1 \\ &= 4(s+t) + 1. \end{aligned}$$

We noted before that $H(1,1) = 9$. We may also calculate directly that $H(1,2) = 13$ and $H(1,3) = 17$, so that the theorem holds for the pairs (1,1), (1,2), and (1,3). Now assume $\gcd(s,t) = 1$, $s \leq t$, and $(s,t) \notin \{(1,1),(1,2),(1,3)\}$. Consider the following collection of subsets of [1,15]:

$$\begin{aligned} \mathcal{C} = \; &\{\{1,2,6\},\{2,3,7\},\{3,4,8\},\{4,5,9\},\{1,3,10\},\{2,4,11\}, \\ &\{3,5,12\},\{1,4,13\},\{2,5,14\},\{1,5,15\},\{6,7,10\},\{7,8,11\}, \\ &\{8,9,12\},\{6,8,13\},\{7,9,14\},\{6,9,15\},\{10,11,13\}, \\ &\{11,12,14\},\{10,12,15\},\{13,14,15\}\}. \end{aligned}$$

We leave it to the reader (Exercise 5.10a) to check that every 2-coloring of [1,15] yields a monochromatic triple from this list of twenty triples in $\mathcal{C}$.

We now make a one-to-one correspondence between the members of $\mathcal{C}$ and twenty different (s,t)-triples that are in $[1, 4(s+t)+1]$ by means of the following associations (that the twenty resulting triples of elements of $[1, 4(s+t)+1]$ are, in fact, distinct (s,t)-triples is left to the reader as Exercise 5.10b):

$1 \leftrightarrow 1$	$2 \leftrightarrow s+1$	$3 \leftrightarrow 2s+1$
$4 \leftrightarrow 3s+1$	$5 \leftrightarrow 4s+1$	$6 \leftrightarrow s+t+1$
$7 \leftrightarrow 2s+t+1$	$8 \leftrightarrow 3s+t+1$	$9 \leftrightarrow 4s+t+1$
$10 \leftrightarrow 2s+2t+1$	$11 \leftrightarrow 3s+2t+1$	$12 \leftrightarrow 4s+4t+1$
$13 \leftrightarrow 3s+3t+1$	$14 \leftrightarrow 4s+3t+1$	$15 \leftrightarrow 4s+4t+1.$

Since every 2-coloring of [1,15] admits a monochromatic member of $\mathcal{C}$, it is clear that every 2-coloring of $[1, 4(s+t)+1]$ must admit a corresponding monochromatic triple from among the twenty (s,t)-triples that result from the above association scheme. This yields the desired upper bound on $H(s,t)$. $\square$

As it turns out, the upper bound given in Theorem 5.16 is known to be the actual value of $H(s,t)$ for a rather broad class of pairs (s,t), although an exact formula that holds for all cases is still unknown. The next theorem provides part of the answer.

Theorem 5.17. *Let $s \le t$ be positive integers. If neither $\frac{s}{\gcd(s,t)}$ nor $\frac{t}{\gcd(s,t)}$ is divisible by 4, then $H(s,t) = 4(s+t)+1$.*

Proof. By Theorem 5.16, we see that it is sufficient to show that $H(s,t) \ge 4(s+t)+1$. We shall split the problem into two cases, in each one providing a specific 2-coloring of $[1, 4(s+t)]$ that avoids monochromatic (s,t)-progressions.

Let $d = \gcd(s,t)$. Of course, not both $\frac{s}{d}$ and $\frac{t}{d}$ are even.

Case 1. $\frac{s}{d}$ and $\frac{t}{d}$ are both odd. As demonstrated in the proof of Theorem 5.16, we may assume $d = 1$. Color $[1, 4(s+t)]$ according to the string

$$\underbrace{1010\ldots10}_{2(s+t)}\underbrace{0101\ldots01}_{2(s+t)}.$$

Assume, for a contradiction, that $\{x < y < z\}$ is a monochromatic (s,t)-progression. Then there exists a positive integer b such that $y = x + bs$ and $z = y + bt$. Let B_1 and B_2 represent the intervals $[1, 2(s+t)]$ and $[2(s+t)+1, 4(s+t)]$, respectively.

In case b is odd, then (since we are under the assumption that s is odd), x and y have different parities (one is even and the other odd). Similarly, y and z have different parities. Since x and y have the same color, yet opposite parity, it is evident from the way the coloring is defined that $x \in B_1$ and $y \in B_2$. Hence $z \in B_2$, from which it follows that y and z cannot have the same color, a contradiction.

If b is even, then x, y, and z are all of the same parity. Hence, either they all belong to B_1 or they all belong to B_2. Thus,

$$b(s+t) = z - x \le 2(s+t),$$

which means that $b = 1$, a contradiction.

Case 2. One of $\frac{s}{d}$ and $\frac{t}{d}$ is even. By the hypotheses, and without loss of generality, we assume that $\frac{s}{d} \equiv 2 \pmod 4$. As in Case 1, we shall assume that $d = 1$.

Let λ be the 2-coloring of $[1, 4(s+t)]$ represented by the string 1100 1100 ... 1100; that is, it consists of $s+t$ consecutive occurrences of the string 1100.

We shall show, by contradiction, that λ avoids monochromatic (s,t)-progressions. Thus, assume that $\{x < y < z\}$ is an (s,t)-progression that is monochromatic under λ. Then there exists $b \ge 1$ such that $y - x = bs$ and $z - y = bt$. Now, since

$$z - x = b(s+t) \le 4(s+t) - 1,$$

we have that $b \le 3$.

First assume $b = 2$. Then $z - x = b(s+t)$ is even. Note that by the definition of λ, we know that the only way that two integers i and j can have an even difference and be of the same color is for $j - i$ to be divisible by 4. Hence 4 divides $z - x$. However, by our assumptions of Case 2, this is impossible since exactly one of s and t is even. Now assume that $b = 1$ or $b = 3$. Because $s \equiv 2 \pmod 4$, $y - x$ is even but is not divisible by 4; this is also impossible, since bs is not a multiple of 4. $\square$

By Lemmas 5.14 and 5.15, and Theorem 5.17, we see that we would know all values of $H(s,t)$ if we knew the value of $H(4j,t)$ when t is odd and $\gcd(j,t) = 1$. For many such pairs it is known that $H(s,t) = 4(s+t)+1$. Meanwhile, it has been conjectured that $H(4j,1) = 4(4j+1)$ for all $m \ge 1$, and it has been proven that $4(4j+1) \le H(4j,1) \le 4(4j+1)+1$ for all positive integers j.

5.3. Sequences of Type $x, x+d, x+2d+b$

One simple way to form a generalization of an arithmetic progression is expressed in the following definition.

Definition 5.18. Let $b \geq 0$. A k-term *augmented progression with tail* b is a sequence of the form

$$\{x, x+d, x+2d, \ldots, x+(k-2)d, x+(k-1)d+b\}$$

for any $x, d \in \mathbb{Z}^+$.

Denote the family of all augmented progressions with tail b by AUG_b. We see that this provides us with a generalization of $w(k)$, since $AUG_0 = AP$ and therefore $w(k) = R(AUG_0, k)$. We shall limit the discussion to the case of $k = 3$, so that we are interested in sequences of the form $\{x, x+d, x+2d+b\}$ and the function $R(AUG_b, 3)$. We are unaware of any significant work that has been done for $k \geq 4$ (this sounds like a research problem that is wide-open for exploration).

An obvious question we should ask is: does $R(AUG_b, 3)$ exist (in other words, is it finite?) for every b? We answer this, and more, in the following theorem.

Theorem 5.19. *For b odd, $R(AUG_b, 3)$ does not exist. For b even, $R(AUG_b, 3) \leq \lceil \frac{9}{4}b \rceil + 9$.*

Proof. First assume b is odd. Consider the coloring of the positive integers represented by the string $101010\ldots$. Then no triple of the form $\{x, x+d, x+2d+b\}$ can be monochromatic because the first and third elements of this triple differ by an odd number.

Now let b be even. We consider two cases.

Case 1. $b \equiv 0 \pmod 4$. Since $R(AUG_0, 3) = w(3) = 9$, the theorem is true when $b = 0$. Hence, it is sufficient to show that for $b \geq 4$, and $m = \lceil \frac{9}{4}b \rceil + 9$, every 2-coloring of $[1, m]$ admits a monochromatic 3-term augmented progression with tail b. Let $\chi : [1, m] \to \{0, 1\}$ be any 2-coloring and assume, by way of contradiction, that χ admits no such monochromatic set.

Let $A = \{i \in [1, m] : \chi(i) = 1\}$ and $B = \{i \in [1, m] : \chi(i) = 0\}$. By the pigeonhole principle, some 3-element subset, S, of $[1, 5]$ is monochromatic. Since there are $\binom{5}{3} = 10$ possibilities for S, we shall consider ten subcases. We will present four of the subcases here, and leave the rest to the reader as Exercise 5.12. Without loss of generality, we shall assume $S \subseteq A$.

Subcase i. $S = \{1, 2, 3\}$. Since $1, 2 \in A$, and since we are assuming that there is no 3-term monochromatic augmented progression with tail b, we must have $3+b \in B$. Likewise, $2, 3 \in A$ implies $4+b \in B$; and $1, 3 \in A$ implies $5+b \in B$. Using the same line of reasoning, we have the following sequence of implications (note that each of the integers occurring in the implications below belongs to $[1, m]$; note also that since $b \geq 4$, in each implication the 3-term augmented progression alluded to is, in fact, a set of 3 *distinct* integers):

$$\begin{aligned}
&3+b, 4+b \in B \text{ implies } 5+2b \in A,\\
&3+b, 5+b \in B \text{ implies } 7+2b \in A,\\
&1, 5+2b \in A \text{ implies } 3+\tfrac{b}{2} \in B,\\
&3, 5+2b \in A \text{ implies } 4+\tfrac{b}{2} \in B,\\
&3, 7+2b \in A \text{ implies } 5+\tfrac{b}{2} \in B,\\
&3+\tfrac{b}{2}, 4+\tfrac{b}{2}, 5+\tfrac{b}{2} \in B \text{ implies } 5+\tfrac{3b}{2}, 7+\tfrac{3b}{2} \in A,\\
&1, 5+\tfrac{3b}{2} \in A \text{ implies } 3+\tfrac{b}{4} \in B,\\
&3, 5+\tfrac{3b}{2} \in A \text{ implies } 4+\tfrac{b}{4} \in B,\\
&3, 7+\tfrac{3b}{2} \in A \text{ implies } 5+\tfrac{b}{4} \in B,\\
&3+\tfrac{b}{4}, 4+\tfrac{b}{4}, 5+\tfrac{b}{4} \in B \text{ implies } 5+\tfrac{5b}{4}, 6+\tfrac{5b}{4} \in A,\\
&5+\tfrac{5b}{4}, 6+\tfrac{5b}{4} \in A \text{ implies } 7+\tfrac{9b}{4} \in B,\\
&3+\tfrac{b}{4}, 7+\tfrac{9b}{4} \in B \text{ implies } 5+\tfrac{3b}{4} \in A,\\
&5+\tfrac{b}{4}, 7+\tfrac{9b}{4} \in B \text{ implies } 6+\tfrac{3b}{4} \in A,\\
&5+\tfrac{3b}{4}, 6+\tfrac{3b}{4} \in A \text{ implies } 7+\tfrac{7b}{4} \in B.
\end{aligned}$$

Thus $\{3+\frac{b}{4}, 5+\frac{b}{2}, 7+\frac{7b}{4}\}$, an augmented progression with tail b, is monochromatic, a contradiction.

Subcase ii. $S = \{1, 2, 4\}$. By Subcase i, we may assume $3 \in B$. Using the same idea as in the proof of Subcase i, we have the following sequence of implications:

$$\begin{aligned}
&1, 2, 4 \in A \text{ implies } 6+b, 7+b \in B,\\
&6+b, 7+b \in B \text{ implies } 8+2b \in A,\\
&2, 4, 8+2b \in A \text{ implies } 5+\tfrac{b}{2}, 6+\tfrac{b}{2} \in B,\\
&3, 5+\tfrac{b}{2} \in B \text{ implies } 7+2b \in A,
\end{aligned}$$

$1, 7+2b \in A$ implies $4+\frac{b}{2} \in B$,
$4+\frac{b}{2}, 5+\frac{b}{2}, 6+\frac{b}{2} \in B$ implies $6+\frac{3b}{2}, 8+\frac{3b}{2} \in A$,
$2, 4, 6+\frac{3b}{2}, 8+\frac{3b}{2} \in A$ implies $4+\frac{b}{4}, 5+\frac{b}{4}, 6+\frac{b}{4} \in B$,
$4+\frac{b}{4}, 5+\frac{b}{4}, 6+\frac{b}{4} \in B$ implies $6+\frac{5b}{4}, 7+\frac{5b}{4} \in A$,
$6+\frac{5b}{4}, 7+\frac{5b}{4} \in A$ implies $8+\frac{9b}{4} \in B$,
$4+\frac{b}{4}, 6+\frac{b}{4}, 8+\frac{9b}{4} \in B$ implies $6+\frac{3b}{4}, 7+\frac{3b}{4} \in A$,
$6+\frac{3b}{4}, 7+\frac{3b}{4} \in A$ implies $8+\frac{7b}{4} \in B$.

Thus $\{4+\frac{b}{4}, 6+\frac{b}{2}, 8+\frac{7b}{4}\}$, an augmented progression with tail b, is monochromatic, a contradiction.

`Subcase iii.` $S = \{1,2,5\}$. By Subcases i and ii, we may assume that $3, 4 \in B$. By the method used in those subcases, it can be shown that there exists a monochromatic augmented progression with tail b whose largest element does not exceed $\max\{8+\frac{9b}{4}, 10+2b\} \leq m$. We leave the details to the reader as Exercise 5.11.

`Subcase iv.` $S = \{2,3,4\}$. That the result holds for this case is a consequence of Subcase i, by a simple translation of length 1 (see Proposition 2.30), because the proof of Subcase i yields a monochromatic augmented progression with tail b that is contained in $[1, m-1]$.

`Case 2.` $b \equiv 2 \pmod 4$. We consider the same ten subcases as for Case 1. As the proofs are quite similar to those of Case 1, we present here only one subcase, and leave the the proofs of the other subcases to the reader.

`Subcase i.` $S = \{1,3,5\}$. Then $5+b, 7+b \in B$. Thus, $9+2b \in A$. We then have the following sequence of implications:

$3, 5, 9+2b \in A$ implies $6+\frac{b}{2}, 7+\frac{b}{2} \in B$,
$6+\frac{b}{2}, 7+\frac{b}{2} \in B$ implies $8+\frac{3b}{2} \in A$,
$1, 3, 5, 8+\frac{3b}{2} \in A$ implies $\frac{9}{2}+\frac{b}{4}, \frac{11}{2}+\frac{b}{4}, \frac{13}{2}+\frac{b}{4} \in B$,
$\frac{9}{2}+\frac{b}{4}, \frac{11}{2}+\frac{b}{4}, \frac{13}{2}+\frac{b}{4} \in B$ implies $\frac{13}{2}+\frac{5b}{4}, \frac{15}{2}+\frac{5b}{4} \in A$,
$\frac{13}{2}+\frac{5b}{4}, \frac{15}{2}+\frac{5b}{4} \in A$ implies $\frac{17}{2}+\frac{9b}{4} \in B$,
$\frac{9}{2}+\frac{b}{4}, \frac{13}{2}+\frac{b}{4}, \frac{17}{2}+\frac{9b}{4} \in B$ implies $\frac{13}{2}+\frac{3b}{4}, \frac{15}{2}+\frac{3b}{4} \in A$,
$\frac{13}{2}+\frac{3b}{4}, \frac{15}{2}+\frac{3b}{4} \in A$ implies $\frac{17}{2}+\frac{7b}{4} \in B$.

Then the augmented progression $\{\frac{11}{2}+\frac{b}{4}, 7+\frac{b}{2}, \frac{17}{2}+\frac{7b}{4}\}$ is contained in B; this is a contradiction since all of the 3-term augmented progressions occurring in the argument are contained in $[1, m]$. □

We next provide a lower bound for $R(AUG_b, 3)$ (for b even). Note that this lower bound agrees precisely with all known values of $R(AUG_b, 3)$ (see Table 5.3 at the end of this section).

Theorem 5.20. *For $b = 2$ and for all even $b \geq 10$,*

$$R(AUG_b, 3) \geq 2b + 10.$$

For $b \in \{0, 4, 6, 8\}$, $R(AUG_b, 3) \geq 2b + 9$.

Proof. The coloring 1100011110000 avoids 3-term mono-chromatic augmented progressions with tail 2, so that $R(AUG_2, 3) \geq 14$. For $b \in \{0, 4, 6, 8\}$, let s_b denote the alternating string $1010\ldots101$ of length $b+3$. It is easy to check that the coloring defined by the string $s_b 00 s_b$, which has length $2b+8$, avoids monochromatic 3-term augmented progressions with tail b.

Now let $b \geq 10$ be even. Define the coloring χ of $[1, 2b+9]$ as follows. Let

$$\begin{aligned}
B_1 &= \{1, 2\},\\
B_2 &= \{3, 4, 5\},\\
B_3 &= [6, b+2],\\
B_4 &= \{b+3\},\\
B_5 &= [b+4, b+7],\\
B_6 &= [b+8, 2b+9].
\end{aligned}$$

Let $\chi(B_1) = \chi(B_3) = \chi(B_5) = 1$ and $\chi(B_2) = \chi(B_4) = \chi(B_6) = 0$. We assume that $P = \{x < y < z\}$ is a monochromatic augmented progression with tail b, and seek a contradiction. Note that no single B_i can contain all elements of P, since $z - x \geq b+2$.

First assume $\chi(P) = 1$. It is clear that $y \neq 2$, since otherwise $z = b+3$, which is of a different color. Thus, if $x \in B_1$, then $y \geq 6$; but then $z \in B_6$, which is not possible. If $x \in B_3$, then again we have $z \in B_6$.

Now assume $\chi(P) = 0$. If $x, y \in B_2$, it then follows that $z \in B_5$, a contradiction. If $x \in B_2$ and $y \notin B_2$, then $z \geq 3b+1 > 2b+9$, which is impossible. Finally, if $x = b+3$, then z is again outside of $[1, 2b+9]$. □

From the discussion above we know that $R(AUG_b, 3; 2)$ exists for every even b (that is, when two colors are used). What happens when we increase the number of colors? As the next theorem shows, the situation is different when $r = 3$, and quite different when $r \geq 4$.

Theorem 5.21. *If $b \not\equiv 0 \pmod 6$, then $R(AUG_b, 3; 3)$ does not exist. Furthermore, for all b, $R(AUG_b, 3; 4)$ does not exist.*

Proof. If b is odd, then this is covered by Theorem 5.19. Hence, we let b be even but not divisible by 6. Assume, for a contradiction, that $R(AUG_b, 3; 3)$ exists. Color the positive integers according to the string 012012012... . Then any monochromatic augmented progression $\{x, x+d, x+2d+b\}$ must have d a multiple of 3; but then b must also be a multiple of 3. Since b is even, it must be a multiple of 6, a contradiction. This proves the first statement.

To see that the second statement holds, color the positive integers with the coloring $\chi = 1^k2^k3^k4^k1^k2^k3^k4^k \ldots$. By the case for three colors, we assume that 6 divides b. Let $(x, y, z) = (x, x+d, x+2d+b)$; we will show that (x, y, z) is not monochromatic. Note that if $s < t$, then $\chi(s) = \chi(t)$ if and only if, for some positive integer i, we have $4ki - (k-1) \leq t - s \leq 4ki + (k-1)$. However, $z - y = d + b$ is in

$$[4kj-(k-1)+6k, 4kj+(k-1)+6k] = [4k(j+1)+k+1, 4k(j+1)+3k-1]$$

for some positive integer j. Hence $\chi(y) \neq \chi(z)$. $\square$

We mention, without proof, that it is known that when b is a multiple of 6, $R(AUG_b, 3; 3) \leq \frac{55}{6}b + 1$, so that $R(AUG_b, 3; 3)$ exists if and only if $b \equiv 0 \pmod 6$.

Recall that in Chapter 4 we discussed a generalization of $w(k)$, denoted $w'(c, k)$, which pertained to arithmetic progressions whose gaps were no less than c. In the same way, we may generalize the function $R(AUG_b, k)$ by defining, for each positive integer c, $R(c, AUG_b, k)$ to be the least positive integer m such that for every 2-coloring of $[1, m]$ there is a monochromatic augmented triple $\{x, x+d, x+2d\}$ with the added restriction that $d \geq c$. Since $R(c, AUG_0, k)$ has the same meaning as $w'(c, k)$, we know by Theorem 4.11 that $R(c, AUG_0, 3) = 8c+1$. By modifying the proofs of Theorems 5.19 and 5.20, it is not all that difficult (although it is a bit tedious) to obtain generalizations

of those two theorems, where the parameter c is included. We do not present the proofs of these generalizations here, but do state the results in the next two theorems. The proof of Theorem 5.22 imitates the proof of Theorem 5.19, except instead of considering the possible colorings of $[1, 5]$, one considers the possible colorings of $\{1, 1+c, 1+2c, 1+3c, 1+4c\}$. Notice that Theorems 5.19 and 5.20 are, indeed, special cases of Theorems 5.22 and 5.23.

Theorem 5.22. *Let $b \geq 0$ be even, $c \geq 1$, and $a = 2b + 9c$. Then*

$$R(c, AUG_b, 3) \leq \begin{cases} \max\left\{\frac{9b}{4} + 8c + 1, a+1\right\} & \text{if } 4|b, \\ \max\left\{\frac{9b}{4} + \frac{17c}{2} + 1, a+1\right\} & \text{if } 4 \nmid b \text{ and } c \text{ is odd}, \\ \max\left\{\frac{9b}{4} + \frac{17c}{2} + \frac{19}{2}, a+10\right\} & \text{if } 4 \nmid b \text{ and } c \text{ is even}. \end{cases}$$

Theorem 5.23. *Let $b \geq 2$ be even, and let $c \geq 1$. Then*

$$R(c, AUG_b, 3) \geq \begin{cases} 2b + 7c + 3 & \text{if } b \geq 6c + 3, \\ 2b + 7c + 2 & \text{if } c + 2 \leq b \leq 6c + 2, \\ b + 8c + 4 & \text{if } 2 \leq b \leq c + 1. \end{cases}$$

The following table gives the known values of $R(c, AUG_b, 3)$. Note that $R(AUG_b, 3) = R(1, AUG_b, 3)$.

$b \setminus c$	1	2	3	4	5	6	7	8	9
0	9	17	25	33	41	49	57	65	73
2	14	22	30	38	46	54	62	70	78
4	17	24	32	40	48	56	64	72	80
6	21	28	35	42	50	58	66	74	?
8	25	32	39	46	53	60	68	?	?
10	30	36	43	50	57	64	71	?	?
12	34	40	47	54	61	68	?	?	?
14	38	44	51	58	65	?	?	?	?
16	42	49	55	62	69	?	?	?	?
18	46	53	59	66	?	?	?	?	?
20	50	57	63	?	?	?	?	?	?
22	54	61	?	?	?	?	?	?	?
24	58	?	?	?	?	?	?	?	?
26	62	?	?	?	?	?	?	?	?

Table 5.3: Values of $R(c, AUG_b, 3)$

5.4. Exercises

5.1 How many $(1,3)$-triples are contained in [1,100]? How many (a,b)-triples are contained in [1,100]? How many are contained in $[1,n]$?

5.2 Prove Theorem 5.3 for the case in which a is odd.

5.3 At the end of the proof of Theorem 5.6 it is claimed that (α,β,γ) is an (a,b)-triple that resides in $[1,m]$. Verify this.

5.4 Complete the proof of Theorem 5.7 by showing that the coloring of $[1,3b^2+5b-4a+3]=[1,m]$ defined by coloring $[b+2,b^2+2b+1]$ with color 1, and its complement in $[1,m]$ with color 0, avoids monochromatic (a,b)-triples.

5.5 By Theorem 5.7, we know that $T(1,b;2)\geq 2b^2+5b+2$ for $b\geq 3$. Show that this lower bound can be tightened to $T(1,b;2)\geq 2b^2+5b+6$ (hint: color

$$S=[1,b+1]\cup\{b+3\}\cup[b^2+2b+4,2b^2+5b+5]$$

with the color 1, and its complement in $[1,2b^2+5b+5]$ with the color 0).

5.6 Use van der Waerden's theorem to prove that for all k and all $(k-1)$-tuples $(s_1,s_2,\ldots,s_{k-1})$ of positive integers, the number $H(s_1,s_2,\ldots,s_{k-1})$ exists (is finite).

5.7 Prove Lemma 5.14.

5.8 Assume that $s_1,s_2,\ldots,s_k$, and c are positive integers and that $H(s_1,s_2,\ldots,s_k)=m$. Prove that

$$H(cs_1,cs_2,\ldots,cs_k)=c(m-1)+1.$$

5.9 Verify that $H(1,2)=13$ and $H(1,3)=17$.

5.10 a) Show that every 2-coloring of [1,15] yields a monochromatic member of the family $\mathcal{C}$ as defined in the proof of Theorem 5.16.
b) Show that the 20 triples of elements of $[1,4(s+t)+1]$ that correspond to the 20 members of $\mathcal{C}$ (via the correspondence between elements of [1,15] and elements of $[1,4(s+t)+1]$ given in the proof of Theorem 5.16) are distinct (s,t)-triples.

5.11 Complete the proof of Subcase iii of Case 1 of Theorem 5.19.

5.12 Prove the subcases of the proof of Theorem 5.19 that were not done in the text.

5.13 For given nonnegative integers a and b, define $\hat{f}(a,b)$ to be the least positive integer such that for every 2-coloring of $[1,\hat{f}(a,b)]$ there is a monochromatic set $\{x,x+d+a,x+2d+b\}$ for some positive integers x and d. Prove that if $b\geq a$, then $\hat{f}(a,b)=\hat{f}(b-a,b)$.

5.14 Using the notation of Exercise 5.13, prove that if $b\geq 2a$, then $\hat{f}(a,b)=R(a+1,AUG_{b-2a},3)$.

5.15 Use Exercises 5.13 and 5.14 to show that if $\frac{b}{a}<a\leq b$, then $\hat{f}(a,b)=R(b-a+1,AUG_{2a-b},3)$.

5.5. Research Problems

5.1 Improve the known bounds for $T(a,a;2)$.
Reference: [**175**]

5.2 Improve the known bounds for $T(a,2a-1;2)$.
Reference: [**175**]

5.3 Improve the known bounds for $T(1,b;2)$.
Reference: [**175**]

5.4 Prove or disprove: (1,1) is the only pair (a,b) such that $dor(a,b)=\infty$.
Reference: [**175**]

5.5 Find the degree of regularity of some pair (a,b) for which the degree of regularity is still unknown (see Table 5.2).
Reference: [**175**]

5.6 Extend the study of $T(a,b;2)$ to $T(a,b,c;2)$ by considering 4-tuples of the form $(x,ax+d,bx+2d,cx+3d)$. Extend it to $T(a_1,a_2,\ldots,a_{k-1};2)$.
Reference: [**175**]

5.7 Let $a>1$ and $r>3$. Define $\tau=\tau(a;r)$ to be the least positive integer such that $dor_\tau(a)\leq r$. Prove or disprove: there exists $s>r$ such that $\tau(a;s)<\tau(a;r)$.
Reference: [**175**]

5.8 Investigate the function $H(s, t, u)$, i.e., for homothetic copies of 4-term sequences.
Reference: [**66**]

5.9 Investigate the function $H(s, t; 3)$; that is, where three colors are used instead of two.
Reference: [**66**]

5.10 Prove or disprove the conjecture that $H(4j, 1) = 4(4j+1)$ for all $j \geq 1$.
Reference: [**66**]

5.11 The only cases for which the exact value of $H(s, t)$ have not been determined are those pairs $(s, t) = (t + e, t)$ such that $0 < e < t < 2e$ and either $t + e$ or t is a multiple of 4. Determine the exact value of $H(s, t)$ for these cases.
Reference: [**66**]

5.12 Investigate the function $R(AUG_b, k)$ for $k > 3$; in particular, consider sequences of the form $\{x, x + d, x + 2d, x + 3d + b\}$.
References: [**39**], [**165**]

5.13 Let $\hat{f}(a, b)$, $a \leq b$, be defined as in Exercise 5.13. Prove or disprove the conjecture that for any fixed even value of b, the maximum of $\hat{f}(a, b)$ occurs when $a = \frac{b}{2}$.
Reference: [**165**]

5.14 Study the Ramsey properties for sequences of the form

$$\{x, x + d, x + 2d + b_1, x + 3d + b_2\}.$$

References: [**39**], [**165**]

5.15 Improve the upper bound for $R(AUG_b, 3; 3)$. Find a lower bound for this function.
References: [**39**], [**165**]

5.6. References

§**5.1**. The results of this section may be found in [**175**]. That article also contains other results on $T(a, b; 2)$, including a quadratic upper bound for $T(a, a; 2)$. It also mentions a result on an extension to k-tuples that is analogous to a result on arithmetic progressions from [**127**], as well as a conjecture on $dor_k(a)$.

§**5.2**. The work on homothetic copies is from [**66**], which contains a more thorough discussion. In [**170**] and [**172**], Ramsey-type functions are considered for the collection of sequences that, for some $t \geq 1$, are homothetic copies of the k-tuple

$$\{1, 2, 2 + t, 2 + t + t^2, \ldots, 2 + t + t^2 + \cdots + t^{k-2}\}.$$

§**5.3**. The function $R(AUG_b, 3; r)$ was first considered in [**39**], where it is shown that $R(AUG_b, 3)$ does not exist for b odd, and that, for b even, $R(AUG_b, 3) \leq \frac{13}{2}b + 1$. The authors of [**39**] also provide proofs of Theorems 5.20 and 5.21, and show that $R(AUG_b, 3; 3) \leq \frac{55}{6}b + 1$ for $b \equiv 0 \pmod 6$. Proofs of Theorems 5.22 and 5.23 are found in [**165**], for which Theorems 5.19 and 5.20 are special cases. That paper also investigates the function $\hat{f}$ (see Exercises 5.13–5.15), and its relationship to the function $R(AUG_b, k)$.

Additional References: An important result, known as the Hales-Jewett theorem [**136**], is a generalization of van der Waerden's theorem.

Chapter 6

Arithmetic Progressions (mod m)

An arithmetic progression is a sequence in which the gaps between successive terms are all equal to some positive integer d. In this chapter we shall consider sequences that are analogous to arithmetic progressions but where, instead of all of the gaps being equal integers, they are all congruent modulo m, where m is some prescribed integer. Another way to think of this analogy is that the gaps of an arithmetic progression are equal elements of $\mathbb{Z}$, whereas the gaps of one of the integer sequences discussed here, although possibly non-identical integers, are equal when considered as elements of the additive group $\mathbb{Z}_m$. We will discover some rather interesting Ramsey properties in this setting.

We begin with some basic definitions and notation.

Definition 6.1. Let $m \geq 2$ and $0 \leq a < m$. A k-term *a(mod m)-progression* is a sequence of positive integers $\{x_1 < x_2 < \cdots < x_k\}$ such that $x_i - x_{i-1} \equiv a \pmod{m}$ for $2 \leq i \leq k$.

For fixed m and a, denote the family of all $a \pmod{m}$-progressions by $AP_{a(m)}$.

Definition 6.2. Let $m \geq 2$. An *arithmetic progression (mod m)* is a sequence that is an $a(\bmod m)$-progression for some $a \in \{1, 2, \ldots, m-1\}$.

Note that in Definition 6.2 we do not allow $a = 0$ (in the context of the congruence classes modulo m, having $a = 0$ is somewhat like having a gap of 0 in an arithmetic progression).

For a given $m \geq 2$, we denote the family of all arithmetic progressions (mod m) by $AP_{(m)}$. Obviously, for any $m \geq 2$,

$$AP_{(m)} = \bigcup_{a=1}^{m-1} AP_{a(m)}.$$

Example 6.3. The sequence $\{1, 7, 33, 44, 70\}$ is a 5-term $1 \pmod 5$-progression. The sequence $\{3, 41, 49, 67\}$ is a member of $AP_{8(10)}$ as well as a member of $AP_{3(5)}$ and $AP_{0(2)}$. Both sequences are members of $AP_{(5)}$.

6.1. The Family of Arithmetic Progressions (mod m)

Van der Waerden's theorem tells us that $R(AP, k; r) < \infty$ for all k and r; in other words, that the family of arithmetic progressions, AP, is regular. In contrast, the next theorem tells us that for every $m \geq 2$, the family of arithmetic progressions (mod m) is not regular (it is not even 2-regular). The proof makes use of elementary group theory.

Theorem 6.4. *Let $m \geq 2$ and $k > \lceil \frac{m}{2} \rceil$. Then $R(AP_{(m)}, k; 2) = \infty$.*

Proof. Let m and k be as in the statement of the theorem. It is sufficient to find a 2-coloring of $\mathbb{Z}^+$ that admits no monochromatic k-term arithmetic progression (mod m).

Let the following string represent $\chi : \mathbb{Z}^+ \to \{0, 1\}$:

$$\underbrace{11\ldots1}_{\lceil \frac{m}{2} \rceil}\underbrace{00\ldots0}_{m-\lceil \frac{m}{2} \rceil}\underbrace{11\ldots1}_{\lceil \frac{m}{2} \rceil}\underbrace{00\ldots0}_{m-\lceil \frac{m}{2} \rceil}\ldots.$$

We will prove that for each a, $1 \leq a \leq m-1$, the maximum size of a monochromatic $a \pmod m$-progression does not exceed $\left\lceil \frac{m}{2\gcd(a,m)} \right\rceil$, which is bounded above by $\lceil \frac{m}{2} \rceil$. It is clear that the theorem follows from this.

Let $a \in \{1, 2, \ldots, m-1\}$ be fixed. Let $d = \gcd(a, m)$ and let $q = \frac{m}{d}$. We will consider the integers $1, 2, \ldots, m$ as the elements of $\mathbb{Z}_m$, where $\mathbb{Z}_m$ is the m-element cyclic group of order m, under the operation of addition modulo m (note: it is more typical to call the identity element 0, but under addition modulo m, using the element m as the identity works just as well; recall that addition in $\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ is defined by $i \oplus j = \overline{i+j}$, where, for any $x \in \mathbb{Z}$, $\bar{x}$ denotes the (unique) integer such that $x \equiv \bar{x} \pmod m$ and $0 \leq \bar{x} \leq m-1$).

From elementary group theory we know that, since q is a divisor of m, there is a unique q-element cyclic subgroup H of $\mathbb{Z}_m$, with $H = \{d, 2d, \ldots, qd = m\}$. Also, since $\gcd(a, m) = d$, H is generated by a, so that

$$H = \{d, 2d, \ldots, qd\} = \left\{\bar{a}, \overline{2a}, \ldots, \overline{qa}\right\}. \tag{6.1}$$

Now assume that $X = \{x_1 < x_2 < \cdots < x_q\}$ is an arbitrary q-term a (mod m)-progression (in $\mathbb{Z}^+$). Therefore, we have, for each i, $1 \leq i \leq q-1$,

$$x_{i+1} = x_i + d_i,$$

where $d_i \equiv a \pmod m$. It follows that

$$\begin{aligned}\{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_q}\} &= \left\{\overline{x_1}, \overline{x_1 + a}, \ldots, \overline{x_1 + (q-1)a}\right\} \\ &= \overline{x_1} + H.\end{aligned}$$

Furthermore, from (6.1) we have that $Y = \overline{x_1} + H$, a subset of $[1, m]$, is an arithmetic progression with gap d and length q. Among the q members of this arithmetic progression, at most $\lceil \frac{q}{2} \rceil$ of them can belong to the interval $[1, \lceil \frac{m}{2} \rceil]$, and the same holds for the interval $[\lceil \frac{m}{2} \rceil + 1, m]$. Hence, by the way χ is defined, no more than $\lceil \frac{q}{2} \rceil$ members of Y can be monochromatic. Further, since $\chi(x_i) = \chi(\overline{x_i})$ for each i, $1 \leq i \leq q$, we see that at most $\lceil \frac{q}{2} \rceil$ members of X can form an arithmetic $a \pmod m$-progression, which gives the desired result. □

We end this section with a table.

$m \setminus k$	2	3	4	5	6	7
2	∞	∞	∞	∞	∞	∞
3	3	∞	∞	∞	∞	∞
4	3	∞	∞	∞	∞	∞
5	3	∞	∞	∞	∞	∞
6	3	9	∞	∞	∞	∞
7	3	9	∞	∞	∞	∞
8	3	9	∞	∞	∞	∞
9	3	9	22	∞	∞	∞
10	3	9	27	∞	∞	∞
11	3	9	22	∞	∞	∞
12	3	9	∞	∞	∞	∞
13	3	9	27	∞	∞	∞
14	3	9	27	∞	∞	∞
15	3	9	27	∞	∞	∞
16	3	9	27	53	∞	∞
17	3	9	27	58	∞	∞
18	3	9	28	54	≥ 97	≥ 101
19	3	9	28	∞	∞	∞
20	3	9	30	∞	∞	∞
21	3	9	32	66	∞	∞
22	3	9	35	∞	∞	∞
23	3	9	32	64	∞	∞
24	3	9	34	67	≥ 100	?
25	3	9	34	79	∞	∞
26	3	9	35	65	≥ 102	?
27	3	9	35	63	≥ 110	?
28	3	9	35	65	?	?
29	3	9	35	65	∞	∞
30	3	9	35	75	∞	∞

Table 6.1: Values and lower bounds of $R(AP_{(m)}, k; 2)$

6.2. A Seemingly Smaller Family is More Regular

It is obvious from Theorem 6.4 that any subset of $AP_{(m)}$ also fails to be 2-regular. In particular, for any fixed $a \in \{1, 2, \ldots, m-1\}$, the collection $AP_{a(m)}$ of all $a \pmod m$-progressions is not 2-regular. This is not unlike the fact that the family of all arithmetic progressions whose gaps must be the fixed integer d (in Chapter 4, we denoted this family by A_D, where $D = \{d\}$) is not 2-regular. Loosely speaking, we might say that families such as $AP_{a(m)}$ $(a \neq 0)$ or $A_{\{d\}}$ are too "small" to be 2-regular. This is not surprising, especially for $A_{\{d\}}$, since such families restrict the allowable gap (or the gap $(\bmod m)$) to just one number. Yet, it turns out that every collection of the form

$$AP_{a(m)} \cup A_{\{m\}} \tag{6.2}$$

is, in fact, 2-regular. This statement seems to run counter to our intuition, since it surely seems that the family $AP_{(m)}$ is, in some sense, much "larger" than the family of (6.2). We proceed to establish the 2-regularity of these "smaller" families.

For families of the form (6.2) in which $a = 0$, it is fairly easy to show regularity. We begin with this case. It is clear from the definitions that any member of $A_{\{m\}}$ is also a member of $AP_{0(m)}$. Hence $AP_{0(m)} \cup A_{\{m\}} = AP_{0(m)}$.

Theorem 6.5. *Let $k, r \geq 1$ and $m \geq 2$. Then*

$$R(AP_{0(m)}, k; r) = rm(k-1) + 1.$$

Proof. Consider any r-coloring of $I = [1, rm(k-1)+1]$. Exactly $r(k-1)+1$ elements of I are congruent to $1 \pmod m$. By the pigeonhole principle, at least k of these $r(k-1)+1$ integers have the same color. Since these k elements are mutually congruent modulo m, they form a $0 \pmod m$-progression. This proves that $rm(k-1)+1$ is an upper bound for $R(AP_{0(m)}, k; r)$.

To show the reverse inequality, let χ be the coloring represented by the string

$$(1^m 2^m \ldots r^m)\,(1^m 2^m \ldots r^m) \ldots (1^m 2^m \ldots r^m)$$

where the string $(1^m 2^m \dots r^m)$ appears $k-1$ consecutive times. Since χ is an r-coloring of $[1, rm(k-1)]$ that avoids k-term monochromatic $0 \pmod{m}$-progressions, the proof is complete. □

We now turn our attention to families of the form (6.2) where $a \neq 0$. For brevity, from now on, we will use the symbol $AP^*_{a(m)}$ to denote $AP_{a(m)} \cup A_{\{m\}}$. We shall obtain upper and lower bounds for $R(AP^*_{a(m)}, k)$. As it turns out, the behavior of $R(AP^*_{a(m)}, k)$ when $\frac{m}{\gcd(a,m)}$ is even is somewhat different from the behavior when $\frac{m}{\gcd(a,m)}$ is odd. Therefore, we shall handle these two cases separately.

We will establish bounds on $R(AP^*_{a(m)}, k)$ by considering a certain generalization of this Ramsey-type function. The generalization involves adding one more parameter. While it is often the case that proving a generalization of a theorem is more difficult than proving the theorem, an extra parameter can also give us some "leeway" in the proof. In this particular instance, proving the generalization appears better suited to the mode of proof than proving the less complex theorem ($k = \ell$ in Theorem 6.7 below). Also, if we can get a more sweeping result without more work, all the better. Here is the generalization of the function.

Definition 6.6. Let $k, \ell, r \geq 2$. For $1 \leq a < m$, let $R(AP^*_{a(m)}, k, \ell; r)$ denote the least positive integer n (if it exists) such that for every r-coloring of $[1, n]$ there is either a k-term monochromatic $a \pmod{m}$-progression or an ℓ-term monochromatic arithmetic progression with gap m.

As usual, $R(AP^*_{a(m)}, k, \ell; 2)$ is abbreviated by $R(AP^*_{a(m)}, k, \ell)$.

We begin with the case in which $\frac{m}{\gcd(a,m)}$ is even.

Theorem 6.7. *Let $1 \leq a < m$ and let $k \geq 2$ and $\ell \geq 3$. Assume $\frac{m}{\gcd(a,m)}$ is even. Let $c = m(k-1)(\ell-1)$ Then*

$$c+1 \leq R(AP^*_{a(m)}, k, \ell) \leq c + a(k-2) + 1. \tag{6.3}$$

Proof. To establish the lower bound, we will present a 2-coloring of $[1, n] = [1, m(k-1)(\ell-1)]$ that avoids both monochromatic k-term $a \pmod{m}$-progressions and monochromatic ℓ-term arithmetic progressions with gap m. To describe this coloring, we first partition $[1, m(k-1)(\ell-1)]$ into the $k-1$ blocks

$$B_i = [m(\ell-1)(i-1)+1, m(\ell-1)i],$$

$1 \leq i \leq k-1$. Letting $d = \gcd(a, m)$, we further partition each B_i into $\frac{m(\ell-1)}{d}$ smaller blocks:

$$B_i = \bigcup_{j=1}^{\frac{m(\ell-1)}{d}} C_{i,j},$$

where

$$C_{i,j} = [m(\ell-1)(i-1) + d(j-1) + 1, m(\ell-1)(i-1) + dj].$$

Note that each $C_{i,j}$ contains d integers and that

$$[1, n] = \bigcup_{j=1}^{\frac{m(\ell-1)}{d}} \bigcup_{i=1}^{k-1} C_{i,j}.$$

Now define the 2-coloring λ of $[1, n]$ as follows. Let $\lambda : C_{1,1} \to \{0, 1\}$ be defined arbitrarily. For $(i, j) \neq (1, 1)$, color $C_{i,j}$ by the following rule: for each $t \in [1, d]$, $\lambda\,(m(\ell-1)(i-1) + d(j-1) + t) = \lambda(t)$ if and only if i and j have the same parity.

Since $|B_i| = m(\ell-1)$ for each i, no single B_i can contain an ℓ-term arithmetic progression with gap m. So if there exists an ℓ-term monochromatic arithmetic progression with gap m, there must exist x, y, i_1, j_1, j_2 with $y - x = m$, $\lambda(x) = \lambda(y)$, $x \in C_{i_1,j_1}$, and $y \in C_{i_1+1,j_2}$. Since $2d$ divides m and $|C_{i,j}| = d$ for all i and j, $j_2 - j_1$ must be even. Therefore, by the definition of λ, $\lambda(x) \neq \lambda(y)$, a contradiction.

To complete the proof of the lower bound, we need to show that, under λ, there is no monochromatic $a \pmod{m}$-progression of length k. Assume, for a contradiction, that such a progression exists. Then for some i, B_i contains at least two terms of this progression. So assume $x, y \in B_i$, with $x < y$, $\lambda(x) = \lambda(y)$, and $y - x \equiv a (\text{mod } m)$. It follows by the definition of λ that if $y \in C_{i,j}$ and $x \in C_{i,j'}$, then $j - j'$ is even. However, since $\frac{m}{d}$ is even, $\frac{a}{d}$ is odd; this implies that $y - x = ud$ where u is odd, contradicting the fact that $j - j'$ is even. This establishes the lower bound.

For the upper bound, we use induction on k. First assume $k = 2$ and let χ be any 2-coloring of $[1, m(\ell - 1) + 1]$. Let

$$S = \{im + 1 : 1 \le i \le \ell - 1\}.$$

If S is not monochromatic, then some member of S has the same color as $m + 1 - a$, and we have a 2-term monochromatic $a \pmod{m}$-progression. If S is monochromatic, and if $\chi(1) = \chi(S)$, then we have an ℓ-term monochromatic arithmetic progression with gap m. Finally, if S is monochromatic, but $\chi(1) \neq \chi(S)$, then we must have either $\chi(m+a+1) = \chi(1)$ or $\chi(m+a+1) = \chi(m+1)$, each of which gives us a monochromatic 2-term $a \pmod{m}$-progression.

Now assume that $k \ge 2$ and that the upper bound of (6.3) holds for k and all $\ell \ge 3$. Let χ be any 2-coloring of

$$I = [1, mk(\ell - 1) + a(k - 1) + 1].$$

We wish to show that, under χ, there is either a $(k + 1)$-term monochromatic $a \pmod{m}$-progression or an ℓ-term monochromatic arithmetic progression with gap m. By the inductive hypothesis, we may assume that there is some k-term monochromatic $a \pmod{m}$-progression X contained in $[1, m(k-1)(\ell-1) + a(k-2) + 1]$ (or else we would have the desired monochromatic ℓ-term arithmetic progression). Let x_k denote the largest member of X. If any member z of $Y = \{x_k + a + im : 0 \le i \le \ell - 1\} \subseteq I$ has the same color as X, then $X \cup \{z\}$ is a monochromatic $(k+1)$-term $a \pmod{m}$-progression, as desired. If no member of Y has the same color as X, then Y is a monochromatic ℓ-term arithmetic progression with gap m, which completes the proof. □

For $k = 2$, Theorem 6.7 gives a precise formula for the associated Ramsey function, which we state as the following corollary.

Corollary 6.8. *Let $1 \le a < m$, and assume $\frac{m}{\gcd(a,m)}$ is even. Then*

$$R(AP^*_{a(m)}, 2, \ell) = m(\ell - 1) + 1.$$

Under a certain strengthening of the hypotheses of Theorem 6.7, it can be shown that the lower bound of (6.3) is the exact value of $R(AP^*_{a(m)}, k, \ell)$. The proof being rather complex, we state this result without proof.

Theorem 6.9. *Let $1 \le a < m$, where a divides m and $\frac{m}{a}$ is even. Let $k, \ell \ge 3$ with $\frac{k-2}{\ell-2} \le \frac{m}{a}$. Then*

$$R(AP^*_{a(m)}, k, \ell) = m(k - 1)(\ell - 1) + 1.$$

For $k = \ell$, we have the following result as an immediate corollary of Theorem 6.9.

Corollary 6.10. *Let $1 \le a < m$ where a divides m and $\frac{m}{a}$ is even. Then for all $k \ge 3$, $R(AP^*_{a(m)}, k) = m(k - 1)^2 + 1$.*

We now examine the situation in which $\frac{m}{\gcd(a,m)}$ is odd. In the next theorem we give an upper bound for the associated Ramsey-type function. We first mention a lemma that will be useful.

Lemma 6.11. *Let $1 \le a < m$. Let $k, \ell \ge 2$ and $c \ge 1$. Let $n = R(AP^*_{a(m)}, k, \ell; r)$. Then*

$$R(AP^*_{ca(cm)}, k, \ell; r) = c(n - 1) + 1.$$

Proof. Notice that for any $c \ge 1$, $X = \{x_1 < x_2 < \cdots < x_k\}$ is an m-a.p. if an only if $1 + cX = \{1 + cx_i : 1 \le i \le k\}$ is an arithmetic progression with gap cm. Also, X is an $a \pmod{m}$-progression if and only if $1 + cX$ is a $ca \pmod{cm}$-progression. Therefore, by the meaning of n, any r-coloring of the set $\{1, c+1, 2c+1, \ldots, (n-1)c+1\}$ must contain a k-term monochromatic $ca \pmod{cm}$-progression or an ℓ-term monochromatic arithmetic progression with gap ca. Hence, $R(AP^*_{ca(cm)}, k, \ell; r) \le c(n - 1) + 1$.

For the reverse inequality, we know there exists an r-coloring χ of $[1, n - 1]$ that avoids both monochromatic k-term $a \pmod{m}$-progressions and monochromatic ℓ-term arithmetic progressions with gap m. Now define the r-coloring χ' of $[1, c(n - 1)]$ as follows: for each j, $1 \le j \le n - 1$,

$$\chi'([c(j - 1) + 1, cj]) = \chi(j).$$

We complete the proof by showing that χ' avoids monochromatic k-term $ca \pmod{cm}$-progressions, and also avoids monochromatic ℓ-term arithmetic progressions with gap cm. Assume, for a contradiction, that $\{s_i : i = 1, 2, \ldots\}$ is a sequence of one of these types that is monochromatic with respect to χ'. Let $t_i = \lceil \frac{s_i}{c} \rceil$ for each i. Then, by the definition of χ', using the reasoning of the previous paragraph, the sequence $\{t_i : i = 1, 2, \dots\} \subseteq [1, n-1]$ is either a k-term $a \pmod m$-progression or an ℓ-term arithmetic progression with gap m that is monochromatic with respect to χ, a contradiction. □

We remark that the above proof also shows that if, for a particular triple $(k, \ell; r)$, one of the two Ramsey-type functions mentioned in the statement of Lemma 6.11 is infinite, then so is the other.

We now give an upper bound for $R(AP^*_{a(m)}, k, \ell)$ when $\frac{m}{\gcd(a,m)}$ is odd.

Theorem 6.12. *Let $1 \le a < m$ and let $k, \ell \ge 2$. Let $d = \gcd(a, m)$ and assume $\frac{m}{d}$ is odd. Then*

$$R(AP^*_{a(m)}, k, \ell) \le m\left((k-2)(\ell-1)+1\right) + ka - d + 1.$$

Proof. Note that it is sufficient to prove the result for $d = 1$. To see this, assume the theorem holds whenever $d = 1$, and let $a' = \frac{a}{d}$ and $m' = \frac{m}{d}$. Note that $\gcd(a', m') = 1$. Hence, using Lemma 6.11, we have

$$\begin{aligned} R(AP^*_{a(m)}, k, \ell) &= d\left(R(AP^*_{a'(m')}, k, \ell) - 1\right) + 1 \\ &\le d\left(m'((k-2)(\ell-1)+1) + ka' - 1\right) + 1 \\ &= m\left((k-2)(\ell-1)+1\right) + ka - d + 1. \end{aligned}$$

We complete the proof by using induction on k. For $k = 2$ we must show that for any $\ell \ge 2$, $R(AP^*_{a(m)}, 2, \ell) \le m + 2a$. Apparently, for $k = 2$ the value of ℓ is irrelevant; this is so, as we will show that $R(AP_{a(m)}, 2) \le m + 2a$. By way of contradiction, assume there exists a 2-coloring χ of $[1, m+2a]$ that avoids 2-term monochromatic $a \pmod m$-progressions. Let $\oplus$ represent the operation of addition modulo $m+2a$ in the group $\mathbb{Z}_{m+2a}$, where we take the identity element to be $m+2a$. Notice that if $m+a < i \le m+2a$, then $i \oplus a = i - m - a$, so that $\{i \oplus a, i\}$ is a 2-term $a \pmod m$-progression. Thus, by our assumption, for every $i \in [1, m+2a]$,

$$\chi(i \oplus a) \ne \chi(i). \tag{6.4}$$

Since $d = 1$, m is odd and therefore $m + 2a$ is odd. Hence, by (6.4) we must have $\chi(1 \oplus (m+2a)a) \ne \chi(1)$. However, $1 \oplus (m+2a)a = 1$, a contradiction.

Now assume that $k \ge 2$ is fixed and that the upper bound holds for k and every $\ell \ge 2$. We will show that it holds for $k+1$. Let $\ell \ge 2$ and let χ be any 2-coloring of $[1, m((k-1)(\ell-1)+1)+(k+1)a]$. By the inductive hypothesis we may assume there is a monochromatic k-term $a \pmod m$-progression X within $[1, m((k-2)(\ell-1)+1)+ka]$ (or else we have the desired ℓ-term progression). The proof is completed in the very same manner as the last part of the proof of Theorem 6.7; we leave the details to the reader as Exercise 6.8. □

We next state, without proof, the best known lower bound for $R(AP^*_{a(m)}, k, \ell)$ when $\frac{m}{\gcd(a,m)}$ is odd.

Theorem 6.13. *Let $1 \le a < m$ and assume that $\frac{m}{\gcd(a,m)}$ is odd. Then for all $k, \ell \ge 3$,*

$$R(AP^*_{a(m)}, k, \ell) \ge (k-2)(m(\ell-2)+a) + 1.$$

From Theorems 6.7, 6.12, and 6.13, the following result is immediate.

Corollary 6.14. *If $1 \le a < m$, then*

$$R(AP^*_{a(m)}, k; 2) = mk^2(1 + o(1)).$$

Note that the magnitude of the Ramsey-type function given by Corollary 6.14 is significantly smaller than that of $w(k; 2)$ (which grows *at least* exponentially – see Theorem 2.18).

6.3. The Degree of Regularity

Van der Waerden's theorem tells us that the family of arithmetic progressions is regular, i.e., the associated Ramsey-type function is finite for all k regardless of the number of colors being used. According to Corollary 6.14, for $1 \le a < m$ the family $AP^*_{a(m)}$ is 2-regular. However, unlike the family of arithmetic progressions, in most cases $AP^*_{a(m)}$ has degree of regularity only two, and never more than three. What makes this especially interesting is that the associated Ramsey function of this family, using two colors, is a much slower growing function than that of the family of arithmetic progressions. The next theorem shows that when $a \neq \frac{m}{2}$, this family is not 3-regular.

Theorem 6.15. *Let $m \geq 3$ and $1 \leq a < m$. Assume $a \neq \frac{m}{2}$. Then $R(AP^*_{a(m)}, k; 3) = \infty$ whenever $k > \lceil \frac{2m}{3} \rceil$.*

Proof. We actually prove a stronger result: for m, a, and k as in the statement of the theorem, $R(AP^*_{a(m)}, k, 2; 3) = \infty$. That is, we show that there exists a 3-coloring of $\mathbb{Z}^+$ that avoids monochromatic k-term $a \,(\mathrm{mod}\, m)$-progressions and monochromatic 2-term arithmetic progressions with gap m.

As was explained in the proof of Theorem 6.12, by Lemma 6.11 we may assume that $\gcd(a, m) = 1$. Let $s = \lceil \frac{2m}{3} \rceil$ and $t = \lceil \frac{4m}{3} \rceil$. Define $\chi : \mathbb{Z}^+ \to \{1, 2, 3\}$ to be the 3-coloring represented by the string

$$1^2 2^{t-2} 3^{2m-t} 1^2 2^{t-2} 3^{2m-t} 1^2 2^{t-2} 3^{2m-t} \dots .$$

Note that χ is periodic with period $2m$.

We next show that χ yields no 2-term monochromatic arithmetic progression with gap m. Since $m \geq 3$, we have $s < m < t$. Thus if $\overline{\jmath} \in [1, s]$, then $s+1 \leq \overline{\jmath+m} \leq 2m$, so that $\chi(j+m) \neq \chi(j)$. Likewise if $\overline{\jmath} \in [s+1, t]$, then $\overline{\jmath+m} \notin [s+1, t]$, and if $\overline{\jmath} \in [t+1, 2m]$, then $\overline{\jmath+m} \notin [t+1, 2m]$. So for every positive integer j, $\chi(j) \neq \chi(j+m)$, i.e., there is no 2-term monochromatic member of $A_{\{m\}}$.

To complete the proof, let S be any $(s+1)$-term a (mod m)-progression; we show that S is not monochromatic under χ. We see that S has the form $\{x, x+a+c_1 m, x+2a+c_2 m, \dots, x+sa+c_s m\}$ with $0 \leq c_1 \leq c_2 \leq \cdots \leq c_s$. Let

$$\overline{S} = \left\{\overline{x}, \overline{x+a+c_1 m}, \overline{x+2a+c_2 m}, \dots, \overline{x+sa+c_s m}\right\}.$$

Note that $\overline{S}$ consists of $s+1$ distinct elements modulo m, for otherwise there exist $i < j$ with $x+ia+c_i m \equiv x+ja+c_j m \,(\mathrm{mod}\, m)$; then, since $\gcd(a, m) = 1$, $i \equiv j \,(\mathrm{mod}\, m)$, contradicting the fact that $0 \leq i < j < m$. Clearly, these $s+1$ elements are also distinct modulo $2m$. By the way χ is defined, and since each of $[1, s]$, $[s+1, t]$, and $[t+1, 2m]$ contains no more than s elements, S cannot be monochromatic under χ. □

In the above proof we needed the fact that $\frac{m}{\gcd(a,m)} \geq 3$. The proof does not work if $\frac{m}{\gcd(a,m)} = 2$, i.e., if $a = \frac{m}{2}$. In this one case, $AP^*_{a(m)}$ is 3-regular. In fact, an exact formula for the associated Ramsey-type function using three colors is known. We state it, without proof, as the next theorem.

Theorem 6.16. *For all $m, k \geq 2$, if $\frac{m}{2} \in \mathbb{Z}^+$ then*

$$R(AP^*_{\frac{m}{2}(m)}, k; 3) = \frac{m}{2}(6k^2 - 13k + 5) + 1.$$

So far we know that the family of sequences $AP^*_{a(m)}$ is regular if $a = 0$, has degree of regularity at least three if $a = \frac{m}{2}$, and has degree of regularity two otherwise. We will know the degree of regularity for this family in all cases if we can determine the degree of regularity when $a = \frac{m}{2}$. The next theorem provides us with the solution: the degree of regularity is exactly three if $a = \frac{m}{2}$. What is even more striking is that although the associated Ramsey function using three colors exists (is finite) for all values of k (the length of the sequence), when four colors are used it does not exist even for $k = 2$.

Theorem 6.17. *Let $1 \leq a < m$ and assume that $\frac{m}{\gcd(a,m)}$ is even. Then $R(AP^*_{a(m)}, 2; 4) = \infty$.*

Proof. By Lemma 6.11 it suffices to prove this when $\gcd(a, m) = 1$. Hence we may assume that m is even and a is odd. To prove the theorem we provide a 4-coloring of $\mathbb{Z}^+$ that admits no monochromatic 2-term $a \,(\mathrm{mod}\, m)$-progressions and no monochromatic 2-term arithmetic progressions with gap m.

Let χ be the 4-coloring of $\mathbb{Z}^+$ represented by

$$\underbrace{1212\dots12}_{m}\underbrace{3434\dots34}_{m}\underbrace{1212\dots12}_{m}\underbrace{3434\dots34}_{m}\dots .$$

It is easy to see that χ admits no monochromatic 2-term arithmetic progression with gap m. Also, since m is even, by the way χ is defined, whenever $|y - x|$ is odd, $\chi(x) \neq \chi(y)$. Thus, there is no monochromatic 2-term $a \,(\mathrm{mod}\, m)$-progression. □

We conclude this chapter with a table that summarizes what is known about the degree of regularity of the families in this chapter.

Throughout the table, we assume that $1 < a < m$ is fixed. Whenever a family is r-regular, we also give the order of magnitude (as a function of the length k of the sequence, with r fixed) of the best known upper bound for the associated function $R(\mathcal{F}, k; r)$. For comparison reasons, we also include the family AP, but do not include Gowers' bounds for $R(AP, k; r)$ (see Section 2.4).

$\mathcal{F}$	Restriction	2-regular	3-regular	r-regular ($r \geq 4$)
AP		yes	yes	yes
$AP_{(m)}$		no	no	no
$AP_{a(m)}$		no	no	no
$AP_{0(m)}$		yes; $2mk$	yes; $3mk$	yes; rmk
$AP^*_a(m)$	$\frac{m}{a} \neq 2$	yes; mk^2	no	no
$AP^*_a(m)$	$\frac{m}{a} = 2$	yes; mk^2	yes; $3mk^2$	no

Table 6.2: Degree of regularity of families of type $AP_{(m)}$

6.4. Exercises

6.1 Let $1 \leq a < m$.
a) Show that there exists a set of positive integers that contains arbitrarily long members of $AP^*_{a(m)}$ but that fails to contain arbitrarily long members of $AP_{a(m)}$.
b) Show that there exists a set of positive integers that contains arbitrarily long members of $AP_{a(m)}$ but does not contain arbitrarily long descending waves (see Definition 3.20).
c) Show that there exists a set of positive integers that contains arbitrarily long arithmetic progressions but does not contain arbitrarily long members of $AP^*_{a(m)}$.

6.2 Can you find a relationship among a, b, and m, so that any set containing arbitrarily long members of $AP^*_{a(m)}$ must also contain arbitrarily long members of $AP^*_{b(m)}$?

6.3 Define $g(n)$ to be the largest value of k such that for every 2-coloring of the group $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ there is a monochromatic k-term arithmetic progression (with distinct members). Note: we are not insisting that the members of the arithmetic progression be increasing, just distinct. Find $g(6)$. Also, If it is known that $g(n) \leq k$, is there anything we can say about $w(k+1)$?

6.4 Define $h(n)$ to be the same as $g(n)$ in Exercise 6.3, except the members of the arithmetic progression are not required to be distinct. Find $h(6)$. Also, find an infinite set S such that if $s \in S$, then $g(s) = h(s)$.

6.5 Prove that if $R(AP_{(m)}, k) > (m-1)(k-1)+1$, then

$$w(k) > (m-1)(k-1)+1.$$

6.6 Prove that if $m \geq \left\lceil \frac{w(k)-1}{k-1} \right\rceil$, then $R(AP_{(m)}, k; 2) \leq w(k)$.

6.7 Prove the following implication: if $R(AP_{(m)}, k; 2) = \infty$ for $k \geq \log m$, then $w(\log m) \geq m \log m$ and hence $w(k) \geq ke^k$.

6.8 Complete the last part of the proof of Theorem 6.12.

6.9 Let $1 \leq a < m$. Prove that $R(AP^*_{a(m)}, 2; 3) \leq 3m$. (Hint: assume there exists a 3-coloring $\chi : [1, 3m] \to \{1, 2, 3\}$ yielding no appropriate monochromatic 2-element set; assume that $\chi(m+1) = 1$ and $\chi(2m+1) = 2$, and arrive at a contradiction.)

6.10 Let $1 \leq a < m$, let $\frac{m}{\gcd(a,m)}$ be odd, and let $k \geq 2$.
a) Prove the following, which is a slight improvement over Theorem 6.12 in certain cases when $\ell = 3$:
$R(AP^*_{a(m)}, k, 3; 2) \leq 2m(k-1) + a(k-2) + 1$.
b) Prove the following, which gives a slight improvement over Theorem 6.12 when $\ell = 2$:
$R(AP^*_{a(m)}, k, 2; 2) \leq m(k-1) + a(k-1) + 1 - \epsilon$,
where $\epsilon = 0$ if a is odd, and $\epsilon = \gcd(a, m)$ is a is even.

6.11 Consider the following generalization of $AP^*_{a(m)}$. For T a set of positive integers, let $AP^*_{a(m),T} = AP_{a(m)} \cup A_T$; that is, it consists of all $a \,(\bmod m)$-progressions and all arithmetic progressions whose gaps are in T. Now define $R(AP^*_{a(m),T}, k, \ell)$ to be as in Definition 6.6, except we replace "ℓ-term member of $A_{\{m\}}$" with "ℓ-term member of A_T". Prove the following.
a) $R(AP^*_{a(m),T}, k) < \infty$ if and only if $cm \in T$ for some $c \in \mathbb{Z}^+$.

b) Let $d = \gcd(a, m)$, and let $c \geq 1$, $k \geq 2$ and $\ell \geq 3$. Assume that $cm \in T$. If $\frac{m}{d}$ is even, then
$R(AP^*_{a(m),T}, k, \ell) \leq cm(k-1)(\ell-1) + (k-2)a + 1$.
c) Under the hypothesis of (b), except assuming $\frac{m}{d}$ is odd,
$R(AP^*_{a(m),T}, k, \ell) \leq cm(k-2)(\ell-1) + m + ak - d + 1$.

6.5. Research Problems

6.1 Investigate which pairs of integers m and k have the property that $R(AP_{(m)}, k) < \infty$. Exercise 6.6 handles the situation in which $m \geq \left\lceil \frac{w(k)-1}{k-1} \right\rceil$. For those values of m and k that do not satisfy this inequality, the situation is unclear. For example, it is known that $R(AP_{(22)}, 5) = \infty$, while $R(AP_{(21)}, 5)$ and $R(AP_{(23)}, 5)$ are both finite. In particular, find a function $h(m)$ (as small as possible) such that $R(AP_{(m)}, h(m)) = \infty$ for all $k \geq h(m)$. (See Table 6.1.)
References: [**65**], [**173**]

6.2 Prove or disprove: if m and k are such that $R(AP_{(m)}, k) < \infty$, then $R(AP_{(m)}, k) \leq w(k)$. (See Table 6.1.)
References: [**65**], [**173**]

6.3 Find bounds for $R(AP_{(m)}, k)$ when it exists. (See Table 6.1.)
References: [**65**], [**173**]

6.4 Computer calculations suggest that the upper bound of $3m$ from Exercise 6.9 is not the best possible. Improve on this bound (or show it *is* the best possible).
References: [**173**], [**176**]

6.5 It is known that if $\frac{m}{\gcd(a,m)}$ is odd, then $R(AP^*_{a(m)}, 2; 6) = \infty$. For certain such pairs a and m the least value of r such that $R(AP^*_{a(m)}, 2; r) = \infty$ is 6; for some it is 5. Determine if the least r is ever 4 (it is never 3 by Exercise 6.9).
Reference: [**176**]

6.6 It has been conjectured that Theorem 6.9 is true when the restriction $\frac{k-2}{\ell-2} \leq \frac{m}{a}$ is removed. Prove or disprove this conjecture. (Note: it is known that this formula does not work if we loosen the requirement to $\frac{m}{\gcd(a,m)}$ being even. For example, computer calculations have shown that $R(AP^*_{9(10)}, 3, 3) = 49$, which is much closer to the upper bound of 50 given by Theorem 6.7 than to the lower bound of 41. The conjecture has been proven when $\frac{m}{a} = 2$, but not for other cases.)
Reference: [**173**]

6.7 Improve the lower bound given by Theorem 6.13. Perhaps it can be tightened to $m(k-2)(\ell-1)+1$.
Reference: [**173**]

6.8 Corollary 6.8 gives a precise formula for $R(AP^*_{a(m)}, k, \ell)$ when $k = 2$ and $\frac{a}{\gcd(a,m)}$ is even. Formulae have also been found for this function when $k = 2$ and $\frac{a}{\gcd(a,m)}$ is odd. No formulae are known for $\ell = 2$ and general k. Find such formulae.
Reference: [**173**]

6.9 Consider the variation of $R(AP^*_{a(m)}, k, \ell)$ that follows. Let $R'(AP^*_{a(m)}, k, \ell)$ denote the least positive integer n so that every 2-coloring of $[1, n]$ either admits an ℓ-term monochromatic arithmetic progression with gap m, or else in *each* color there is a k-term monochromatic a (mod m)-progression. It has been shown that $R'(AP^*_{1(2)}, k, \ell) \leq 2(k-1)(\ell-1)+k-2$. Does this generalize to arbitrary a and m?
Reference: [**173**]

6.10 Families analogous to type $AP_{a(m)}$ have been considered, where more than one congruence class is allowed. For example, the families $AP_{a(m),b(n)} = AP_{a(m)} \cup AP_{b(n)}$ were studied, and a characterization was given for those pairs of congruence classes $a \pmod{m}$ and $b \pmod{n}$ for which such a family is 2-regular. An upper bound was also given when 2-regularity occurs. As one special case, if m is even and $b = (c-1)m$ for some $c \geq 2$, then $R(AP_{a(m),b(cm)}, 3) \leq 4cm$. Prove or disprove the conjecture that under the same hypotheses, $R(AP_{a(m),b(cm)}, 3) = 4(c-1)m + 1$.
Reference: [**65**]

6.11 Given $1 \leq a < m$ fixed, determine the Ramsey properties of sequences of the form $\{x_1 < x_2 < \cdots < x_k\}$ such that $x_i - x_1 \equiv a \pmod{m}$ for all $i = 2, \ldots, k$.
References: [**65**], [**160**], [**173**]

6.12 Let $1 \leq a < m$. Consider the family of all sequences of positive integers of the form $\{x_1 < x_2 < \cdots < x_k\}$ satisfying $x_i - x_{i-1} = t_i d$, with $t_i \equiv a \pmod m$ for each i, and where d is some fixed positive integer. For example, if $t_i = 1$ for all i, then these are the arithmetic progressions. For $m = 2$ and $a = 1$, it has been calculated that for this family (call it $\mathcal{C}$), $R(\mathcal{C}, 3) = 9$, $R(\mathcal{C}, 4) = 15$, and $R(\mathcal{C}, 5) = 21$. Study $\mathcal{C}$. One specific question is this: what happens for $a = 1$ and arbitrary m?
Reference: [**65**], [**160**], [**173**]

6.6. References

§**6.1**. The proof of Theorem 6.4 is from [**65**], which also discusses the Ramsey-type functions for $AP_{a(m)} \cup AP_{b(n)}$, as well as unions of more than two such families. It also contains work related to Exercise 6.11.

§**6.2**. The family $AP^*_{a(m)}$ is introduced by Landman and Long [**173**], and its 2-color Ramsey-type functions are studied. This paper contains proofs of Theorems 6.5, 6.7, 6.9, 6.12, and 6.13, and of Lemma 6.11. It also gives an exact formula for $R(AP^*_{a(m)}, 2, \ell)$. It discusses several computer-generated values and patterns which leave us with several unanswered, but intriguing questions.

§**6.3**. Proofs of Theorems 6.15–6.17 are in [**176**]; that paper also includes more work on $R(AP^*_{a(m)}, 2; r)$. The generalization of $AP^*_{a(m)}$ to families of the type $AP_{a(m)} \cup A_T$ is examined in [**160**], which also includes a table summarizing what is known about the regularity of these and related families, as well as the asymptotic values of their associated Ramsey-type functions. For an extended version of Table 6.2, see [**176**].

Additional References: Arithmetic progressions contained in the group $\mathbb{Z}_n$ (where the elements of the arithmetic progression are distinct in $\mathbb{Z}_n$) are studied in [**254**] (we believe that the very last inequality in the article is incorrect) with upper and lower bounds given for the associated Ramsey-type functions. A relationship between such Ramsey functions and the classical van der Waerden numbers yields a lower bound for $w(6)$.

Chapter 7

Other Variations on van der Waerden's Theorem

The notion of an arithmetic progression, being so basic, has naturally led to many intriguing mathematical questions. Here we mention a few selected topics dealing with arithmetic progressions, not covered in the previous chapters, that fall under the general heading of Ramsey theory on the integers.

7.1. The Function $\Gamma_m(k)$

In Chapter 3 the function $\Gamma_m(k)$ was introduced. Here is a reminder of its definition.

Definition 7.1. For $m, k \geq 2$, $\Gamma_m(k)$ denotes the least positive integer s such that for every set $S \subseteq \mathbb{Z}^+$ with $S = \{x_1, x_2, \ldots, x_s\}$ and $x_i - x_{i-1} \in \{1, 2, \ldots, m\}$, $2 \leq i \leq s$, S contains a k-term arithmetic progression.

We showed in Chapter 3 that $\Gamma_m(k)$ exists for all m and k. We also discussed the relevance of $\Gamma_m(k)$ to the goal of finding an upper bound on the van der Waerden numbers: $w(k) \leq SP_m(\Gamma_m(k))$, where SP_m is the Ramsey-type function associated with semi-progressions of scope m. In this section we look more closely at the Γ_m function.

We first define a related function.

Definition 7.2. For $m, k \geq 2$, $\Omega_m(k)$ is the least positive integer n such that whenever $X = \{x_1, x_2, \ldots, x_n\}$ and $x_i \in [(i-1)m, im-1]$, there is a k-term arithmetic progression in X.

We consider an example.

Example 7.3. Let $m = 2$ and $k = 3$. We wish to find the least integer n so that every sequence $x_1, x_2, \ldots, x_n$ satisfying the conditions $x_1 \in \{0,1\}$, $x_2 \in \{2,3\}$, $\ldots$, $x_n \in \{2n-2, 2n-1\}$, will contain a 3-term arithmetic progression. The sequence $\{0, 2, 5, 6, 9, 11\}$, since it does not contain a 3-term arithmetic progression, shows that $\Omega_2(3) > 6$. If we check directly, we find that every 7-term sequence $x_1, x_2, \ldots, x_7$ with each $x_i \in \{2i-2, 2i-1\}$ does contain a 3-term arithmetic progression. Therefore $\Omega_2(3) = 7$.

One obvious question we want to ask is whether $\Omega_m(k)$ always exists. The next theorem, which describes a fundamental relationship between the values of the Ω functions and $w(k;r)$, answers this question in the affirmative.

Theorem 7.4. *For all $k, r \geq 1$,*

$$\Omega_r(k) \leq w(k;r) \leq \Omega_r(r(k-1)+1).$$

Proof. Let $w = w(k;r)$. We first show that $\Omega_r(k) \leq w$. To this end, let $X = \{x_1, x_2, \ldots, x_w\}$ with $x_n \in [(n-1)r, nr-1]$ for $n = 1, 2, \ldots, w$. Thus, for each $n \in [1, w]$, $x_n = (n-1)r + c_n$, where $c_n \in [0, r-1]$. Now r-color $[1, w]$ as follows: $\chi(n) = j$ if and only if $c_n = j$. From the definition of w, there is some monochromatic k-term arithmetic progression under χ. So assume $\{a + id : 0 \leq i \leq k-1\} \subseteq [1, w]$ has color j_0 for some $a, d \geq 1$. Hence, $c_{a+id} = j_0$ for each $i \in [0, k-1]$, so that

$$\begin{aligned} x_{a+id} &= (a+id-1)r + c_{a+id} \\ &= (a+id-1)r + j_0 \\ &= (a-1)r + j_0 + idr, \end{aligned}$$

for $0 \leq i \leq k-1$. Therefore, $\{x_{a+id} : 0 \leq i \leq k-1\}$ is a k-term arithmetic progression contained in X, and hence, $\Omega_r(k) \leq w(k;r)$.

Now let $m = \Omega_r(r(k-1)+1)$. To complete the proof, we show that every r-coloring of $[0, m-1]$ yields a monochromatic k-term arithmetic progression (this is equivalent to having $w(k;r) \leq m$). Let $\lambda : [0, m-1] \to \{0, 1, \ldots, r-1\}$ be an arbitrary r-coloring of $[0, m-1]$. Define the sequence $A = \{a_i : 0 \leq i \leq m-1\}$ by letting $a_i = ir + \lambda(i)$. Notice that for each i, $a_i \in [ir, (i+1)r-1]$. By the definition of m, A must contain an arithmetic progression of length $r(k-1)+1$, say

$$B = \{a_{i_j} : 0 \leq j \leq r(k-1)\} = \{a_{i_0} + jd : 0 \leq j \leq r(k-1)\}$$

for some $d \geq 1$. Then the set $\{a_{i_{jr}} : 0 \leq j \leq k-1\}$ is a k-term arithmetic progression with gap rd. Hence, for each j, $1 \leq j \leq k-1$, we have

$$rd = a_{i_{jr}} - a_{i_{(j-1)r}} = r(i_{jr} - i_{(j-1)r}) + \lambda(i_{jr}) - \lambda(i_{(j-1)r}). \tag{7.1}$$

Since the range of λ is $\{0, 1, \ldots, r-1\}$, and since the right-hand side of (7.1) must be a multiple of r, we have that $\lambda(i_{jr}) = \lambda(i_{(j-1)r})$ for each j, $1 \leq j \leq k-1$. Hence, by (7.1), $i_{jr} - i_{(j-1)r} = d$ for each $j \in \{0, 1, \ldots, k-1\}$. Thus, $\{i_{jr} : 0 \leq j \leq k-1\}$ is a k-term arithmetic progression that is monochromatic, and is contained in $[1, a_{r(k-1)}] \subseteq [1, m]$. □

In the next theorem we show that $\Gamma_r(k)$ is bounded above by a simple function involving $\Omega_r(k)$.

Theorem 7.5. $\Gamma_r(k) \leq r\Omega_r(k)$ *for all $k, r \geq 2$.*

Proof. Let $m = \Omega_r(k)$. It is necessary to show that every sequence $\{x_1 < x_2 < \cdots < x_{rm}\}$ satisfying $x_i - x_{i-1} \leq r$ for each i, $2 \leq i \leq rm$, contains a k-term arithmetic progression. Let X be such a sequence. We begin with the case in which $x_1 = 1$, so that for each j, $1 \leq j \leq m$, the interval $[(j-1)r, jr-1]$ must contain some element y_j of X. By the definition of m, the sequence $Y = \{y_j : 1 \leq j \leq m\}$ contains a k-term arithmetic progression. Since $Y \subseteq X$, we are done.

Now assume $x_1 > 1$. Let $x_i' = x_i - x_1 + 1$ for each i, $1 \leq i \leq rm$. Since $x_1' = 1$, we know that $\{x_i' : 1 \leq i \leq rm\}$ contains a k-term arithmetic progression $\{y_j : 1 \leq j \leq k\}$. Hence,

$$\{y_j + x_1 - 1 : 1 \leq j \leq k\}$$

is a k-term arithmetic progression contained in X, which completes the proof. □

Theorem 7.5 bounds Γ_r from above by an expression involving the Ω function. The Ω function may also be used to bound the Γ function from below. The proof being rather straightforward, we leave it as Exercise 7.2.

Theorem 7.6. *$\Omega_r(k) \leq \Gamma_{2r-1}(k)$ for all $k, r \geq 2$.*

We are now able to tie the van der Waerden numbers more closely to the Γ function.

Theorem 7.7. *Let $k, r \geq 2$. Then*

$$\Gamma_r(k) \leq w(k;r) \leq \Gamma_{2r-1}(r(k-1)+1).$$

Proof. We first show that $\Gamma_r(k) \leq w(k;r)$. Let $w = w(k;r)$ and let $X = \{x_i : 1 \leq i \leq w\}$ with $x_i - x_{i-1} \in \{1, 2, \ldots, r\}$. We wish to show that X contains a k-term arithmetic progression. As in the proof of Theorem 7.5, it suffices to assume that $x_1 = 1$.

Define the coloring χ on $[1, w]$ as follows: $\chi(n) = j$ if and only if $j = \min\{x_i - n : x_i \geq n\}$. Obviously, $w \leq x_w$. Therefore, χ is an r-coloring of $[1, w]$ using the colors $\{0, 1, \ldots, r-1\}$. By the definition of w, there is a monochromatic k-term arithmetic progression under χ; say $\{a+jd : 0 \leq j \leq k-1\}$ has color t. Note that for any $n \in [1, w]$ such that $\chi(n) = t$, we have $t + n = x_i$ for some i, $1 \leq i \leq w$. So for each j, $0 \leq j \leq k-1$, we have $t + a + jd = x_{i_j} \in X$. Thus X contains a k-term arithmetic progression, as desired.

The second inequality follows immediately from Theorems 7.4 and 7.6, since

$$w(k;r) \leq \Omega_r(r(k-1)+1) \leq \Gamma_{2r-1}(r(k-1)+1).$$

□

Example 7.8. Taking $r = 2$ and using Theorems 7.4–7.6, we obtain

$$\frac{1}{2}\Gamma_2(k) \leq \Omega_2(k) \leq w(k;2) \leq \Omega_2(2k-1) \leq \Gamma_3(2k-1). \tag{7.2}$$

In particular, for $k = 3$, the leftmost two inequalities agree with our previous calculations of $w(3;2) = 9$, $\Omega_2(3) = 7$ (Example 7.3), and $\Gamma_2(3) = 5$ (Example 3.28), while the other two inequalities tell us that $9 \leq \Omega_2(5) \leq \Gamma_3(5)$.

The above series of theorems, although providing a good picture of how the different functions are interrelated, does not actually provide any specific upper or lower bound for any of these functions. We next mention, without proof, a lower bound for $\Gamma_r(k)$ due to Alon and Zaks. Note that by (7.2) this also gives a lower bound for $\Omega_2(k)$

Theorem 7.9. *For every $r \geq 2$, there exists a constant $c > 0$ (dependent upon r) such that*

$$\Gamma_r(k) > r^{k - c\sqrt{k}}$$

for all $k \geq 3$.

7.2. Monochromatic Sets $a(S+b)$

In Chapter 2 we gave several equivalent forms of van der Waerden's theorem. One of these is the following: for every r-coloring of $\mathbb{Z}^+$ and every finite $S \subseteq \mathbb{Z}^+$, there exist integers $a, b \geq 1$ such that $aS + b$ is monochromatic. We can think of $aS + b$ as being derived from S via the operations, in order, of multiplication and addition. What happens if we reverse the order of these two operations? More explicitly, is it true that for every finite S and every r-coloring of $\mathbb{Z}^+$, there must be a monochromatic set of the form $a(S+b)$? As we shall see, the answer is no. This suggests the question: for which S and r does this Ramsey property hold?

We adopt the following terminology and notation.

Definition 7.10. For S a finite set of positive integers and $r \geq 2$, we say that S is *reverse r-regular* if for every r-coloring of $\mathbb{Z}^+$ there exist $a \geq 1$ and $b \geq 0$ such that $a(S+b)$ is monochromatic. We say S is *reverse regular* is S is reverse r-regular for all $r \geq 2$.

Notation. If S is reverse r-regular, denote by $RR(S;r)$ the least positive integer m such that for every r-coloring of $[1, m]$ there is a monochromatic set $a(S+b)$ for some $a \geq 1$ and $b \geq 0$.

Interestingly enough, the regularity properties of sets of the form $a(S+b)$ are far different from those of the form $aS+b$. As mentioned above, by van der Waerden's theorem, the latter type are always regular. The following theorem describes, quite plainly, which sets are reverse regular. We omit the proof, but we do prove Theorem 7.12, below, which has the "if" part of Theorem 7.11 as an immediate corollary.

Theorem 7.11. *A set S of positive integers is reverse regular if and only if $|S| \leq 2$.*

It is obvious that every 1-element set is reverse regular. For sets S such that $|S| = 2$, it is not difficult to give a formula for $RR(S) = RR(S; 2)$.

Theorem 7.12. *Let $s, t \geq 1$ with $s < t$. Then $RR(\{s,t\}; 2) = 2t$.*

Proof. For convenience we will denote $RR(\{s,t\}; 2)$ more simply by $RR(s,t)$. To show that $RR(s,t) > 2t-1$, let $d = t - s$ and consider the 2-coloring χ of $[1, 2t-1]$ defined by $\chi(i) = 1$ if $\lfloor \frac{i}{d} \rfloor$ is odd and $\chi(i) = 0$ if $\lfloor \frac{i}{d} \rfloor$ is even. Clearly, if $a(S+b) \subseteq [1, 2t-1]$ with $b \geq 0$, then $a = 1$ and $b \leq t-1$. So it suffices to show that there is no $0 \leq b \leq t-1$ such that $S + b$ is monochromatic. Assume, by way of contradiction, that $\{s+b, t+b\}$ is monochromatic. Then $\left\lfloor \frac{s+b}{t-s} \right\rfloor$ and $\left\lfloor \frac{t+b}{t-s} \right\rfloor$ have the same parity. That is, $\lfloor \frac{s+b}{d} \rfloor$ and $\lfloor \frac{s+d+b}{d} \rfloor$ have the same parity. Since $\lfloor \frac{s+d+b}{d} \rfloor = \lfloor \frac{s+b}{d} \rfloor + 1$, we have arrived at a contradiction.

To show that $RR(s,t) \leq 2t$, consider any 2-coloring of $[1, 2t]$. At least two elements of $\{2s, s+t, 2t\}$ must have the same color. We consider three cases, depending on which two of these elements are monochromatic. If $2s$ and $s+t$ have the same color, then taking $b = s$ and $a = 1$, we have $a(\{s,t\} + b)$ monochromatic. We leave the other two cases as Exercise 7.4. □

Although the only reverse regular sets (those that are reverse r-regular for every r) are those with two or less elements, the next theorem tells us that for any finite set S and any positive integer r, there exists some positive integer c such that cS is reverse r-regular. That is to say, for every finite S, for arbitrarily large r there is a multiple of S that is reverse r-regular (*which* multiple of S this is, however, depends on r).

Theorem 7.13. *Let S be a finite set of positive integers, and let $r \geq 1$. Then there exists a positive integer c (depending on S and r) such that cS is reverse r-regular.*

Proof. Let S and r be fixed. From the equivalent form of van der Waerden's theorem mentioned in the first paragraph of this section, by using the compactness principle, we know there is a positive integer m such that for every r-coloring of $[1, m]$ there is a monochromatic set of the form $aS + b$, $a \geq 1$, $b \geq 0$. Let $c = \text{lcm}\{1, 2, \ldots, m\}$.

Let χ be any r-coloring of $\mathbb{Z}^+$. We wish to show that there exist $a' \geq 1$ and $b' \geq 0$ such that $a'(cS + b')$ is monochromatic under χ. Consider the r-coloring α of $[1, m]$ defined by $\alpha(i) = \chi(ci)$. We know that there exist $a > 0$, $b \geq 0$ such that $aS + b$ is monochromatic under α. Therefore, $caS + cb$ is monochromatic under χ. Clearly, $a \leq m$, and hence a divides c. Let $c = da$. Then $caS + dab = a(cS + db)$ is monochromatic under χ. Hence, letting $a' = a$ and $b' = db$, the proof is complete. □

7.3. Having Most Elements Monochromatic

In this section we look at a modification of van der Waerden's theorem by loosening the requirement that there is a k-term arithmetic progression having *all* of its terms be of the same color. What if we instead require only that, in any 2-coloring, there be an arithmetic progression that is predominantly of one color? We formalize this idea in the following definition.

Definition 7.14. Let $k \geq 1$ and let $0 \leq j < k$. Let $w^*(k,j)$ be the least positive integer w such that for every $\chi : [1, w] \to \{0, 1\}$ there is a k-term arithmetic progression with the property that the number of elements of color 0 and the number of elements of color 1 differ by more than j.

It is clear from van der Waerden's theorem that $w^*(k,j)$ always exists.

Example 7.15. If $j = k-1$, then $w^*(k,j)$ has the same meaning as $w(k)$, since there must be k elements of one color and none of the other color. At the opposite extreme, if k is any odd positive integer, then $w^*(k,0)=k$, because in any 2-coloring of $[1,k]$ (which itself is a k-term arithmetic progression), the number of integers of one color must exceed the number of the other color.

We saw in Example 7.15 that the case in which k is odd and $j=0$ is rather trivial. This is not so when k is even and $j=0$.

Theorem 7.16. *If $k\ge 2$ is even, then $w^*(k,0)=2^j(k-1)+1$, where j is the largest positive integer such that 2^j divides k.*

Proof. Let $m=2^j(k-1)+1$, and let $\chi:[1,m]\to\{1,-1\}$ be an arbitrary 2-coloring of $[1,m]$. To show that $w^*(k,0)\le m$, assume, for a contradiction, that, under χ, every k-term arithmetic progression has exactly $\frac{k}{2}$ of its elements in each color. In particular, for each $a\in[1,m-k]$,

$$\sum_{i=0}^{k-1}\chi(a+i)=0 \tag{7.3}$$

and

$$\sum_{i=1}^{k}\chi(a+i)=0. \tag{7.4}$$

Likewise,

$$\sum_{i=0}^{k-1}\chi\left(1+i2^j\right)=0. \tag{7.5}$$

Note that from (7.3) and (7.4) we have

$$\chi(a)=\chi(a+k) \text{ for all } a\in[1,m-k]. \tag{7.6}$$

We next show that (7.5) and (7.6) lead to a contradiction.

Let $q=\frac{k}{2^j}$. Note that from the meaning of j, q must be odd. Since $k-1=q2^j-1$, we have

$$\begin{aligned}\sum_{i=0}^{k-1}\chi\left(1+i2^j\right) &= \sum_{s=0}^{2^j-1}\left(\sum_{i=sq}^{(s+1)q-1}\chi\left(1+i2^j\right)\right)\\ &= \sum_{s=0}^{2^j-1}\left(\sum_{i=0}^{q-1}\chi(1+(i+sq)2^j)\right)\\ &= \sum_{s=0}^{2^j-1}\left(\sum_{i=0}^{q-1}\chi\left(1+i2^j+sk\right)\right).\end{aligned}$$

By (7.6), for all s and i such that $0\le i\le q-1$ and $0\le s\le 2^j-2$, we have

$$\chi\left(1+i2^j+sk\right)=\chi\left(1+i2^j+(s+1)k\right).$$

Therefore

$$\sum_{i=0}^{k-1}\chi\left(1+i2^j\right)=2^j\sum_{i=0}^{q-1}\chi\left(1+i2^j\right). \tag{7.7}$$

From (7.5) and (7.7), we obtain

$$\sum_{i=0}^{q-1}\chi\left(1+i2^j\right)=0,$$

which gives us a contradiction because q is odd.

We still must show that $w^*(k,0)\ge 2^j(k-1)+1$. It suffices to find one 2-coloring $\lambda:[1,m-1]\to\{-1,1\}$ such that for every arithmetic progression $\{a,a+d,\ldots,a+(k-1)d\}\subseteq[1,m-1]$, we have

$$\sum_{i=1}^{k}\lambda(a+(i-1)d)=0.$$

Define λ as follows. Let $\lambda(x)=1$ for $1\le x\le\frac{k}{2}$, and let $\lambda(x)=-1$ for $\frac{k}{2}<x\le k$. Finally, for $x>k$, let $\lambda(x)=\lambda(\overline{x})$, where $\overline{x}\in[1,k]$ and $x\equiv\overline{x}\,(\mathrm{mod}\,k)$.

Let $\{a+(i-1)d:1\le i\le k\}$ be any arithmetic progression contained in $[1,m-1]$. Let $e=\gcd(d,k)$ and $k'=\frac{k}{e}$. Since $d<2^j$,

k' is even. Therefore, by the way λ is defined and the fact that $k'd$ is a multiple of k,

$$\begin{aligned}\sum_{i=1}^{k}\lambda(a+(i-1)d) &= \sum_{s=0}^{e-1}\left(\sum_{i=sk'+1}^{(s+1)k'}\lambda(a+(i-1)d)\right)\\ &= e\sum_{i=1}^{k'}\lambda(a+(i-1)d)\\ &= e\sum_{i=1}^{\frac{k'}{2}}\left(\lambda(a+(i-1)d)+\lambda\left(a+(i-1)d+\frac{k'd}{2}\right)\right)\\ &= 0,\end{aligned}$$

because $\frac{k'd}{2} \equiv \frac{k}{2} \pmod{k}$. This completes the proof. □

Although very little is known about $w^*(k,j)$ for $j \geq 1$, progress has been made on a related question. Before stating the question, we adopt the following notation. For a 2-coloring $\chi : [1,n] \to \{1,-1\}$, denote by $[1,n]_1$ and $[1,n]_{-1}$ the sets of elements of $[1,n]$ having the colors 1 and -1, respectively. The question we wish to ask may now be stated this way: for n a positive integer, can we determine some minimum number j, depending on n, such that for every 2-coloring $\chi : [1,n] \to \{1,-1\}$, there must be some arithmetic progression A so that $A \cap [1,n]_1$ and $A \cap [1,n]_{-1}$ differ in size by at least j? The following theorem gives an asymptotic lower bound for such a j as a function of n. We omit the proof, which is beyond the scope of this book.

Theorem 7.17. *Let*

$$j(n) = \min_{\chi}\max_{a,d,k}\left\{\left|\sum_{i=0}^{k}\chi(a+id)\right| : a,d,k \in \mathbb{Z}^+ \text{ and } a+kd \leq n\right\},$$

where the minimum is taken over all 2-colorings $\chi : [1,n] \to \{1,-1\}$. *Then* $j(n) \geq O(n^{1/4})$.

Example 7.18. To help get some insight into what the above theorem is saying, consider $j(12)$. For a given coloring of $[1,12]$, we want the most "unbalanced" arithmetic progression A, in the sense that the difference in the sizes of the sets $A \cap [1,12]_1$ and $A \cap [1,12]_{-1}$ is maximized. For example, let χ^* be the coloring such that $[1,12]_1 = \{3,4,5,6,8,9,10\}$ and $[1,12]_{-1} = \{1,2,7,11,12\}$. Then it is easy to check that the arithmetic progression $A = \{3,4,5,6,7,8,9,10\}$ is the arithmetic progression we seek because the difference between the sizes of $A \cap [1,12]_1$ and $A \cap [1,12]_{-1}$ is six, and no other arithmetic progression yields as great a difference (check this). Using the notation of Theorem 7.17, this may be stated (equivalently) as

$$\max_{a,d,k}\left\{\left|\sum_{i=0}^{k}\chi^*(a+id)\right| : a+kd \leq 12\right\} = 6.$$

Denoting this maximum by $M(\chi^*)$, we have $M(\chi^*) = 6$, and this is obtained with $a = 3$, $d = 1$, and $k = 7$ (it is, of course, possible that there is more than one arithmetic progression that gives the maximum). Now, if we do this for each 2-coloring χ of [1,12], we get its associated maximum $M(\chi)$. Then by taking the minimum, $j(12)$, over all 2-colorings χ, we are saying that, for each coloring χ, there will always be some arithmetic progression A in [1,12] having the property that the difference between the number of members of A with color 1 and the number of members of A with color -1 is at least $j(12)$. We leave it to the reader (Exercise 7.7) to determine the value of $j(12)$.

7.4. Permutations Avoiding Arithmetic Progressions

In this section we are concerned with the question of whether permutations (i.e., arrangements) of certain sequences (possibly infinite) contain subsequences of a desired length that form an arithmetic progression. To clarify, we begin with a definition.

Definition 7.19. A sequence $(a_1, a_2, \ldots, a_n)$ or $(a_1, a_2, \ldots)$ has a k-term *monotone arithmetic progression* if there is a set of indices $\{i_1 < i_2 < \cdots < i_k\}$ such that the subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ is either an increasing or a decreasing arithmetic progression.

Example 7.20. The sequence $(3,1,4,5)$ contains a 3-term monotone arithmetic progression (namely, 3, 4, 5), but none of four terms. The sequence $(4,2,3,1)$ has no 3-term monotone arithmetic progression.

The sequence $(1,9,7,6,2,4,5,3,8)$ has a 4-term monotone arithmetic progression (the subsequence $9,7,5,3$ is one), but none of length 5.

Here is one simple question: are there any positive integers n such that *every* permutation of the sequence $(1,2,\ldots,n)$ has a 3-term monotone arithmetic progression? This is relatively easy to answer.

Theorem 7.21. *Let $n \geq 1$. There is a permutation of $(1,2,\ldots,n)$ that does not contain a 3-term monotone arithmetic progression.*

Proof. First note that if $1 \leq n_1 < n_2$, and if the theorem holds for n_2, then it also holds for n_1 (simply take the permutation that works for n_2, and delete those integers greater than n_1). Hence, it is sufficient to prove the theorem for $n \in \{2^k : k \geq 0\}$. We do this by induction on k.

Clearly, the result holds for $n = 2^0$. Now assume that $k \geq 0$, that $n = 2^k$, and that there is a permutation $(c_1, c_2, \ldots, c_n)$ of $(1,2,\ldots,n)$ that does not contain a 3-term monotone arithmetic progression. Consider the sequence

$$S = (2c_1 - 1, 2c_2 - 1, \ldots, 2c_n - 1, 2c_1, 2c_2, \ldots, 2c_n).$$

Note that S is a permutation of the sequence $(1,2,\ldots,2n = 2^{k+1})$, and that the first 2^k terms of S are odd and the other 2^k terms of S are even.

In any 3-term monotone arithmetic progression, the first and third terms must be of the same parity. Therefore, if S contains a 3-term monotone arithmetic progression P, then either

$$P \subseteq \{2c_1 - 1, 2c_2 - 1, \ldots, 2c_n - 1\} = S_1$$

or

$$P \subseteq \{2c_1, 2c_2, \ldots, 2c_n\} = S_2.$$

This is impossible, for if $\{a,b,c\} \subseteq [1,n]$ is not a monotone arithmetic progression, then neither is $\{2a-1, 2b-1, 2c-1\}$ nor $\{2a, 2b, 2c\}$. Therefore, S is a permutation of $\{1,2,\ldots,2^{k+1}\}$ that contains no 3-term monotone arithmetic progression, and the proof is complete. □

Now that we know that for each n there is some permutation of $(1,2,\ldots,n)$ containing no 3-term monotone arithmetic progression, a

natural question to ask is: how many such permutations are there? Clearly, for $n \geq 3$, not all $n!$ permutations of $[1,n]$ avoid 3-term monotone arithmetic progressions (for example, $(1,2,\ldots,n)$). Let us denote by $\theta(n)$ the number of permutations of $(1,2,\ldots,n)$ containing no 3-term monotone arithmetic progressions. No formula for $\theta(n)$ is known. However, both upper and lower bounds are known for $\theta(n)$. The following theorem gives a lower bound.

Theorem 7.22. *For all $n \geq 1$, $\theta(n) \geq 2^{n-1}$.*

Proof. We use induction on n. First note that $\theta(1) = 1$, $\theta(2) = 2$, and $\theta(3) = 4$, so the inequality holds for $n \leq 3$. Now assume $n \geq 4$, and that it holds for any $n' < n$. We consider two cases.

Case 1. $n = 2m$ is even. By the reasoning used in the proof of Theorem 7.21, if S_1 and S_2 are any permutations of $[1,m]$ that avoid 3-term monotone arithmetic progressions, then $((2S_1 - 1), 2S_2)$ and $(2S_2, (2S_1 - 1))$ are permutations of $[1,n]$ that also avoid 3-term monotone arithmetic progressions. Therefore,

$$\theta(2m) \geq (\theta(m))^2 + (\theta(m))^2 = 2(\theta(m))^2 \geq 2\left(2^{m-1}\right)^2 = 2^{n-1}.$$

Case 2. $n = 2m+1$ is odd. As in Case 1, if S_1 and S_2 are permutations of $[1,m]$ and $[1,m+1]$, respectively, that avoid 3-term monotone arithmetic progressions, then $(2S_1, (2S_2 - 1))$ and $((2S_2 - 1), 2S_1)$ are permutations of $[1,n]$ that do likewise. Therefore,

$$\theta(2m+1) \geq 2\theta(m)\theta(m+1).$$

From the induction hypothesis, if $n = 2m$, then

$$\theta(n) \geq 2(2^{m-1})^2 = 2^{n-1},$$

and if $n = 2m+1$, then

$$\theta(n) \geq 2(2^{m-1})(2^m) = 2^{n-1}.$$

□

We now turn to upper bounds for $\theta(n)$.

Theorem 7.23. *Let $n \geq 1$. If $n = 2m-1$, then $\theta(n) \leq (m!)^2$. If $n = 2m$, then $\theta(n) \leq (m+1)(m!)^2$.*

Proof. Denote by $\Theta(t)$ the set of permutations of $[1,t]$ that avoid 3-term monotone arithmetic progressions. Note that each member S' of $\Theta(n+1)$ may be obtained from some member S of $\Theta(n)$ by the insertion of $n+1$ somewhere into $S=(a_1,a_2,\ldots,a_n)$. If a_i is such that

$$\left\lceil \frac{n+3}{2} \right\rceil \le a_i \le n, \tag{7.8}$$

then $\{n+1, a_i, 2a_i-n-1\}$ is an arithmetic progression that, from the meaning of $\Theta(n+1)$, cannot occur as a monotone progression in S'. Therefore, for each a_i that satisfies (7.8), S' cannot have $n+1$ to the immediate right of a_i if $2a_i-n-1$ is to the left of a_i. Likewise, $n+1$ cannot be to the immediate left of a_i if $2a_i-n-1$ is to the right of a_i.

Now, if $i<n$ and if $n+1$ is allowed neither to the right of a_i nor to the left of a_{i+1}, S cannot be extended to an element of $\Theta(n+1)$. Similarly, S cannot be extended if $i=1$ and $n+1$ is not allowed to be placed to the left of a_i; and S cannot be extended if $i=n$ and $n+1$ is not allowed to be placed to the right of a_i. Therefore, each of the $n-\lfloor\frac{n+3}{2}\rfloor+1$ values of a_i satisfying (7.8) eliminates at least one of the $n+1$ positions in S' as a possible location for $n+1$. Subtracting this quantity from $n+1$ leaves us with at most $\lfloor\frac{n+3}{2}\rfloor$ positions where $n+1$ may be placed. Hence

$$\theta(n+1) \le \left\lfloor \frac{n+3}{2} \right\rfloor \theta(n). \tag{7.9}$$

By a straightforward induction argument, using the fact that $\theta(1)=1$ and $\theta(2)=2$, (7.9) implies the theorem. We leave the details to the reader as Exercise 7.9. $\square$

We now turn our attention to permutations of $\mathbb{Z}^+$, i.e., of the infinite sequence $(1,2,3,\ldots)$. For example, $S=(2,1,4,3,6,5,8,7,\ldots)$ and $(12,1,2,3,4,\ldots)$ are such permutations. Theorem 7.21 tells us that not all permutations of the interval $[1,n]$ yield 3-term monotone arithmetic progressions. Is this also the case for the permutations of $\mathbb{Z}^+$? The following theorem tells us the answer.

Theorem 7.24. *There is no permutation of $\mathbb{Z}^+$ that avoids 3-term monotone arithmetic progressions.*

Proof. Let $S=(a_1,a_2,\ldots)$ be any permutation of $\mathbb{Z}^+$. Let j be the least positive integer such that $a_j>a_1$. Consider $a_k=2a_j-a_1$. Since $a_j>a_1$, we have $a_k>a_j$. Also, $a_k-a_j=a_j-a_1$, so that $A=\{a_1,a_j,a_k\}$ is an arithmetic progression. By the definition of j, we know that $k>j$, and therefore A is a monotone arithmetic progression in S. $\square$

As for guaranteeing monotone arithmetic progressions of more than three terms in the permutations of $\mathbb{Z}^+$, the following theorem is known, the proof of which we omit.

Theorem 7.25. *There exist permutations of $\mathbb{Z}^+$ that avoid 5-term monotone arithmetic progressions*

An intriguing question, whose answer is still unknown, is whether there are any permutations of $\mathbb{Z}^+$ that avoid 4-term monotone arithmetic progressions.

7.5. Exercises

7.1 Determine the value of $\Omega_3(3)$.

7.2 Prove Theorem 7.6.

7.3 Verify, by direct computation, the observation in Example 7.8 that $9\le\Omega_2(5)\le\Gamma_3(5)$.

7.4 Complete the proof of Theorem 7.12 by considering the two remaining ways in which $\{2s,s+t,2t\}$ may be colored.

7.5 What is the value of $w^*(3,2)$? of $w^*(3,1)$? How do these examples generalize to arbitrary values of k (the length of the arithmetic progression)?

7.6 Determine the value of $w^*(4,1)$.

7.7 Determine the value of $j(12)$ (see Example 7.18).

7.8 Find $\theta(4)$ and $\theta(5)$.

7.9 Complete the proof of Theorem 7.23 by using induction to show that (7.9) implies the statement of the theorem.

7.6. Research Problems

7.1 Improve the lower bound of Theorem 7.9.
References: [**53**], [**63**]

7.2 Run a computer program to calculate values of $\Gamma_2(k)$. Try to find an upper bound on $\Gamma_2(k)$.
References: [**53**], [**63**], [**195**], [**209**]

7.3 Run a computer program to calculate values of $\Gamma_3(k)$. Along the lines of Theorem 7.9, try to find a lower bound for $\Gamma_3(k)$.
References: [**53**], [**63**], [**195**], [**209**]

7.4 Is it true that $RR(S,;r) < \infty$ if and only if, for every prime p, every r-coloring of $[1, p-1]$, every $a \geq 1$, and every $b \geq 0$, $a(S+b) \pmod{p}$ is monochromatic?
Reference: [**72**]

7.5 Characterize those pairs (S, r) for which $RR(S;r)$ exists.
Reference: [**72**]

7.6 Find an upper bound and/or a lower bound on $w^*(k,1)$. A computer program to calculate values may give interesting results.
References: [**23**], [**92**], [**99**], [**257**]

7.7 Determine if $\lim_{k\to\infty}[w^*(k,1)]^{1/k} < \infty$.
References: [**23**], [**92**], [**99**], [**257**]

7.8 Determine if $\lim_{k\to\infty}[w^*(k,\sqrt{k})]^{1/k} < \infty$.
References: [**23**], [**92**], [**99**], [**257**]

7.9 Improve on Theorem 7.17, i.e., find an improved lower bound.
References: [**84**], [**92**], [**99**], [**228**], [**257**]

7.10 Improve the best known upper bound for $j(n)$ (see the References section below).
References: [**99**], [**257**]

7.11 Determine if there exists a permutation of $\mathbb{Z}^+$ that contains no 4-term monotone arithmetic progressions.
References: [**43**], [**44**], [**76**], [**200**], [**223**], [**244**]

7.12 Does there exist $c > 0$ such that for all $n \in \mathbb{Z}^+$, $[\theta(n)]^{1/n} \leq c$? Does $\lim_{n\to\infty}[\theta(n)]^{1/n}$ exist?
Reference: [**76**]

7.13 Investigate the question of avoiding monotone arithmetic progressions by permutations of $\mathbb{Z}$ (rather than $\mathbb{Z}^+$).
Reference: [**76**]

7.14 Determine if it is possible to partition $\mathbb{Z}^+$ into two sets, each of which can be permuted to avoid 3-term monotone arithmetic progressions. What if we replace $\mathbb{Z}^+$ with $\mathbb{Z}$?
Reference: [**76**]

$*$ **7.15** Let $k \geq 1$. Let $\{a_1 < a_2 < \cdots\}$ be a sequence of positive integers such that $a_{i+1} - a_i \leq k$ for all $i \geq 1$. Must there be a 3-term arithmetic progression a_x, a_y, a_z such that x, y, z is also an arithmetic progression? This is known to be true for $k \leq 4$.

7.7. References

§**7.1**. Theorems 7.4–7.7 are due to Nathanson [**195**]. Earlier, Rabung [**209**] showed that the existence of $\Gamma_r(k)$ for all $r, k \geq 1$ is equivalent to van der Waerden's theorem. Theorem 7.9 is due to Alon and Zaks [**16**].

§**7.2**. Theorems 7.11–7.13 are from [**72**], which also includes more on reverse regular sets.

§**7.3**. Theorem 7.16 is due to Spencer [**260**]. Roth [**228**] proved Theorem 7.17. In the other direction, Spencer [**257**] showed that $j(n) < c\sqrt{n}\frac{\ln \ln n}{\ln n}$ for some constant c. Further work on $j(n)$ may be found in [**22**], [**99**], [**234**], [**235**], [**236**], [**237**]. Valko [**269**] generalizes the known upper and lower bounds on $j(n)$ to higher dimensions. Some other results related to $j(n)$ are found in [**84**], [**88**], [**238**].

§**7.4** The proof of Theorem 7.21 is from [**202**]. Theorems 7.22–7.25 and their proofs appear in [**76**]. Sidorenko [**250**] gives a permutation of $\mathbb{Z}^+$ in which there are no 3 terms such that both their values and positions form arithmetic progressions. Modular analogs of some of this work are consider in [**196**].

Additional References: This chapter covered only a few of the many interesting problems related to van der Waerden's theorem that have been considered. A stronger form of van der Waerden's theorem states that for every r-coloring of $\mathbb{Z}^+$, for each k there is a monochromatic arithmetic progression $A = \{a + id : 0 \le i \le k-1\}$ such that d has the same color as A. For proofs of this theorem and additional information, see [**3**], [**51**], [**127**], [**215**]. A related result of Graham, Spencer, and Witsenhausen [**129**] determines how large a subset of $[1, n]$ can be, that fails to contain both a k-term arithmetic progression and its gap d (the answer is a function of n and k).

Another question is this: what is the maximum number s of subsets $C_1, C_2, \ldots, C_s$ of $[1, n]$ such that for all $i \neq j$, $C_i \cap C_j$ is an arithmetic progression? If we include the empty set as an arithmetic progression, then it is known that

$$s \le \binom{n}{3} + \binom{n}{2} + \binom{n}{1} + 1,$$

and that this is the best possible upper bound on s [**128**]. If we do not allow the empty set to be an arithmetic progression, then it is known that there is a constant c such that $s < cn^2$ [**252**]. For other interesting variations involving arithmetic progressions, see [**18**], [**60**], and [**205**].

Chapter 8

Schur's Theorem

Until now we have devoted most of our study to Ramsey-type theorems dealing with variations of van der Waerden's theorem. There are many other interesting aspects of Ramsey theory on the integers that we may explore, and we do so in the remaining chapters. We begin with a result that came before van der Waerden's theorem: Schur's theorem.

Van der Waerden's theorem proves the existence, in particular, of $w(3; r)$. In other words, any r-coloring of $\mathbb{Z}^+$ must admit a monochromatic 3-term arithmetic progression $\{a, a+d, a+2d\}$ for some $a, d \ge 1$. Letting $x = a$, $y = a + 2d$, and $z = a + d$, this may also be described as a monochromatic solution to $x + y = 2z$, where $x, y, z \in \mathbb{Z}^+$ and $x \neq y$. Since $x + y = 2z$ is the equation of a plane, it is natural to ask other questions about coloring points in a plane.

Consider the equation of the simple plane $z = x+y$. Let P be the set of the points in this plane whose coordinates are positive integers. Thus, for example, $(1, 1, 2)$ and $(3, 4, 7)$ are in P (note that we are not insisting that x and y be distinct). Next, using any finite set of colors, assign a color to each positive integer.

Now, for each $(a, b, c) \in P$, perform the following. If the colors of a, b, and c are identical, then color (a, b, c) (in the plane) with that color. Otherwise, mark (a, b, c) (in the plane) with an X. The question is: can all of the points in the plane be marked with an X, or must there be a colored point? We would like to know whether *every* coloring yields a colored point; in other words, is it possible to finitely color $\mathbb{Z}^+$ so that no point $(x, y, z) \in P$ is colored?

This question was answered by Issai Schur in 1916, and is one of the first Ramsey-type theorems. However, Erdős and Szekeres' rediscovery of Ramsey's theorem in 1935 is due most of the credit for popularizing the subject.

Schur was not motivated by the idea of coloring points in the plane, but rather by perhaps the most famous and elusive of all mathematical problems, Fermat's Last Theorem (which was not officially a theorem until Wiles proved it in 1995). His result, which states that the answer to the above question is "no, there must be a colored point," has become known as Schur's theorem, but was only used as a lemma for the main theorem in his paper, which is given below as Theorem 8.1. We will prove this theorem after we have acquired a necessary tool.

Theorem 8.1. *Let $n \geq 1$. There exists a prime q such that for all primes $p \geq q$ the congruence $x^n + y^n \equiv z^n$ (mod p) has a solution in the integers with $xyz \not\equiv 0$ (mod p).*

From Theorem 8.1 we can garner some insight into why Fermat's Last Theorem was so difficult to prove, and why no elementary proof (if one exists) has surfaced yet: Fermat's Last Theorem is false if we replace the equation by a congruence. Because of Schur's result, we know that we cannot prove Fermat's Last Theorem just by considering congruences.

8.1. The Basic Theorem

We now state and prove Schur's theorem. We make use of Ramsey's theorem (see Section 1.2). We remind the reader of the following terminology.

Given a coloring χ of a set of positive integers and an equation $\mathcal{E}$ in the variables $x_1, x_2, \ldots, x_n$, we say that $\mathcal{E}$ has a monochromatic solution under χ if there exist values of $x_1, x_2, \ldots, x_n$ that satisfy $\mathcal{E}$ and that are monochromatic under χ.

Theorem 8.2 (Schur's Theorem). *For any $r \geq 1$, there exists a least positive integer $s = s(r)$ such that for any r-coloring of $[1, s]$ there is a monochromatic solution to $x + y = z$.*

Proof. Ramsey's theorem states, in particular, that for any $r \geq 1$ there exists an integer $n = R(3; r)$ such that for any r-coloring of K_n (the complete graph on n vertices) there is a monochromatic triangle. We will use a specific coloring, described as follows. Number the vertices of K_n by $1, 2, \ldots, n$. Next, arbitrarily partition $\{1, 2, \ldots, n-1\}$ into r sets. In other words, randomly place each $x \in \{1, 2, \ldots, n-1\}$ into exactly one of the r sets. These sets will correspond to the r colors. Color the edge that connects vertices i and j according to the set of which $|j-i|$ is a member. By Ramsey's theorem, a monochromatic triangle must exist. Let the vertices of this monochromatic triangle be $a < b < c$. Hence, $b-a, c-b$, and $c-a$ are all the same color. To finish the proof, let $x = b-a$, $y = c-b$, and $z = c-a$, and notice that $x + y = z$. □

Definition 8.3. We call the numbers that satisfy Schur's theorem the *Schur numbers* and denote them by $s(r)$. (Some books define the Schur number as the maximal number $m = m(r)$ such that there exists an r-coloring of $[1, m]$ that avoids monochromatic solutions to $x + y = z$. We prefer to use the same type of notation as that of the van der Waerden numbers.)

Definition 8.4. A triple $\{x, y, z\}$ that satisfies $x + y = z$ is called a *Schur triple.*

The only values known for the Schur numbers are for $r = 1, 2, 3, 4$: $s(1) = 2$, $s(2) = 5$, $s(3) = 14$, and $s(4) = 45$. In the next example we show that $s(2) = 5$.

Example 8.5. In Section 1.4 we showed that $s(2) \geq 5$ (color 1 and 4 red; color 2 and 3 blue). We now show that $s(2) \leq 5$. Consider any 2-coloring of $[1, 5]$. Without loss of generality we may assume that 1 is colored red. Assume, by way of contradiction, that there is no monochromatic Schur triple. Since $1 + 1 = 2$, we must color 2 blue. Since $2 + 2 = 4$, we must color 4 red. Since $1 + 4 = 5$, we must color 5 blue. All that remains is to color 3. However, if 3 is red, then $\{1, 3, 4\}$

is a red Schur triple and, if 3 is blue, then $\{2,3,5\}$ is a blue Schur triple.

Having Theorem 8.2 under our belt, we are now in a position to prove Theorem 8.1. The proof uses elementary group theory.

Proof of Theorem 8.1. Let $p > s(n)$ be a prime and let

$$\mathbb{Z}_p^* = \{1,2,\ldots,p-1\}$$

be the group (under multiplication) of nonzero residues modulo p. Let $S = \{x^n \pmod p) \colon x \in \mathbb{Z}_p^*\}$. Notice that S is a subgroup of $\mathbb{Z}_p^*$. Hence, we can write $\mathbb{Z}_p^*$ as a union of cosets $\mathbb{Z}_p^* = \bigcup_{i=1}^k a_i S$, where $k = \frac{n}{\gcd(n,p-1)}$.

Next, define a k-coloring of $\mathbb{Z}_p^*$ by assigning the element $t \in \mathbb{Z}_p^*$ the color j if and only if $t \in a_j S$. Since $k \le n$ and $p-1 \ge s(n)$, by Schur's theorem there exists a monochromatic triple $\{a,b,c\} \subseteq \mathbb{Z}_p^*$ such that $a+b=c$. That is, for some i, $1 \le i \le k$, there exist $a,b,c \in a_i S$ with $a+b=c$. Hence, there exist $x,y,z \in \mathbb{Z}_p^*$ such that $a_i x^n + a_i y^n \equiv a_i z^n \pmod p$. Multiplying through by a_i^{-1} completes the proof. □

Although Schur's original proof of Theorem 8.2 did not involve Ramsey numbers (in fact, Ramsey's theorem was not proved until 1928, while Schur proved his result in 1916), it is clear from our proof that the Schur numbers and Ramsey numbers are related. In particular, we have the following corollary. (Recall that $R_r(3)$ denotes $R(3,3,\ldots,3)$, where we are using r colors.)

Corollary 8.6. *For $r \ge 1$, $s(r) \le R_r(3) - 1$.*

Proof. Using the coloring given in the proof of Theorem 8.2, we have a correspondence between edgewise colorings of K_n and colorings of $[1, n-1]$. More explicitly, for any r-coloring of $[1, n-1]$ we derive an r-coloring of K_n by numbering the vertices of K_n and considering the differences between all pairs of vertices. By Ramsey's theorem, for $n = R_r(3)$ we are guaranteed a monochromatic triangle. By the definition of our coloring, this monochromatic triangle corresponds to a monochromatic Schur triple. Hence, if $n = R_r(3)$, we have $s(r) \le n-1 = R_r(3)-1$. □

Corollary 8.6 gives us an upper bound. However, the upper bound is not very useful unless we have an explicit upper bound for $R_r(3)$. Fortunately, we do.

Lemma 8.7. *For $r \ge 1$, $R_r(3) \le 3r!$.*

Proof. For $r=1$ we have $R_1(3) = 3$, so that the given bound is true. We now assume that $r \ge 2$. For $1 \le i \le r$, denote by $R_r^i(3)$ the Ramsey number $R(\underbrace{3,3,\ldots,3}_{i-1},2,\underbrace{3,3,\ldots,3}_{r-i})$. We begin by showing that

$$R_r(3) \le \sum_{i=1}^{r} R_r^i(3). \tag{8.1}$$

To prove (8.1) we will use the same method of proof as that employed to prove Theorem 1.15. Let $m = \sum_{i=1}^r R_r^i(3)$ and consider any r-coloring of the edges of K_m. Select one vertex, say v. Let the colors be $1,2,\ldots,r$, and let C_i, $i=1,\ldots,r$, denote the set of vertices connected to v by an edge of color i. By the pigeonhole principle there must exist j such that $|C_j| \ge R_r^j(3)$. Hence, the complete graph on C_j must contain either a K_2 of color j or a monochromatic triangle of color $c \in \{1,2,\ldots,j-1,j+1,j+2\ldots,r\}$. If it contains a monochromatic triangle, we are done. If it contains a K_2 of color j, then these two vertices, together with v, create a monochromatic triangle of color j, thereby proving (8.1).

Next, we show that for any $i \in \{1,2,\ldots,r\}$,

$$R_r^i(3) = R_{r-1}(3). \tag{8.2}$$

Let $n = R_r^i(3)$. Clearly, $n \ge R_{r-1}(3)$. It remains to show that $n \le R_{r-1}(3)$. By the definition of n, there must exist an r-coloring of the edges of K_{n-1} that avoids monochromatic triangles of color $c \in \{1,2,\ldots,i-1,i+1,i+2\ldots,r\}$ and avoids a K_2 of color i. Since we must avoid a K_2 of color i, no edge may have color i. Hence, we actually have an $(r-1)$-coloring of the edges of K_{n-1} that avoids monochromatic triangles. Thus $R_{r-1}(3) \geq n$, and we conclude that (8.2) holds.

Using (8.2), we see that $\sum_{i=1}^{r} R_r^i(3) = rR_{r-1}(3)$. Hence, by (8.1), $R_r(3) \leq rR_{r-1}(3)$ for $r \geq 2$. By repeated application of this inequality and the fact that $R_2(3) = 6$, we get $R_r(3) \leq 3r!$. □

From Corollary 8.6 and Lemma 8.7, we have the following result.

Theorem 8.8. *For $r \geq 1$, $s(r) \leq 3r! - 1$.*

Now that we have an upper bound on the Schur numbers, we turn our attention to a lower bound.

Theorem 8.9. *For $r \geq 1$, $s(r) \geq \frac{3^r+1}{2}$.*

Proof. Let $n \geq 1$ and assume $\chi : [1,n] \to \{1,2,\ldots,r\}$ is an r-coloring of $[1,n]$ that has no monochromatic Schur triple. Define an $(r+1)$-coloring $\hat{\chi} : [1,3n+1] \to \{1,2,\ldots,r+1\}$ that extends χ as follows. For all $x \in [n+1, 2n+1]$, let $\hat{\chi}(x) = r+1$; and for all $x \in [1,n] \cup [2n+2, 3n+1]$, let $\hat{\chi}(x) = \chi(y)$, where $x \equiv y \pmod{2n+1}$.

We now argue that $[1,3n+1]$ contains no monochromatic Schur triple under $\hat{\chi}$. Let $\{x,y,z\}$ with $x \leq y$ be a Schur triple. First, consider the color $r+1$. Since $2(n+1) > 2n+1$, $\{x,y,z\}$ cannot be a monochromatic Schur triple of color $r+1$. Now consider any color $j \neq r+1$. Since $\hat{\chi}$ is identical to χ on $[1,n]$, $\{x,y,z\} \subseteq [1,n]$ cannot be a monochromatic Schur triple of color j. Furthermore, since $(2n+2)+(2n+2) = 4n+4 > 3n+1$, it is not possible that $x, y \in [2n+2, 3n+1]$. Thus, $[2n+2, 3n+1]$ does not contain a Schur triple of color j. Hence, any Schur triple of color j must have $x \in [1,n]$ and $y \in [2n+2, 3n+1]$.

However, if $\{x,y,z\}$ is such a Schur triple of color j, by taking $y', z' \in [1,n]$ with $y' \equiv y \pmod{(2n+1)}$ and $z' \equiv z \pmod{(2n+1)}$, we see that $\{x, y', z'\}$ is a Schur triple of color j contained in $[1,n]$, a contradiction.

Thus, we have shown that if $s(r) \geq n+1$, then $s(r+1) \geq 3n+2$. Hence,

$$s(r+1) \geq 3s(r) - 1. \tag{8.3}$$

The proof is completed by induction on r. Clearly, $s(1) = 2 \geq \frac{3^1+1}{2}$. Now assume that $r \geq 1$ and that $s(r) \geq \frac{3^r+1}{2}$. Then, by (8.3), $s(r+1) \geq 3s(r) - 1 \geq 3\left(\frac{3^r+1}{2}\right) - 1 = \frac{3^{r+1}+1}{2}$. □

Since we have the existence of at least one monochromatic Schur triple in any r-coloring of $[1, s(r)]$, we must have an infinite number of Schur triples in any r-coloring of the natural numbers. To see this, note that if $[1,n]$ contains a monochromatic Schur triple, then so does $k[1,n] = \{k, 2k, \ldots, nk\}$, for any positive integer k. This holds by the trivial observation that if $x+y=z$ then $kx+ky=kz$. Now consider the sets $S_j = s(r)^j[1, s(r)]$ for $j = 0,1,2,\ldots$. For each j we have a monochromatic Schur triple that resides in S_j. Consequently, any r-coloring of the positive integers must contain an infinite number of monochromatic Schur triples.

With regards to the infinitude of monochromatic Schur triples, we consider the following problem: find the minimum number of monochromatic Schur triples that a 2-coloring of $[1,n]$ must have. We will answer this question in Theorem 8.15, below. As is often the case with such questions, the answer is given asymptotically, and so we will be using the $O(n)$ notation (see Section 1.5). To solve the above problem, we have need of the following notation and lemmas.

Notation. We denote by $\Delta_{\{a<b<c\}}$ a triangle on vertices a, b, c. We denote by $M_\chi(n)$ the number of monochromatic Schur triples under χ, where χ is a given 2-coloring of $[1,n]$.

Lemma 8.10. *In every edgewise 2-coloring of the complete graph K_n, there are at least $\frac{n^3}{24} + O(n^2)$ monochromatic triangles.*

The proof of Lemma 8.10 is Exercise 1.10.

Lemma 8.11. *Over all 2-colorings of $[1,n]$, the minimum number of monochromatic Schur triples is $O(n^2)$, i.e.,*

$$\min_\chi (M_\chi(n)) = O(n^2).$$

Proof. We note first that for any 2-coloring χ, $M_\chi(n) \leq cn^2 + O(n)$ for some positive constant c which is independent of χ (we leave this to the reader; see Exercise 8.2). To complete the proof, we use Lemma

8.10 to show that $M_\chi(n) \geq \frac{n^2}{48}+O(n)$ for any 2-coloring χ. Examining the proof of Schur's theorem, we see that there is a natural connection between triangles and Schur triples: with each triangle $\Delta_{\{a<b<c\}}$ we can associate the Schur triple $\{b-a, c-b, c-a\}$.

Define $a \oplus b = j$, where $a+b \equiv j \pmod n$ and $1 \leq j \leq n$.

Next, we notice that, for $j = 1, \ldots, n$, the triangles

$$\Delta_{\{a\oplus j<b\oplus j<c\oplus j\}} \text{ and } \Delta_{\{a\oplus j<(a+c-b)\oplus j<c\oplus j\}}$$

have the same associated Schur triple. Consequently, each Schur triple corresponds to at most $2n$ triangles (why?). Combining this with Lemma 8.10, we have at least $\frac{n^2}{48}+O(n)$ monochromatic Schur triples, thereby proving the lemma. □

Lemma 8.12. *Let $c > 0$ and let n be sufficiently large. Let χ be a 2-coloring of $[1, n+1]$ and let $\bar{\chi}$ be χ restricted to $[1, n]$. If $M_{\bar{\chi}}(n) = cn^2 + O(n)$, then $M_\chi(n+1) = cn^2 + O(n)$. Hence, we may take n to be even when determining the rate of growth of $M_\chi(n)$.*

Proof. The only monochromatic Schur triples counted in $M_\chi(n+1)$ that are not counted in $M_{\bar{\chi}}(n)$ are of the form $\{x, y, n+1\}$ with $x+y = n+1$. There are only $\lceil \frac{n}{2} \rceil$ possibilities, since $\{x, y\}$ must belong to $\left\{\{1, n\}, \{2, n-1\}, \ldots, \left\{\lfloor \frac{n+1}{2} \rfloor, n+1-\lfloor \frac{n+1}{2} \rfloor\right\}\right\}$. Hence,

$$0 \leq M_\chi(n+1) - M_{\bar{\chi}}(n) \leq \frac{n}{2}+1,$$

so the conclusion of the lemma holds. □

Lemma 8.13. *Let $\chi : [1, n] \to \{red, blue\}$ be described by R, the set of red integers under χ, and B, the set of blue integers under χ. Let N^+ be the set of non-monochromatic pairs $\{a, b\} \subseteq [1, n]$ such that $a+b > n$. Then*

$$2M_\chi(n) = \binom{n}{2} - 2|R||B| + |N^+|. \tag{8.4}$$

Proof. To justify (8.4), we will view the Schur triples as ordered sets, so that we consider $(a, b, a+b)$ and $(b, a, a+b)$ to be distinct (for $a \neq b$). Thus, we will actually count each Schur triple twice. This is the reason why $2M_\chi(n)$ is on the left-hand side of (8.4), instead of just $M_\chi(n)$.

First, note that $\binom{n}{2}$ counts all ordered Schur triples by choosing y and z in $x+y = z$. Next, let $a \in R$ and $b \in B$. Then the ordered Schur triples $(|b-a|, a, b)$ and $(a, b, a+b)$ are not monochromatic, and thus we want to remove these from the set of $\binom{n}{2}$ Schur triples. Hence, we subtract $2|R||B|$ from $\binom{n}{2}$. However, the ordered Schur triples $(a, b, a+b)$ with $a+b > n$ were not counted in $\binom{n}{2}$. Hence, since these were subtracted once in $-2|R||B|$, we need to add $|N^+|$. This establishes (8.4). □

We are now in a position to give the asymptotic minimum number of monochromatic Schur triples over all 2-colorings of $[1, n]$. We start with Theorem 8.14, which gives an upper bound on this minimum.

Theorem 8.14. *Over all 2-colorings of $[1, n]$, the minimum number of monochromatic Schur triples is at most $\frac{n^2}{22} + O(n)$.*

Proof. It is sufficient to prove the result for those n that are multiples of 11, since if $11 \nmid n$ we may apply Lemma 8.12 at most 10 times to get the desired result. Hence, assume $11 \mid n$ and consider the 2-coloring of $[1, n]$ defined by $R = [\frac{4n}{11}, \frac{10n}{11}]$ and $B = [1, \frac{4n}{11}-1] \cup [\frac{10n}{11}+1, n]$, where R is the set of red integers and B is the set of blue integers. To prove this theorem we will show that this coloring admits, asymptotically, only $\frac{n^2}{22}+O(n)$ monochromatic Schur triples. By letting $n = 11k$, our coloring is $R = [4k, 10k]$ and $B = [1, 4k-1] \cup [10k+1, 11k]$.

We start by counting the red Schur triples. Thus, we will count the number of solutions to $x+y = z$ in the interval $[4k, 10k]$. First, note that $z \geq 8k$. It follows that the red Schur triples are the triples

$$\left\{\{4k+c, z-4k-c, z\} : 8k \leq z \leq 10k, 0 \leq c \leq \tfrac{z-8k}{2}\right\}.$$

(So that we do not count any red Schur triple more than once, we require $4k+c \leq z-4k-c$, i.e., $0 \leq c \leq \frac{z-8k}{2}$.) Now, by summing over all possible values for z we have

$$\begin{aligned}\sum_{z=8k}^{10k} \left(\left\lfloor \tfrac{z-8k}{2} \right\rfloor + 1\right) &= 2\sum_{z=1}^{k} z + (k+1) \\ &= k(k+1) + (k+1) \\ &= k^2 + 2k + 1\end{aligned}$$

red Schur triples.

Next, we count the blue Schur triples. The same analysis as above shows that within the interval $[1, 4k-1]$ there are $4k^2 + O(k)$ blue Schur triples. The details for this are left to the reader as Exercise 8.3. Furthermore, there is obviously no blue Schur triple within the interval $[10k+1, 11k]$. However, we have not counted all of the blue Schur triples yet; for example, $\{1, 10k+1, 10k+2\}$ has not been counted.

In order to count the remaining blue Schur triples $\{x, y, z\}$, let $x \in [1, 4k-1]$ and let $z \in [10k+1, 11k]$. Notice that we must have $y \in [10k+1, 11k]$ in order to have $x + y = z$. Writing $x = 1 + c$ and $y = z - c - 1$ with $c \geq 0$, we must have $z - 1 - c \geq 10k + 1$, i.e., $c \leq z - 10k - 2$. Hence, we wish to count the number of triples $\{1 + c, z - c - 1, z\}$ with $0 \leq c \leq z - 10k - 2$ and $z \in [10k+1, 11k]$. Summing over all possible values of z, we have

$$\begin{aligned}\sum_{z=10k+2}^{11k} (z - 10k - 1) &= \tfrac{1}{2}\left((11k)^2 - (10k)^2\right) - 10k^2 + O(k) \\ &= \tfrac{k^2}{2} + O(k)\end{aligned}$$

remaining blue Schur triples.

Adding all cases together, we have

$$\frac{11}{2}k^2 + O(k) = \frac{11}{2}\left(\frac{n}{11}\right)^2 + O(n) = \frac{n^2}{22} + O(n)$$

monochromatic Schur triples. □

We now show that the upper bound given in Theorem 8.14 is actually the minimum.

Theorem 8.15. *Over all 2-colorings of $[1, n]$, the minimum number of monochromatic Schur triples is $\frac{n^2}{22} + O(n)$.*

Proof. Denote by R and B the sets of red integers and blue integers, respectively, in any red-blue coloring χ. Let N^+ be the set of non-monochromatic pairs $\{a, b\} \subseteq [1, n]$ such that $a + b > n$, and let N^- be the set of non-monochromatic pairs $\{a, b\} \subseteq [1, n]$ such that $a + b \leq n$.

From Lemma 8.13, we have $2M_\chi(n) = \binom{n}{2} - 2|R||B| + |N^+|$. Now, since $|R| + |B| = n$, we have

$$|R| = n\left(\tfrac{1}{2} + \alpha\right) \text{ and } |B| = n\left(\tfrac{1}{2} - \alpha\right) \tag{8.5}$$

for some $\alpha \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Notice that for α and n fixed, $|R|$ and $|B|$ are determined, so that by minimizing $|N^+|$ we are minimizing $M_\chi(n)$. Hence, our first goal is to minimize $|N^+|$ for α and n fixed. To this end, notice that

$$|N^+| + |N^-| = |R||B| = n^2\left(\tfrac{1}{4} - \alpha^2\right). \tag{8.6}$$

Our next step is to find an upper bound for $|N^-| - |N^+|$. Such an upper bound, when coupled with (8.6), will give a lower bound for $|N^+|$, thereby bounding the number of monochromatic Schur triples from below. From Lemma 8.12, we may assume that n is even, so that to obtain an upper bound for $|N^-| - |N^+|$ we may decompose $[1, n]$ into disjoint 2-element subsets $\{x, n+1-x\}$, $x = 1, 2, \ldots, \frac{n}{2}$.

For the pair of sets $X = \{a, n+1-a\}$ and $Y = \{b, n+1-b\}$ with $a \neq b$, we will count the contribution of the elements of these sets to N^+ and to N^-. In the following we will only consider $x+y$ with $x \in X$ and $y \in Y$. Thus, we do not count pairs of the form $\{x, n+1-x\}$. The reason for this is that each such pair only contributes one to N^+, and since the number of such pairs is $\frac{n}{2} = O(n)$, by Lemma 8.11 we may safely ignore these pairs in the asymptotic calculation.

We consider four cases. In each case, let

$$P = \{(x, y) : x \in X, y \in Y\}.$$

We will determine the number of pairs in P contributing to N^- and the number contributing to N^+. By doing so, we will find a function $f(n)$ such that $|N^-| - |N^+| \leq f(n)$. Such a function, together with (8.6), will provide a lower bound for $|N^+|$.

Case 1. X and Y are monochromatic of different colors. These two sets produce four non-monochromatic pairs in P whose sums are $a + b, n + 1 + a - b, n + 1 + b - a$, and $2n + 2 - a - b$. Clearly two of these sums exceed n and the other two are less than n. Consequently, $|P \cap N^+| = |P \cap N^-|$, i.e., the number of pairs in P contributing to N^+ is equal to the number of pairs in P contributing to N^-.

Case 2. X and Y are non-monochromatic with a and b being different colors. These sets produce two non-monochromatic pairs with sums $a+b$ and $2n+2-a-b$. Hence, $|P \cap N^+| = |P \cap N^-|$.

Case 3. X and Y are non-monochromatic with a and b being the same color. These sets produce two non-monochromatic pairs with sums $n+1+b-a$ and $n+1+a-b$. Thus, $|P \cap N^+| = |P \cap N^-|$.

Case 4. X is monochromatic and Y is non-monochromatic with a and b different colors. These sets produce two non-monochromatic pairs with sums $a+b$ and $n+1+b-a$. Since in our decomposition $a+b<n$, both of these sums are at most n. In this situation the contribution of these pairs to N^- is 2 greater than their contribution to N^+, i.e., $|P \cap N^+|+2 = |P \cap N^-|$.

Since our goal is to find some $f(n)$ such that $|N^-|-|N^+| \leq f(n)$, we note that only Case 4 gives us a situation where $|N^-|$ is larger than $|N^+|$ (so that we must have $f(n) > 0$ for $n > 0$). We now analyze Case 4 to determine $f(n)$.

Let βn be the number of non-monochromatic sets $\{x, n+1-x\}$. Then exactly one of $x, n+1-x$ is in R. Thus, using (8.5), we see that the number of monochromatic sets $\{x, n+1-x\}$ in R is $\frac{1}{2}\left(n\left(\frac{1}{2}+\alpha\right)\right) - \frac{\beta n}{2} = \left(\frac{1}{2}+\alpha-\beta\right)\frac{n}{2}$. Looking at the details of Case 4, we see that for every non-monochromatic set $\{b, n+1-b\}$, a monochromatic set $\{a, n+1-a\}$ with a and b the same color will have sums that contribute exactly 2 more to N^- than to N^+. Therefore,

$$\begin{aligned} |N^-|-|N^+| &= 2\beta n\left(\tfrac{1}{2}+\alpha-\beta\right)\left(\tfrac{n}{2}\right)+O(n) \\ &\leq \frac{\left(\frac{1}{2}+\alpha\right)^2}{4}n^2+O(n), \end{aligned} \tag{8.7}$$

where the second expression equals the third expression precisely when $\beta = \frac{1}{2}\left(\frac{1}{2}+\alpha\right)$. Thus, we have found an upper bound on $|N^-|-|N^+|$ as sought.

Combining (8.6) and (8.7), we get

$$|N^+| \geq \left(\frac{\frac{1}{4}-\alpha^2}{2} - \frac{\left(\frac{1}{2}+\alpha\right)^2}{8}\right)n^2+O(n). \tag{8.8}$$

Using (8.4) with (8.8), we have

$$2M_\chi(n) \geq \left(\frac{11\left(2\alpha-\frac{1}{11}\right)^2}{32}+\frac{1}{11}\right)n^2+O(n). \tag{8.9}$$

We can easily see that (8.9) is minimized when $\alpha = \frac{1}{22}$, giving $2M_\chi(n) \geq \frac{n^2}{11}+O(n)$, i.e., $M_\chi(n) \geq \frac{n^2}{22}+O(n)$.

The proof is complete, since the upper bound given in Theorem 8.14 matches this lower bound. □

Now that we see that the coloring in Theorem 8.14 attains the minimum, we remark that it can be shown that this coloring is essentially the only coloring that attains the minimum number of monochromatic Schur triples.

We have an extension of Theorem 8.14 that gives the following upper bound for the analogous r-colored question ($r \geq 2$).

Corollary 8.16. *Let $r \geq 2$. The minimum number of monochromatic Schur triples over all r-colorings of $[1,n]$ is no greater than $\frac{n^2}{11\cdot 2^{2r-3}}+O(n)$.*

Proof. Extending the coloring defined in the proof of Theorem 8.14, we define, for $n \geq 2^r$, the coloring $\chi : [1,n] \to \{1,2,\ldots,r\}$ by

$$\chi(i) = \begin{cases} j & \text{if } \frac{n}{2^j} < i \leq \frac{n}{2^{j-1}} \text{ for } 1 \leq j \leq r-2, \\ r-1 & \text{if } 1 \leq i \leq \frac{4n}{2^{r-2}11} \text{ or } \frac{10n}{2^{r-2}11} < i \leq \frac{n}{2^{r-2}}, \\ r & \text{if } \frac{4n}{2^{r-2}11} < i \leq \frac{10n}{2^{r-2}11}. \end{cases}$$

The calculation showing that χ admits only $\frac{n^2}{11\cdot 2^{2r-3}}+O(n)$ monochromatic Schur triples is left to the reader as Exercise 8.6. □

8.2. A Generalization of Schur's Theorem

We turn our attention to a generalization of Schur's theorem, given below as Theorem 8.17. Another, more powerful, generalization was done by Richard Rado, a student of Schur, and is investigated in Chapter 9.

It will be convenient to use the following notation.

Notation. Let $\mathcal{L}(t)$ represent the equation $x_1+x_2+\cdots+x_{t-1}=x_t$, where $x_1,x_2,\ldots,x_t$ are variables.

Theorem 8.17. *Let $r\geq 1$ and, for $1\leq i\leq r$, assume that $k_i\geq 3$. Then there exists a least positive integer $S=S(k_1,k_2,\ldots,k_r)$ such that for every r-coloring of $[1,S]$ there is a solution to $\mathcal{L}(k_j)$ of color j for some $j\in\{1,2,\ldots,r\}$.*

The numbers $S(k_1,k_2,\ldots,k_r)$ are called the *generalized Schur numbers.* We denote the case $k_1=k_2=\cdots=k_r=k$ more simply by $S_r(k)$.

To fully understand Theorem 8.17, consider the following examples.

Example 8.18. Any red-blue-green coloring of $[1,S(3,4,5)]$ must contain a solution to $\mathcal{L}(3)$ consisting of only red integers, or a solution to $\mathcal{L}(4)$ consisting of only blue integers, or a solution to $\mathcal{L}(5)$ consisting of only green integers. Theorem 8.17 tells us that $S(3,4,5)$ is the least positive integer such that the above condition is met.

Example 8.19. If $k_i=3$ for $1\leq i\leq r$ we have $S(3,3,\ldots,3)=s(r)$, the r-colored Schur number we have already investigated. Considering $S_r(k)=S(k,k,\ldots,k)$, Theorem 8.17 states that any r-coloring of $[1,S_r(k)]$ must contain a monochromatic solution to $\mathcal{L}(k)$.

Example 8.20. We will determine $S(4,5)$. To see that $S(4,5)\geq 14$, consider the 2-coloring of $[1,13]$ defined by $R=\{1,2,12,13\}$ and $B=[3,11]$, where R is the set of red integers and B is the set of blue integers. It is easy to check that this coloring admits no red solution to $\mathcal{L}(4)$ and no blue solution to $\mathcal{L}(5)$.

To show that $S(4,5)\leq 14$, let R be the set of red integers and B be the set of blue integers in a given red-blue coloring of $[1,14]$. We must show that there is either a red solution to $\mathcal{L}(4)$ or a blue solution to $\mathcal{L}(5)$.

We proceed by contradiction, so assume that no red solution to $\mathcal{L}(4)$ and no blue solution to $\mathcal{L}(5)$ exist in the given coloring. First consider the case in which $1\in R$. Since $1+1+1=3$ we must have $3\in B$. Next, since $3+3+3+3=12$ we must have $12\in R$. This implies that $10,14\in B$, for otherwise $1+1+10=12$ or $1+1+12=14$ would be a red solution. Next, since $2+2+3+3=10$, we must have $2\in R$. This implies that we must have $5\in B$ to avoid the red solution $1+2+2=5$, which gives $3+3+3+5=14$ as a blue solution, a contradiction. The case when $1\in B$ is left to the reader in Exercise 8.7.

Just as the proof of Schur's theorem follows easily from Ramsey's theorem, so does the proof of Theorem 8.17.

Proof of Theorem 8.17. Let $n=R(k_1,k_2,\ldots,k_r)$ be the r-colored Ramsey number. Consider the same edgewise coloring as in the proof of Schur's theorem. That is, number the vertices of K_n by $1,2,\ldots,n$ and arbitrarily partition the set $\{1,2,\ldots,n-1\}$ into r subsets, with each of these subsets corresponding to a different color. Color the edge connecting vertices i and j according to the subset of which $|j-i|$ is a member.

By Ramsey's theorem, this coloring of K_n must admit a monochromatic K_{k_j} subgraph for some color $1\leq j\leq r$. For ease of notation let $k=k_j$. Let the vertices of this monochromatic subgraph be $\{v_0,v_1,\ldots,v_{k-1}\}$, and define the differences $d_i=v_i-v_0$. By ordering and renaming the vertices we may assume that $d_{i-1}<d_i$ for $2\leq i\leq k-1$. Since K_k is monochromatic of color j, we have that the edges $\overline{v_{i-1}v_i}$, $i=1,2,\ldots,k-1$, and $\overline{v_{k-1}v_0}$ must all have color j. Since $v_i-v_{i-1}=d_i-d_{i-1}$, for $i=2,3,\ldots,k-1$, we see that d_i-d_{i-1}, $2\leq i\leq k-1$, d_1, and d_{k-1} must all have color j. Hence we have the solution to $\mathcal{L}(k)$ of color j given by $d_1+\sum_{i=2}^{k-1}(d_i-d_{i-1})=d_{k-1}$ in any r-coloring of $\{1,2,\ldots,n-1\}$. $\square$

Now that we have extablished the existence of generalized Schur numbers, we investigate the values and bounds for some of these numbers. We start with Theorem 8.21, which gives the exact values for all 2-colored generalized Schur numbers. We give part of the proof of Theorem 8.21, and leave the remainder to the reader as Exercise 8.8.

Theorem 8.21. *Let $k, \ell \geq 3$. Then*

$$S(k,\ell) = \begin{cases} 3\ell - 4 & \text{if } k = 3 \text{ and } \ell \text{ is odd,} \\ 3\ell - 5 & \text{if } k = 3 \text{ and } \ell \text{ is even,} \\ k\ell - \ell - 1 & \text{if } 4 \leq k \leq \ell. \end{cases}$$

Proof. For all colorings below, we denote by R the set of red integers and by B the set of blue integers.

We start by showing that the given expressions serve as lower bounds for their respective cases. To do this, for each case we exhibit a valid 2-coloring, i.e., one that avoids both a red solution to $\mathcal{L}(k)$ and a blue solution to $\mathcal{L}(\ell)$.

`Case 1.` $k = 3$ and $\ell \geq 3$ is odd. Consider the coloring of $[1, 3\ell - 5]$ given by

$R = \{n : 1 \leq n \leq \ell - 2,\ n \text{ odd}\} \cup \{n : 2\ell - 2 \leq n \leq 3\ell - 5,\ n \text{ even}\}$;

$B = [1, 3\ell - 5] - R$.

We will first establish that there is no red solution to $\mathcal{L}(3)$. Let $x_1 \leq x_2 < x_3 \leq 3\ell - 5$ be red integers. If $\{x_1, x_2\} \subseteq [1, \ell - 2]$ then $x_1 + x_2$ belongs to $[2, 2\ell - 4]$ and is even. Thus $x_1 + x_2$ is colored blue, so that $\{x_1, x_2, x_3\}$ is not a red solution to $\mathcal{L}(3)$. Hence, we assume that $x_1 \in [1, \ell - 2]$ and $x_2 \in [2\ell - 2, 3\ell - 5]$. Here we have $x_1 + x_2 \in [2\ell - 1, 4\ell - 7]$ and odd. This shows that either $x_1 + x_2$ is colored blue or out of bounds, so that $\{x_1, x_2, x_3\}$ is not a red solution to $\mathcal{L}(3)$. Finally, we assume $\{x_1, x_2\} \subseteq [2\ell - 2, 3\ell - 5]$. This gives $x_1 + x_2 \geq 4\ell - 4 > 3\ell - 5$. In this situation, since the sum is out of bounds, $\{x_1, x_2, x_3\}$ cannot be a red solution to $\mathcal{L}(3)$.

Next, we show that there is no blue solution to $\mathcal{L}(\ell)$. Assume $x_1 \leq x_2 \leq \cdots \leq x_{\ell-1} < x_\ell \leq 3\ell - 5$ are ℓ integers all colored blue. If $\{x_1, x_2, \ldots, x_{\ell-1}\} \subseteq [2, \ell - 3]$ then $\sum_{i=1}^{\ell-1} x_i \geq 2\ell - 2$ and is even. This implies that $\sum_{i=1}^{\ell-1} x_i$ is either colored red or is out of bounds. Hence $\{x_1, x_2, \ldots, x_\ell\}$ is not a blue solution to $\mathcal{L}(\ell)$. If, on the other hand, there exists $j \in \{1, 2, \ldots, \ell - 1\}$ such that $x_j \notin [2, \ell - 3]$, then $x_1, x_2, \ldots, x_{j-1} \geq 2$ implies that $\sum_{i=1}^{j} x_i \geq 2(j-1) + \ell - 3 = 3j - 5$. However, since $x_\ell \leq 3\ell - 5$, we may assume that $\sum_{i=1}^{\ell-1} x_i = 3l - 5$, which is colored red. This shows that $\{x_1, x_2, \ldots, x_\ell\}$ cannot be a blue solution to $\mathcal{L}(\ell)$.

This completes the proof of the lower bound for $k = 3$ and ℓ odd. The lower bounds for the other cases are left to the reader in Exercise 8.8.

We now move on to the upper bounds. We prove one case and leave the other cases to the reader in Exercise 8.8. We will consider $4 \leq k \leq \ell$ and $1 \in R$. We prove the equivalent statement:

$$S(k+1, \ell+1) \leq k\ell + k - 1 \quad \text{for } 3 \leq k \leq \ell.$$

Assume, for a contradiction, that there exists a 2-coloring of $[1, k\ell + k - 1]$ that avoids both a red solution to $\mathcal{L}(k+1)$ and a blue solution to $\mathcal{L}(\ell + 1)$.

Since $1 \in R$, it follows that $k \in B$, and hence $k\ell \in R$, which in turn implies that $\ell \in B$. Since $1, k\ell \in R$, we have $k\ell + k - 1 \in B$. We deduce from this that $2k - 1 \in R$ (or else the $(\ell+1)$-tuple $\{k, k, \ldots, k, 2k-1, k\ell+k-1\}$ would be a blue solution). Since $2k-1$ is red, we must have $2 \in B$ or else the $(k+1)$-tuple $\{1, 2, 2, \ldots, 2, 2k-1\}$ would be red. This implies that $3\ell - 2 \in R$ in order to avoid the blue solution given by the $(\ell+1)$-tuple $\{2, 2, \ldots, 2, \ell, 3\ell - 2\}$. This in turn shows that we must have $3\ell + k - 3 \in B$ in order not to have the $(k+1)$-tuple $\{1, 1, \ldots, 1, 3\ell - 2, 3\ell + k - 3\}$ be red.

We next show that $\ell + 1 \in B$. First note that $3 \in R$, for otherwise $\{3, 3, \ldots, 3, k, 3\ell + k - 3\}$ would be blue. Since $1, 3 \in R$, it follows that $k + 2 \in B$. Now assume, for a contradiction, that $\ell + 1 \in R$. Under this assumption, we have $2\ell + k \in B$, so that the $(k+1)$-tuple $\{1, 1, \ldots, 1, \ell + 1, \ell + 1, 2\ell + k\}$ is not red. This leads to the blue solution to $\mathcal{L}(\ell + 1)$ given by $\{2, 2, \ldots, 2, k + 2, 2\ell + k\}$, the desired contradiction. Thus, $\ell + 1 \in B$.

We have shown that $2, k, \ell + 1$ and $3\ell + k - 3$ are all blue, so that the $(\ell+1)$-tuple $\{2, 2, \ldots, 2, k, \ell + 1, 3\ell + k - 3\}$ gives a blue solution to $\mathcal{L}(\ell + 1)$, a contradiction. □

We conclude this section with upper and lower bounds for the generalized Schur number $S_r(k)$. We start with an upper bound, the proof of which is very similar to the proof of Corollary 8.6.

Theorem 8.22. *Let $r \geq 2$. If $k \geq 3$, then $S_r(k) \leq R_r(k) - 1$, where $R_r(k)$ is the r-colored Ramsey number.*

Proof. The proof of Theorem 8.17 gives a correspondence between the edgewise r-colorings of K_n and the r-colorings of $[1, n-1]$. Hence, if $n = R_r(k)$, we have $S_r(k) \leq n - 1 = R_r(k) - 1$. □

As we did in the proof of Lemma 8.7, it is possible to use Ramsey's theorem to obtain an upper bound independent of the Ramsey numbers. However, the formula is rather cumbersome, and we will not present it here.

On the other hand, we do have a nice formula for a lower bound. The lower bound is a generalization of Theorem 8.9. The proof is quite similar to that of Theorem 8.9 and we leave much of it to the reader as Exercise 8.9.

Theorem 8.23. *Let $r \geq 2$. If $k \geq 3$, then $S_r(k) \geq \frac{k^{r+1}-2k^r+1}{k-1}$.*

Proof. Let $\chi : [1, n] \to \{1, 2, \ldots, r\}$ be an r-coloring of $[1, n]$ with no monochromatic solution to $\mathcal{L}(k)$. Define an $(r+1)$-coloring

$$\hat{\chi} : [1, kn + k - 1] \to \{1, 2, \ldots, r + 1\}$$

that extends χ as follows: for $x \in [n+1, (k-1)n+k-2]$ let $\hat{\chi}(x) = r+1$; otherwise let $\hat{\chi}(x) = \chi(y)$, where $x \equiv y \pmod{((k-1)n + k - 2)}$. We leave it to the reader to deduce that under $\hat{\chi}$, $[1, kn + k - 1]$ contains no monochromatic solution to $\mathcal{L}(k)$. Thus, we have that if $S_r(k) \geq n+1$ then $S_{r+1}(k) \geq kn+k-1$. Hence, $S_{r+1}(k) \geq kS_r(k)-1$. Noting that $S_1(k) = k - 1$, we leave it to the reader to show that $S_r(k) \geq \frac{k^{r+1}-2k^r+1}{k-1}$. □

8.3. Refinements of Schur's Theorem

Schur's theorem (Theorem 8.2) tells us that any r-coloring of $[1, s(r)]$ must contain a monochromatic solution to $x + y = z$. However, x and y need not be distinct, and in fact this was crucial in showing that $s(2) = 5$ (see Example 8.5). You may have asked yourself: does Schur's theorem hold if we require that x and y be distinct? This is obviously a stricter condition and, as we have seen, this can turn

Ramsey-type statements into false statements. However, in this case, the resulting Ramsey-type statement is true.

Theorem 8.24. *For $r \geq 1$, there exists a minimal integer $\hat{s} = \hat{s}(r)$ such that every r-coloring of $[1, \hat{s}]$ admits a monochromatic solution to $x + y = z$ with x and y distinct.*

Proof. For $r = 1$ we have $\hat{s}(1) = 3$. We proceed by induction on r. Let $r \geq 2$ and assume that $\hat{s}(r-1)$ exists. We will show that $\hat{s}(r) \leq w(2\hat{s}(r-1) + 1; r)$, where $w(k; r)$ is the van der Waerden function.

For any r-coloring of $[1, w(2\hat{s}(r-1) + 1; r)]$ there is a monochromatic arithmetic progression $\{a, a + d, a + 2d, \cdots, a + 2\hat{s}(r-1)d\}$ of color, say, red. If there exists i, $1 \leq i \leq \hat{s}(r-1)$, such that id is red, then $x = id$, $y = a + id$, and $z = a + 2id$ are all red, so that we have $x + y = z$ with x and y distinct. Hence, assume that $d[1, \hat{s}(r-1)] = \{id : 1 \leq i \leq \hat{s}(r-1)\}$ is void of the color red. Then, $d[1, \hat{s}(r-1)]$ is $(r-1)$-colored. By the inductive assumption and Proposition 2.30 we have a monochromatic solution to $x+y = z$ with x and y distinct. □

From previous chapters we have seen that the best known upper bounds on the van der Waerden numbers are very large. Hence, the upper bound for $\hat{s}(r)$ provided by the proof of Theorem 8.24 is not very useful. For example, the proof gives $\hat{s}(2) \leq w(7; 2)$. Gowers' bound (Theorem 2.21) gives the smallest known (at least to our knowledge) upper bound for $w(7; 2)$:

$$2^{2^{2^{2^{65536}}}},$$

a number whose decimal representation cannot be written down because it exceeds the number of atoms in the universe (estimated as 10^{78} atoms).

As it turns out, much better upper bounds for $\hat{s}(r)$ are known. We can fairly easily provide one such bound; this bound is given in Theorem 8.25. The best known upper bound is given in Theorem 8.26, which we state without proof.

Theorem 8.25. *For $r \geq 1$, $\hat{s}(r) \leq 2^{3r!-1}$.*

Proof. Let $n = 3r!$. Let $\{a_i\}_{i=1}^{n}$ be an increasing sequence of positive integers with no arithmetic progression of length 3.

We first claim that any r-coloring of $[1, a_n]$ must contain a monochromatic solution to $x+y=z$ with $x \neq y$. To see this, we show that there exist $a_i < a_j < a_k$ such that $a_k - a_j, a_k - a_i, a_j - a_i \in [1, a_n]$ all have the same color.

Recall that $R_r(3)$ is an r-color Ramsey number. From Lemma 8.7 we have $R_r(3) \leq n$. Hence, any edgewise r-coloring of K_n must contain a monochromatic triangle. Label the vertices of K_n with the a_i's. We use the same coloring as defined in the proof of Schur's theorem. Hence, we color the edge between vertices $a_i < a_j$ depending upon $a_j - a_i$. As a result, we have a monochromatic triangle on three vertices, say $a_i < a_j < a_k$.

Now, since $a_k - a_i = (a_k - a_j) + (a_j - a_i)$ and $a_k - a_j \neq a_j - a_i$ (because a_i, a_j, a_k are not in arithmetic progression), we have proven the claim.

To complete the proof, let $a_i = 2^{i-1}$, $i = 1, 2, \ldots, n$, and note that $\{2^i\}_{i=0}^{n-1}$ contains no arithmetic progression of length three. □

Theorem 8.26. *For $r \geq 1$, $\hat{s}(r) \leq \lfloor r!re \rfloor + 1$.*

There is still room for improvement on the upper bound of Theorem 8.26. Via a computer search it can be proved that $\hat{s}(2) = 9$; the bound from Theorem 8.26 gives $\hat{s}(2) \leq 11$.

We conclude this chapter with a result similar to Theorem 8.24 for the generalized Schur numbers. Recall that we denote by $\mathcal{L}(t)$ the equation $x_1 + x_2 + \cdots + x_{t-1} = x_t$.

Theorem 8.27. *For any $r \geq 1$, there exists a least positive integer $\hat{S} = \hat{S}(k_1, k_2, \ldots, k_r)$ such that for any r-coloring of $[1, \hat{S}]$ there is a monochromatic solution to $\mathcal{L}(k_j)$ of color j for some $j \in \{1, 2, \ldots, r\}$ with $x_1, x_2, \cdots, x_{k_j}$ distinct.*

Proof. Let $n = R(k_1, k_2, \ldots, k_r)$. Let $\{a_i\}_{i=1}^{n}$ be an increasing sequence of positive integers such that

$$a_m - a_\ell \neq a_j - a_i \quad \text{for all } 1 \leq i < j \leq \ell < m \leq n.$$

We first claim that any r-coloring of $[1, a_n]$ must contain a monochromatic solution to $\mathcal{L}(k_j)$ of color j with $x_1, x_2, \ldots, x_{k_j}$ distinct. To establish this claim, we show that for some $j \in \{1, 2, \ldots, r\}$ there exist $a_{i_1} < a_{i_2} < \cdots < a_{i_{k_j}}$ such that $\{a_{i_{t+1}} - a_{i_t} : 1 \leq t \leq k_j - 1\} \subseteq [1, a_n]$ is monochromatic.

By the definition of n, any edgewise r-coloring of K_n must contain a j-colored K_{k_j} for some $1 \leq j \leq r$. Label the vertices of K_n with the a_i's. For all pairs of vertices $a_s < a_t$, coloring the edge between a_s and a_t with the color $a_t - a_s$, we have, for some $j \in [1, r]$, a monochromatic K_{k_j} of color j on $k = k_j$ vertices, say $a_{n_1} < a_{n_2} < \cdots < a_{n_k}$.

Now, since $a_{n_k} - a_{n_1} = \sum_{i=1}^{k-1} (a_{n_{i+1}} - a_{n_i})$ and all the differences $a_t - a_s$, $1 \leq s < t \leq n$, are distinct, we have proven the claim.

To complete the proof, let $a_i = 2^{i-1}$, $i = 1, 2, \ldots, n$. To show that the differences $a_t - a_s$, $1 \leq s < t \leq n$, are distinct, assume (for a contradiction) that there exist $0 \leq w < x \leq y < z \leq n-1$ such that $2^z - 2^y = 2^x - 2^w$. Then $2^y(2^{z-y} - 1) = 2^w(2^{x-w} - 1)$ and hence

$$2^{y-w}(2^{z-y} - 1) = 2^{x-w} - 1,$$

a contradiction since the left side is even and the right side is odd. □

Remark 8.28. We remind the reader of Definition 1.23. We say that an equation $\mathcal{L}$ is regular if, for all $r \geq 1$, for every r-coloring of $\mathbb{Z}^+$ there is a monochromatic solution to $\mathcal{L}$. From Theorem 8.17, we see that $x_1 + x_2 + \cdots + x_{k-1} = x_k$ is regular.

8.4. Exercises

8.1 Regarding the discussion in the introduction of this chapter, explain why the planes $x + y = z$ and $x + y = 2z$ must, under any given finite coloring, contain an infinite number of colored points with positive integer coordinates.

8.2 The maximum number, asymptotically, of monochromatic Schur triples over all r-colorings of $[1, n]$ is known to be $cn^2 + O(n)$, for some $c > 0$. Find c.

8.3 Use Exercise 8.2 to complete Theorem 8.14 by showing that within $[1, 4k - 1]$ there are $4k^2 + O(k)$ Schur triples.

8.4 Use Lemma 8.11 to show that, under any 2-coloring of $[1, n]$ the number of monochromatic 3-term arithmetic progressions is $O(n^2)$.

8.5 Show that the 2-coloring of $[1, 1000m] = [1, n]$, $m \in \mathbb{Z}^+$, given by

$$1^{52m}0^{10m}1^{52m}0^{68m}1^{106m}0^{212m}1^{212m}0^{106m}1^{68m}0^{52m}1^{10m}0^{52m}$$

yields only $.053382n^2 + O(n)$ monochromatic 3-term arithmetic progressions. (A computer may be helpful here.) (Note: this is slightly more than $\frac{4}{75}n^2 + O(n)$, which the second-named author conjectures to be the minimum number, over all 2-colorings of $[1, n]$, of monochromatic 3-term arithmetic progressions.)

8.6 Show that the coloring given in Corollary 8.16 admits only $\frac{n^2}{11 \cdot 2^{2r-3}} + O(n)$ monochromatic Schur triples.

8.7 Finish Example 8.20 by

a) showing that the given coloring of $[1, 13]$ is indeed a valid coloring, i.e., that there is no red solution to $\mathcal{L}(4)$ and no blue solution to $\mathcal{L}(5)$, and

b) concluding that any 2-coloring of $[1, 14]$ with $1 \in B$ must admit either a red solution to $\mathcal{L}(4)$ or a blue solution to $\mathcal{L}(5)$.

8.8 Finish the proof of Theorem 8.21 as follows:

a) Show that the following colorings provide lower bounds for the stated cases:

i) For $\ell \geq 4$ and even, the 2-coloring of $[3\ell - 6]$ given by $R = R_1 \cup R_2$ and $B = [1, 3\ell - 6] - R$, where $R_1 = \{n : 1 \leq n \leq \ell - 3,\ n \text{ odd}\}$ and $R_2 = \{n : 2\ell - 2 \leq n \leq 3\ell - 6,\ n \text{ even}\}$;

ii) For $4 \leq k \leq \ell$, the 2-coloring of $[k\ell - \ell - 2]$ given by $R = R_1 \cup R_2$ and $B = [1, k\ell - \ell - 2] - R$, where $R_1 = \{n : (k-1)(\ell - 1) \leq n \leq k\ell - \ell - 2\}$ and $R_1 = \{n : 1 \leq n \leq k - 2\}$.

b) Finish the upper bound $4 \leq k < \ell$ for $1 \in B$. (Hint: in the case where $1 \in R$, interchange all occurrences of the set R with the set B, of the word 'red' with the word 'blue', and of the value k with the value ℓ. Check to make sure that the resulting argument is correct.)

c) Deduce the upper bounds for $k = 3$ and $\ell \geq 4$ and even, and for $k = 3$ and $\ell \geq 3$ and odd, in a fashion similar to the case presented in the proof of Theorem 8.21. (Hint: begin by considering two subcases: $1 \in R$ and $1 \in B$.)

8.9 Finish the proof of Theorem 8.23 by showing that

a) $\hat{\chi}$ admits no monochromatic Schur triple, and

b) $S_r(k) \geq \frac{k^{r+1} - 2k^r + 1}{k-1}$.

8.5. Research Problems

8.1. Prove or disprove the conjecture that $s(5) = 160$.
References: [**106**], [**110**]

***8.2.** For $r \geq 3$, find the minimum number, asymptotically, over all r-colorings of $[1, n]$, of monochromatic Schur triples.
References: [**75**], [**225**], [**248**]

***8.3.** Find the minimum number, asymptotically, over all 2-colorings of $[1, n]$, of monochromatic 3-term arithmetic progressions. (The 2-coloring of $[1, n]$ that gives the smallest upper bound (to date) for this minimum is given in Exercise 8.5.)
References: [**203**], [**225**], [**248**]

***8.4.** Find the minimum number, asymptotically, over all 2-colorings of $[1, n]$, of monochromatic solutions to $x + ay = z$, for $a \neq 1$. (The existence of such monochromatic solutions is implied by Rado's theorem, given in the next chapter.)
References: [**225**], [**248**]

8.5. Determine new bounds and/or values for the r-colored generalized Schur numbers, for $r \geq 3$.
References: [**36**], [**224**]

8.6. Determine the exact value of the 2-colored *strict* generalized Schur numbers (see Theorem 8.27), i.e., determine the least positive integer $G(k, \ell)$ such that any 2-coloring of $[1, G(k, \ell)]$ must admit either a red solution to $x_1 + \cdots + x_{k-1} = x_k$ with $x_1 < x_2 < \cdots < x_k$, or a blue solution to $x_1 + \cdots + x_{\ell-1} = x_\ell$ with $x_1 < x_2 < \cdots < x_\ell$.
References: [**41**], [**127**, p. 77], [**224**]

8.7. Improve upon the upper bound given for the *strict* Schur numbers $\hat{s}(r)$ (from Theorem 8.25).
Reference: [**45**]

8.6. References

§**8.1**. Theorem 8.1, Schur's theorem, and Theorem 8.9 can be found in [**247**]. See [**273**] for a brief summary of Wiles' proof of Fermat's Last Theorem. Erdős and Szekeres' rediscovery of Ramsey's theorem can be found in [**101**]. Goodman's result on the number of monochromatic triangles in any 2-coloring of the edges of K_n can be found in [**120**]. The problem of finding the asymptotic minimum number of Schur triples over all 2-colorings of $[1, n]$ was first posed by Graham, Rödl, and Ruciński in [**125**]. It was first solved, independently, by Robertson and Zeilberger [**225**] and Schoen [**248**]. The proof of Lemma 8.11 can be found in [**125**]. The proof of Theorem 8.15 is due to Datskovsky [**75**]. The coloring given in Theorem 8.14 is due to Zeilberger. The fact that the coloring in Theorem 8.14 is essentially the only minimal coloring is proved in [**248**]. Corollary 8.16 is from [**225**].

§**8.2**. The proof of Theorem 8.17 is from [**222**]. Theorem 8.17 proves the existence, in particular, of $S_r(k)$; however, this is implied by Rado's theorem (Theorem 9.2), which was established before Theorem 8.17. The values for the 2-color generalized Schur numbers $S(k, \ell)$ were determined for $k = \ell$ in [**36**] and for all k and ℓ in [**224**].

§**8.3**. The proof of Theorem 8.24 is adapted from the proof of Theorem 2 in [**127**, p. 70]. The proof of Theorem 8.26 is found in [**45**], which improves upon Irving's bound given in [**149**].

§**8.4**. Exercise 8.5's coloring was discovered by Pablo Parrilo [**203**].

Additional References: For a tribute to Schur and an overview of his work, see [**190**]. Exoo [**106**] and Fredricksen and Sweet [**110**] give the current best known lower bounds for the Schur numbers $s(5)$, $s(6)$, and $s(7)$. Bergelson, in [**27**], uses ergodic theory to prove a density statement which generalizes and strengthens Schur's theorem. A combinatorial proof of Bergelson's result is provided in [**109**]. Additional density results appear in [**31**]. In [**149**], Irving considers $m = M(r, k)$,

the minimum number such that any r-coloring of $[1, m]$ must admit a monochromatic solution to $x_1 + x_2 + \cdots + x_{k-1} = x_k$ with all x_i distinct, and provides an upper bound for m. The case $r = 2$ can be found in [**251**]. Schaal [**239**], and Schaal and Bialostocki [**41**], consider variations of Schur numbers that are not guaranteed to exist. The 1892 result of Hilbert [**142**] can be specialized to give Schur's theorem. Work on the numbers associated with Hilbert's theorem can be found in [**57**].

Chapter 9

Rado's Theorem

The previous chapter, on Schur's theorem, included the extension of Schur's theorem to equations of the form $\sum_{i=1}^{k-1} x_i = x_k$, where $k \geq 4$. In other words, we found that for any finite coloring of $\mathbb{Z}^+$ there is a monochromatic solution to an equation of this form. The extension can be taken further, and was taken further, by one of Schur's Ph.D. students, Richard Rado.

In a series of papers in the 1930's, Rado determined, in particular, exactly which equations $\sum_{i=1}^{k} c_i x_i = 0$ are guaranteed to have monochromatic solutions under any finite coloring of the positive integers. In fact, part of this result is contained in his Ph.D. thesis.

9.1. Rado's Single Equation Theorem

Before stating the goal of this section, we remind the reader that a linear homogeneous equation is any equation of the form

$$\sum_{i=1}^{k} c_i x_i = 0,$$

where each $c_i \in \mathbb{Z}$ is a nonzero constant and each x_i is a variable. Since we will be considering only linear equations, when we use "homogeneous" we will mean "linear homogeneous."

We saw in Chapter 8 (see Remark 8.28) that

$$x_1 + x_2 + \cdots + x_{k-1} = x_k$$

is regular. In this section, we use this result and classify those homogeneous equations that, under any finite coloring of $\mathbb{Z}^+$, have a monochromatic solution ($x_1, x_2, \ldots, x_k \in \mathbb{Z}^+$ all are of the same color).

Considering Remark 8.28, we expand upon Definition 1.23 with the following definition.

Definition 9.1. Let $\mathcal{S}$ be a system of linear homogeneous equations. We say that $\mathcal{S}$ is *r-regular* if, for $r \geq 1$, for every r-coloring of $\mathbb{Z}^+$ there is a monochromatic solution to $\mathcal{S}$. If $\mathcal{S}$ is r-regular for all $r \geq 1$, we say that $\mathcal{S}$ is *regular*.

We will find a condition on the constants c_i in the equation

$$\sum_{i=1}^{k} c_i x_i = 0 \tag{9.1}$$

so that (9.1) is regular. Clearly, some restriction on the constants is needed. For example, if all of the constants are positive, then there is no solution in the positive integers, much less a monochromatic one. So, at least one of the constants must be negative. We also know, from Theorem 8.17, that if $c_i = c$ for $i = 1, 2, \ldots, k-1$ and $c_k = -c$, then we are guaranteed a monochromatic solution (by factoring out the c). This heads us in the right direction.

In this section we will be dealing with systems that consist of only one equation. However, in Section 9.3 we consider systems in general.

Theorem 9.2 (Rado's Single Equation Theorem). *Let $k \geq 2$. Let $c_i \in \mathbb{Z} - \{0\}, 1 \leq i \leq k$, be constants. Then (9.1) is regular if and only if there exists a nonempty $D \subseteq \{c_i : 1 \leq i \leq k\}$ such that $\sum_{d \in D} d = 0$.*

Proof. We first prove the 'if' part. We show that for all $r \geq 1$, if there exists a nonempty subset D as above, then (9.1) is r-regular.

We use induction on r, starting with the case of $r = 1$. Without loss of generality, let $D = \{c_1, \ldots, c_m\}$ with $c_1 > 0$. If $m = k$, then we may take $x_i = 1$ for $1 \leq i \leq k$ as our monochromatic solution. Hence, we assume that $m < k$. Thus, $c_{m+1} + c_{m+2} \cdots + c_k$ is a nonempty sum. Let $s = c_{m+1} + c_{m+2} + \cdots + c_k$ and note that $s \neq 0$.

Let $x_2 = x_3 = \cdots = x_m$ and $x_{m+1} = x_{m+2} = \cdots = x_k$. Equation (9.1) reduces to

$$c_1 x_1 + x_2(c_2 + c_3 + \cdots + c_m) + x_{m+1}(c_{m+1} + c_{m+2} + \cdots + c_k) = 0.$$

Since $c_1 + c_2 + \cdots + c_m = 0$, we have

$$c_1(x_1 - x_2) + s x_{m+1} = 0. \tag{9.2}$$

Any positive integers x_1 and x_2 such that $x_2 - x_1 = s$, together with $x_{m+1} = c_1$, provide a (monochromatic) solution to (9.2), completing the case when $r = 1$.

Now let $r \geq 2$ and assume that the result holds for $1 \leq t \leq r-1$. We show that it holds for r. For each $t \leq r-1$, let $n(t)$ be the least positive integer such that for every t-coloring of $[1, n(t)]$ there is a monochromatic solution to (9.1) ($n(t)$ exists by the induction hypothesis).

Assume, without loss of generality, that $c_1 + c_2 + \cdots + c_m = 0$, with m maximal and $c_1 > 0$. Again, we may assume that $m \neq k$, so that $s = c_{m+1} + c_{m+2} + \cdots + c_k$ is a nonempty sum. Note that $s \neq 0$.

Let $b = \sum_{i=1}^{k} |c_i|$ and let $n = n(r-1)$. We will show that $n(r) \leq bw(n+1; r)$, where w represents the usual van der Waerden function. We will show that every r-coloring of $[1, bw(n+1; r)]$ admits a monochromatic solution to (9.1).

Let $x_2 = x_3 = \cdots = x_m$ and $x_{m+1} = x_{m+2} = \cdots = x_k$. As in the case when $r = 1$, (9.1) reduces to (9.2).

Let χ be an r-coloring of $[1, bw(n+1; r)]$. We shall now find x_1, x_2, and x_{m+1} satisfying (9.2), with $\chi(x_1) = \chi(x_2) = \chi(x_{m+1})$. We see that by Theorem 4.9, χ must yield an $(n+1)$-term monochromatic arithmetic progression with gap a multiple of $|s|$ (note that we have $1 \leq |s| < b$). That is, we have that $\{a, a+d|s|, a+2d|s|, \ldots, a+nd|s|\}$ is monochromatic for some $a, d \geq 1$, with $a + nd|s| \leq bw(n+1; r)$.

If there exists $j \in [1,n]$ such that $\chi(jdc_1) = \chi(a)$, then by letting $|x_2 - x_1| = jd|s|$ and $x_{m+1} = jdc_1$, we have a monochromatic solution to (9.2) and are done. If, on the other hand, for all $j \in [1,n]$ we have $\chi(jdc_1) \neq \chi(a)$, then $\{dc_1, 2dc_1, \ldots, ndc_1\} = dc_1[1,n]$ is $(r-1)$-colored, and by Proposition 2.30 we are done.

For the 'only if' part of the theorem, we prove the contrapositive. Let $c_1, c_2, \ldots, c_k$ be fixed with no subset summing to zero. We prove the existence of an r-coloring of $\mathbb{Z}^+$, for some $r \geq 1$, that admits no monochromatic solution to (9.1).

Choose a prime p such that for any $C \subseteq \{c_i : 1 \leq i \leq k\}$ we have $p \nmid \sum_{c \in C} c$. Since C is a finite set, such a choice is obviously possible.

We now define a $2(p-1)$-coloring $\chi : \mathbb{Z}^+ \to \{1, 2, \ldots, 2(p-1)\}$. For $i \not\equiv 0 \pmod{p}$, let $\chi(i) = \bar{\imath}$ where $i \equiv \bar{\imath} \pmod{p}$ and $1 \leq \bar{\imath} \leq p-1$. For $i \equiv 0 \pmod{p}$, color i as follows. Let $\mu(i)$ be the maximal integer such that $\frac{i}{p^{\mu(i)}} \in \mathbb{Z}^+$. Thus, we may write $i = \sum_{j \geq \mu(i)} a_j p^j$, where the a_j's are positive integers. Now let $\chi(i) = a_{\mu(i)} + p - 1$ (note that $1 \leq a_{\mu(i)} < p$).

To prove that χ admits no monochromatic solution to (9.1), assume, for a contradiction, that $Y = \{y_1, y_2 \ldots, y_k\}$ is a monochromatic solution under χ. We consider two cases.

`Case 1.` $\chi(Y) = u < p$. Then we have $\sum_{i=1}^k c_i y_i = 0$, which implies that $u \sum_{i=1}^k c_i \equiv 0 \pmod{p}$. Since u and $\sum_{i=1}^k c_i$ are both nonzero, p divides $u \sum_{i=1}^k c_i$. This is a contradiction, since $p \nmid u$ and $p \nmid \sum_{i=1}^k c_i$.

`Case 2.` $\chi(Y) = u \geq p$. Write $y_i = \sum_{j \geq \mu(y_i)} a_j p^j$, and define $y_i' = y_i - ap^{\mu(y_i)}$ for $1 \leq i \leq k$, where $a = \chi(y_i) - p + 1$. Let $\mu^\star = \min\{\mu(y_1), \mu(y_2), \ldots, \mu(y_k)\}$.

By assumption we have $\sum_{i=1}^k c_i y_i = 0$. Hence

$$\sum_{i=1}^k c_i y_i' + ap^{\mu^\star} \sum_{i=1}^k c_i p^{\mu(y_i) - \mu^\star} = 0.$$

Note that $p^{\mu^\star + 1}$ divides $\sum_{i=1}^k c_i y_i'$. Thus,

$$ap^{\mu^\star} \sum_{i=1}^k c_i p^{\mu(y_i) - \mu^\star} \equiv 0 \pmod{p^{\mu^\star + 1}}.$$

Therefore, p divides $a \sum_{i=1}^k c_i p^{\mu(y_i) - \mu^\star}$. Since $1 \leq a < p$, this implies that p divides $\sum_{i=1}^k c_i p^{\mu(y_i) - \mu^\star}$. By the definition of $\mu^\star$, at least one of the $\mu(y_i)$'s must equal $\mu^\star$. Letting $M = \{t : \mu(t) = \mu^\star\}$, we have

$$\sum_{i=1}^k c_i p^{\mu(i) - \mu^\star} = \sum_{i \in M} c_i + \sum_{i \notin M} c_i p^{\mu(i) - \mu^\star},$$

where the first sum on the right-hand side is nonempty.

Since p divides $\left(\sum_{i \notin M} c_i p^{\mu(i) - \mu^\star}\right)$ and we require p to divide $\left(\sum_{i \in M} c_i + \sum_{i \notin M} c_i p^{\mu(i) - \mu^\star}\right)$, we see that p divides $\left(\sum_{i \in M} c_i\right)$, contradicting our choice of p. $\square$

Example 9.3. Theorem 8.17 with $k_1 = k_2 = \cdots = k_r$ follows from Rado's single equation theorem, since $x_1 + x_2 + \cdots + x_{k-1} - x_k = 0$ satisfies the necessary subset requirement $(c_{k-1} + c_k = 0)$.

Theorem 9.2 can be strengthened as seen in the following theorem, which we offer without proof.

Theorem 9.4. *Let $r \geq 1$. For every r-coloring of $\mathbb{Z}^+$ there is a monochromatic solution $(b_1, b_2, \ldots, b_k)$ to (9.1), where the b_i's are distinct, if and only if (9.1) is regular and there exist distinct integers (not necessarily positive) $y_1, y_2, \ldots, y_n$ such that $(y_1, y_2, \ldots, y_n)$ satisfies (9.1).*

Example 9.5. Since $x + y = 2z$ is another way of describing 3-term arithmetic progressions, provided $x \neq y$, we see that Theorem 9.4 proves the existence of a monochromatic 3-term arithmetic progression in any finite coloring of $\mathbb{Z}^+$. However, Theorem 9.4 does not imply the same result for arithmetic progressions of length more than 3. For this we need Rado's "full" theorem, which is presented later in this chapter as Theorem 9.27.

Rado's single equation theorem deals only with homogeneous equations. An obvious question is: is there a similar result concerning nonhomogeneous linear equations, i.e., equations of the form $\sum_i c_i x_i = b$, where b is a nonzero integer? As we will see, it is possible in this setting to have a nonempty subset of the coefficients summing to 0 while not being guaranteed a monochromatic solution. Knowing this, any regularity result will most probably be dependent upon the value of b. In the following proposition, we state what is known about one of the simplest nonhomogeneous cases. Before doing so, we introduce the following notation.

Notation. Let $\mathcal{E}$ be a linear equation. Denote by $r(\mathcal{E};s)$ the minimal integer, if it exists, such that any s-coloring of $[1,r(\mathcal{E};s)]$ must admit a monochromatic solution to $\mathcal{E}$. For $s=2$ we write simply $r(\mathcal{E})$.

Definition 9.6. The numbers $r(\mathcal{E};s)$ are called *s-color Rado numbers for equation $\mathcal{E}$.*

Proposition 9.7. *Let $b \in \mathbb{Z}-\{0\}$. Then $r(x-y=b;2)$ does not exist.*

Proof. Consider the 2-coloring of $\mathbb{Z}^+$ defined by $0^{|b|}1^{|b|}0^{|b|}1^{|b|}\dots$. Clearly, under this coloring, no two positive integers having the same color can have their difference equal to b. Hence, we cannot have a monochromatic solution to $x-y=b$. □

We next present a theorem that generalizes the equation of Proposition 9.7.

Theorem 9.8. *Let $b \geq 1$, $k \geq 3$, and let $\mathcal{E}(b)$ represent the equation $x_1+x_2+\cdots+x_{k-1}-x_k=-b$. Then $r(\mathcal{E}(b))$ does not exist precisely when k is even and b is odd. Furthermore, we have*

$$r(\mathcal{E}(b)) = k^2+(b-1)(k+1)$$

whenever $r(\mathcal{E}(b);2)$ exists.

Proof. We start by showing that $r(\mathcal{E}(b))$ does not exist if k is even and b is odd. We do this by giving a 2-coloring of $\mathbb{Z}^+$ with no monochromatic solution to $\mathcal{E}(b)$. To this end, color all even integers with one color and all odd integers with the other color. If all of the x_i are the same color, then $x_1+x_2+\cdots+x_{k-1}-x_k$ is even. Since b is odd, we see that a monochromatic solution cannot exist under this coloring.

Now assume that k is odd or b is even, and let

$$n = k^2+(b-1)(k+1).$$

We show that n is a lower bound for $r(\mathcal{E}(b))$ by providing a 2-coloring $\chi : [1,n-1] \to \{0,1\}$ that does not admit a monochromatic solution to $\mathcal{E}(b)$. Define χ as follows:

$$\chi(j)=0 \text{ if and only if } j \in [k+b-1, k^2+(b-2)k].$$

Assume, by way of contradiction, that $Y=\{y_i : 1 \leq i \leq k\}$ is a monochromatic solution.

If $\chi(Y)=0$, then $y_i \geq k+b-1$ for $1 \leq i \leq k-1$. Hence

$$b+\sum_{i=1}^{k-1} y_i \geq (k-1)(k+b-1) = k^2+(b-2)k+1,$$

a contradiction since $y_k \leq k^2+(b-2)k$.

Now assume $\chi(Y)=1$. If $Y-\{y_k\} \subseteq [1,k+b-2]$, then

$$k+b-1 \leq b+\sum_{i=1}^{k-1} y_i \leq k^2+(b-3)k+2.$$

Note that since $k \geq 2$ we have $k^2+(b-3)k+2 \leq k^2+(b-2)k$. Hence, $b+\sum_{i=1}^{k-1} y_i \in [k+b-1,k^2+(b-2)k]$, a contradiction since this interval is of color 0. Therefore, if $\chi(Y)=1$ we must have $y_{k-1} \in [k^2+(b-2)k+1,n-1]$. Then

$$b+\sum_{i=1}^{k-1} y_i \geq b+(k-2)+k^2+(b-2)k+1 = n,$$

and again we have a contradiction, since $y_k \leq n-1$.

We conclude that χ is a valid 2-coloring of $[1,n-1]$, so that $r(\mathcal{E}(b)) \geq k^2+(b-1)(k+1)$.

For the upper bound, let $m \in \mathbb{Z}^+$ and let $\chi : [1,m] \to \{0,1\}$ be a 2-coloring that admits no monochromatic solution to $\mathcal{E}(b)$. Define the set

$$A(\chi) = \{x : x \in [1,m-1] \text{ and } \chi(x) \neq \chi(x+1)\}.$$

Note that $|A(\chi)|$ is the number of times the color changes as we proceed from 1 to m.

We finish the proof via a series of claims.

`Claim 1.` If $|A(\chi) \cap [1,k+b-2]|$ is even, then $m \leq k+b-2$.

Proof of Claim 1. Assume, for a contradiction, that $m \geq k+b-1$. By the definition of $A(\chi)$ we have $\chi(1) = \chi(k+b-1)$. Hence,

$$x_i = 1,\ 1 \leq i \leq k-1;$$
$$x_k = k+b-1$$

is a monochromatic solution to $\mathcal{E}(b)$, a contradiction.

`Claim 2.` If $|A(\chi) \cap [1, k+b-2]| = 1$, then $m \leq k^2 + (b-1)(k+1) - 1$.

Proof of Claim 2. Without loss of generality, assume there exists $a_1 \geq 1$ such that $\chi([1, a_1]) = 0$ and $\chi([a_1+1, k+b-1]) = 1$. Note that

$$a_1 \leq k+b-2. \tag{9.3}$$

Since

$$x_i = k+b-1,\ 1 \leq i \leq k-1;$$
$$x_k = k^2 + (b-2)k + 1$$

is a solution to $\mathcal{E}(b)$, either $|A(\chi)| = 1$ and $m \leq k^2 + (b-2)k$, or $|A(\chi)| \geq 2$. If $|A(\chi)| = 1$ we are done, so assume $|A(\chi)| \geq 2$.

Since $|A(\chi)| \geq 2$, we know there exists $a_2 \geq k+b-1$ such that $\chi([1, a_1]) = 0$, $\chi([a_1+1, a_2]) = 1$, and $\chi(a_2+1) = 0$. Now, since $\chi(a_1+1) = 1$, we have that $\chi((k-1)(a_1+1)+b) = 0$ or $m < (k-1)(a_1+1)+b$. In either case we have

$$a_2 \leq (k-1)(a_1+1)+b-1. \tag{9.4}$$

Next, since

$$x_i = 1,\ 1 \leq i \leq k-2;$$
$$x_{k-1} = a_2 + 1;$$
$$x_k = a_2 + k + b - 1$$

is a solution to $\mathcal{E}(b)$, either $m \leq a_2+k+b-2$ or $\chi(a_2+k+b-1) = 1$.

We show that $\chi(a_2+k+b-1) = 1$ contradicts the fact that χ is valid. Since $[1, a_1]$ has color 0, we see that $[k+b-1, (k-1)a_1+b]$ has color 1. Thus, $a_2 \geq (k-1)a_1 + b$. This implies that

$$a_2 + k + b - 1 \geq (k-1)(a_1+1) + 2b.$$

Since $a_2 \geq 2$, we have $a_2 \geq \frac{k-1}{k-2}$, i.e., $(k-2)a_2 \geq k-1$. From this we see that $a_2 + k + b - 1 \leq (k-1)a_2 + b$. Hence,

$$(k-1)(a_1+1) + b \leq a_2 + k + b - 1 \leq (k-1)a_2 + b.$$

By the above equation and the fact that the interval $[a_1+1, a_2]$ has color 1, there exist $x_1, x_2, \ldots, x_{k-1} \in [a_1+1, a_2]$ for which

$$\sum_{i=1}^{k-1} x_i + b = a_2 + k + b - 1,$$

and we have found a monochromatic solution to $\mathcal{E}(b)$, a contradiction. Hence, we cannot have $\chi(a_2+k+b-1) = 1$. We conclude that

$$m \leq a_2 + k + b - 2. \tag{9.5}$$

Combining (9.5) with (9.3) and (9.4), we see that

$$m \leq k^2 + (b-1)(k+1) - 1,$$

thereby completing the proof of Claim 2.

`Claim 3.` If k is odd and $|A(\chi) \cap [1, k+b-2]| \geq 3$, then we have $m \leq k^2 + (b-2)k - 2$.

Proof of Claim 3. Without loss of generality, assume there exist a_1 and a_2 such that $1 < a_1 < a_2 \leq k+b-2$, $\chi(1) = 0$, $\chi(a_1) = 1$, $\chi([a_1+1, a_2]) = 0$, and $\chi(a_2+1) = 1$. Since

$$x_i = a_1 + 1,\ 1 \leq i \leq \tfrac{k-1}{2};$$
$$x_i = a_2,\ \tfrac{k+1}{2} \leq i \leq k-1;$$
$$x_k = \tfrac{k-1}{2}(a_2 + a_1 + 1) + b$$

satisfies $\mathcal{E}(b)$, we have either

$$m < \left(\tfrac{k-1}{2}\right)(a_2 + a_1 + 1) + b \tag{9.6}$$

or

$$\chi\left(\left(\tfrac{k-1}{2}\right)(a_2 + a_1 + 1) + b\right) = 1. \tag{9.7}$$

However, if (9.7) holds, then we have the monochromatic solution:

$$x_i = a_1,\ 1 \leq i \leq \tfrac{k-1}{2};$$
$$x_i = a_2 + 1,\ \tfrac{k+1}{2} \leq i \leq k-1,$$

a contradiction. Hence, (9.6) holds. Since $a_1 + 1 \le a_2 \le k + b - 2$, this gives

$$\begin{aligned} m &\le (k-1)(k+b-2) + b - 1 \\ &= k^2 - k + (b-2)(k-1) + (b-2) + 1 \\ &\le k^2 + (b-2)k - 2, \end{aligned}$$

as required. This completes the proof of Claim 3.

`Claim 4.` If k and b are both even and $|A(\chi) \cap [1, k+b-2]| = i \ge 3$ with i odd, then $m \le k^2 + (b-2)k - 1$.

Proof of Claim 4. Let

$$A(\chi) \cap [1, k+b-2] = \{a_1 < a_2 < \cdots < a_i\}.$$

We assume, without loss of generality, that $\chi(a_j) = 0$ and $\chi(a_j+1) = 1$ for $j = 1, 3, 5, \ldots, i$; and that $\chi(a_j) = 1$ and $\chi(a_j + 1) = 0$ for $j = 2, 4, \ldots, i-1$. If there exists $t \in [1, i-1]$ such that $a_{t+1} \ne a_t + 1$, then we have the following two solutions to $\mathcal{E}(b)$:

$$\begin{aligned} x_i &= a_t, \ 1 \le i \le \tfrac{k}{2}; \\ x_i &= a_{t+1} + 1, \ \tfrac{k}{2} + 1 \le i \le k-1; \\ x_k &= \tfrac{k}{2}(a_t) + (\tfrac{k}{2} - 1)(a_{t+1} + 1) + b - 1 \end{aligned}$$

and

$$\begin{aligned} x_i &= a_t + 1, \ 1 \le i \le \tfrac{k}{2}; \\ x_i &= a_{t+1}, \ \tfrac{k}{2} + 1 \le i \le k-2; \\ x_{k-1} &= a_{t+1} - 1; \\ x_k &= \tfrac{k}{2}(a_t) + (\tfrac{k}{2} - 1)(a_{t+1} + 1) + b - 1. \end{aligned}$$

One of these solutions must be monochromatic unless

$$m \le \frac{k}{2}(a_t) + \left(\frac{k}{2} - 1\right)(a_{t+1} + 1) + b - 2.$$

Since $a_t < a_{t+1} + 1 \le k + b - 1$, this implies that

$$m \le (k-1)(k+b-1) + b - 2 = k^2 + (b-2)k - 1.$$

Thus, we may assume that $a_{j+1} = a_j + 1$ for all $j \in [1, i-1]$. Since $k + b - 2$ is even, either $a_1 \ne 1$ or $a_i \ne k + b - 2$.

If $a_1 \ne 1$, then we can assume, without loss of generality, that $\chi(a_1 - 1) = \chi(a_1) = 0$. Hence, the following solutions both satisfy $\mathcal{E}(b)$:

$$\begin{aligned} x_i &= a_1 + 1, \ 1 \le i \le k-1; \\ x_k &= (k-1)(a_1+1) + b \end{aligned}$$

and

$$\begin{aligned} x_1 &= a_1 - 1; \\ x_i &= a_1, \ 2 \le i \le \tfrac{k}{2} - 1; \\ x_i &= a_1 + 2, \ \tfrac{k}{2} \le i \le k-1; \\ x_k &= (k-1)(a_1+1) + b, \end{aligned}$$

one of which is monochromatic unless $m \le (k-1)(a_1+1) + b - 1$. Since $a_1 + 1 \le k + b - 3$ and $k \ge 3$, this implies that

$$m \le k^2 + (b-4)k + 2 \le k^2 + (b-2)k - 4.$$

If $a_i \ne k+b-2$, we may also assume that $\chi(a_i - 1) = 0$, $\chi(a_i) = 1$, and $\chi([a_i + 1, k + b - 1]) = 0$. The following solutions both satisfy $\mathcal{E}(b)$:

$$\begin{aligned} x_i &= a_i - 1, \ 1 \le i \le \tfrac{k}{2}; \\ x_i &= a_i + 1, \ \tfrac{k}{2} + 1 \le i \le k-2; \\ x_{k-1} &= a_i + 2; \\ x_k &= (k-1)a_i + b \end{aligned}$$

and

$$\begin{aligned} x_i &= a_i, \ 1 \le i \le k-1; \\ x_k &= (k-1)a_i + b, \end{aligned}$$

one of which is monochromatic unless $m \le (k-1)a_i + b - 1$. Since $a_i \le k + b - 3$, this gives

$$m \le k^2 + (b-4)k + 2 \le k^2 + (b-2)k - 4.$$

This completes the proof of Claim 4.

To complete the proof of the theorem, it suffices to show that the hypothesis of one of the above claims is true. For the cases when $|A(\chi)| \in \{0, 1, 2, 4, 6, \ldots\}$, we see that the hypothesis of either Claim 1 or Claim 2 holds. If $|A(\chi)| \ge 3$ and $|A(\chi)|$ and k are both odd, then the hypothesis of Claim 3 holds. Finally, if $|A(\chi)| \geq 3$ with $|A(\chi)|$ odd and k even, then the hypothesis of Claim 4 holds (according to our earlier assumption, since k is even, b must be even).

We can now conclude that for any valid 2-coloring of $[1, m]$ we have $m \leq k^2 + (b-1)(k+1) - 1$. This inequality, along with the lower bound for $r(\mathcal{E}(b))$ given earlier, yields $r(\mathcal{E}(b)) = k^2 + (b-1)(k+1)$. □

A result analogous to, and as complete as, Theorem 9.8, but with b negative, has yet to be found. However, some progress has been made. In particular, for $k = 3$, we have the following result. We omit the proof, which is is similar to that of Theorem 9.8; see also Exercise 9.6.

Theorem 9.9. *For $b \geq 1$, $r(x + y - z = b) = b - \lceil \frac{b}{5} \rceil + 1$.*

We noted before that any regularity result concerning nonhomogeneous equations will most likely be dependent upon the constant b in

$$\sum_{i=1}^{k} c_i x_i = b.$$

Let $\mathcal{E}(b)$ represent the above equation, and let $\bar{\mathcal{E}}(b)$ denote this same equation with the restrictions $c_1 = c_2 = \cdots = c_{k-1} = 1$ and $c_k = -1$. From Proposition 9.7, we have that $\bar{\mathcal{E}}(b)$ is not regular if $k = 2$ (for $b \neq 0$). Theorem 9.8 specifies those values (with the restriction that $b < 0$) of k and b for which $\bar{\mathcal{E}}(b)$ is 2-regular and those values for which $\bar{\mathcal{E}}(b)$ is not 2-regular. Ultimately, we would like to determine those values of k and b (if any) for which $\mathcal{E}(b)$ is regular (i.e., r-regular for all $r \geq 1$).

One reason we see more regularity with Theorem 9.8 than Proposition 9.7 is that the more variables the equation has, the easier it is to find solutions to $\mathcal{E}(b)$ in $\mathbb{Z}^+$. At the same time, the more colors we use to color $\mathbb{Z}^+$, the harder it is to be guaranteed a monochromatic solution to $\mathcal{E}(b)$. Fortunately, we have the following result, which completely determines which equations are regular.

Theorem 9.10. *Let $k \geq 2$ and let $b, c_1, c_2, \ldots, c_k$ be nonzero integers. Let $\mathcal{E}(b)$ represent the equation*

$$\sum_{i=1}^{k} c_i x_i = b, \tag{9.8}$$

and let $s = \sum_{i=1}^{k} c_i$. Then $\mathcal{E}(b)$ is regular if and only if one of the following two conditions holds:

(i) $\frac{b}{s} \in \mathbb{Z}^+$;

(ii) $\frac{b}{s}$ is a negative integer and $\mathcal{E}(0)$ is regular.

Proof. Let $r \geq 1$. We first prove the 'if' portion of the theorem. If (i) holds, then $x_i = \frac{b}{s}$, $1 \leq i \leq k$, is a monochromatic solution under any r-coloring of $\mathbb{Z}^+$. If (ii) holds, let χ be any r-coloring of $\mathbb{Z}^+$. Now, define γ, an r-coloring of $\{|\frac{b}{s}|, |\frac{b}{s}| + 1, |\frac{b}{s}| + 2, \ldots\}$, by $\gamma(i - \frac{b}{s}) = \chi(i)$. By Proposition 2.30 and Rado's single equation theorem, we have a monochromatic solution to $\mathcal{E}(0)$ under γ. We may write $y_1 - \frac{b}{s}, y_2 - \frac{b}{s}, \ldots, y_k - \frac{b}{s}$ for such a solution. Since

$$\sum_{i=1}^{k} c_i \left(y_i - \frac{b}{s} \right) = 0,$$

we have that

$$\sum_{i=1}^{k} c_i y_i = b.$$

By the definition of γ, we know that $y_1, y_2, \ldots, y_k$ is a monochromatic (under χ) solution to (9.8). This completes the proof of the 'if' portion of the theorem.

For the 'only if' part of the theorem, we will prove its contrapositive. We first assume that $s \nmid b$.

Note that we may rewrite (9.8) as

$$\sum_{i=2}^{k} c_i (x_i - x_1) = b - s x_1. \tag{9.9}$$

We consider two cases.

Case 1. $s=0$. Let p be a prime such that $p>b$. By (9.9), we have

$$\sum_{i=2}^{k} c_i(x_i - x_1) = b. \tag{9.10}$$

Define $\chi : \mathbb{Z}^+ \to \{0,1,\ldots,p-1\}$ by $\chi(i) = \bar{\imath}$, where $i \equiv \bar{\imath} \,(\mathrm{mod}\, p)$. For any monochromatic solution to (9.10) we have, by the definition of χ, that p divides the left-hand side of (9.10). Hence, p must divide b. However, this contradicts our choice of p. Hence, (9.8) is not p-regular.

Case 2. $s \neq 0$. We assume, without loss of generality, that $s>0$. Define $\gamma : \mathbb{Z}^+ \to \{0,1,\ldots,s-1\}$ by $\gamma(i) = \bar{\imath}$, where $i \equiv \bar{\imath} \,(\mathrm{mod}\, s)$. Assume, for a contradiction, that $y_1, y_2, \ldots, y_k$ is a monochromatic solution to (9.9). By the definition of γ, s divides $\sum_{i=2}^{k} c_i(y_i - y_1)$. Hence, we must have

$$b - sy_1 \equiv 0 \,(\mathrm{mod}\, s),$$

i.e., $b \equiv 0 \,(\mathrm{mod}\, s)$. Thus, s divides b, a contradiction. Hence, (9.8) is not $|s|$-regular.

Since we have shown that (9.8) is not p-regular in Case 1 and not $|s|$-regular in Case 2, we have shown that if $s \nmid b$, then (9.8) is not regular.

To finish proving the contrapositive, assume that $\frac{b}{s}$ is a negative integer, but that $\mathcal{E}(0)$ is not regular. Then there exist a finite t and a t-coloring χ of $\mathbb{Z}^+$ such that, under χ, there is no monochromatic solution to $\mathcal{E}(0)$. Assume, for a contradiction, that (9.8) is regular. Define the t-coloring γ by $\gamma(i) = \chi\left(i - \frac{b}{s}\right)$. Since $\frac{b}{s}$ is a negative integer, γ is well-defined. By assumption, there exists, under γ, a monochromatic solution to (9.8), say $y_1, y_2, \ldots, y_k$. By the definition of γ, this means that $y_1 - \frac{b}{s}, y_2 - \frac{b}{s}, \ldots, y_k - \frac{b}{s}$ are monochromatic under χ. Since

$$\begin{aligned}\sum_{i=1}^{k} c_i\left(y_i - \frac{b}{s}\right) &= \sum_{i=1}^{k} c_i y_i - \frac{b}{s}\sum_{i=1}^{k} c_i \\ &= b - b \\ &= 0,\end{aligned}$$

we have that $y_1 - \frac{b}{s}, y_2 - \frac{b}{s}, \ldots, y_k - \frac{b}{s}$ is a monochromatic (under χ) solution to $\mathcal{E}(0)$, a contradiction. This concludes the proof. □

9.2. Some Rado Numbers

In this section we present several values and bounds for some 2-color Rado numbers associated with various linear equations in 3 variables. Theorems 9.11, 9.12, and 9.15 deal with equations that, by Rado's single equation theorem and Theorem 9.4, are regular.

Theorem 9.11. *Let $a, b \geq 1$ with $\gcd(a,b) = 1$. Then*

$$r(ax+by=bz) = \begin{cases} a^2+3a+1 & \text{if } b=1, \\ b^2 & \text{if } a<b, \\ a^2+a+1 & \text{if } 2 \leq b < a. \end{cases}$$

Proof. We separate the proof into the three obvious cases.

Case 1. $b=1$. Let $n = a^2+3a+1$. Color $[a+1, a(a+2)]$ one color and its complement in $[1, n-1]$ the other color. We leave it to the reader in Exercise 9.7 to verify that this is a valid 2-coloring of $[1, n-1]$, which implies that $r(ax+y=z) \geq n$.

For the upper bound, assume, for a contradiction, that there exists $\chi : [1,n] \to \{0,1\}$ that is valid. We may assume that $\chi(1) = 0$, and hence $\chi(a+1) = 1$, which in turn implies that $\chi\left((a+1)^2\right) = 0$. Since $(x,y,z) = \left(a+2, 1, (a+1)^2\right)$ satisfies the equation, we must have $\chi(a+2) = 1$. So that $\left(a+2, a+1, a^2+3a+1\right)$ is not monochromatic we must have $\chi\left(a^2+3a+1\right) = 0$. This gives us the monochromatic solution $\left(1, (a+1)^2, a^2+3a+1\right)$, a contradiction.

Case 2. $a<b$. Let $n = b^2$. We leave it to the reader in Exercise 9.7 to verify that a valid 2-coloring of $[1, n-1]$ is given by coloring $b[1, b-1]$ one color and its complement the other color.

For the upper bound, assume, for a contradiction, that there exists $\chi : [1,n] \to \{0,1\}$ that is valid.

Assume, without loss of generality, that $\chi(b) = 0$. We now show that $\chi(ib) = 0$ for $1 \leq i \leq b$ by considering the following solutions (which require $a < b$) for $1 \le i \le b-1$:

$$s_1(i) = (ib, i(b-a), ib),$$
$$s_2(i) = (b, ib, ib+a),$$
$$s_3(i) = ((i+1)b, i(b-a), ib+a).$$

Since none of these solutions is monochromatic (by assumption), we have, for $i = 1, \dots, b-1$, in order, the following sequence of implications:

$s_1(i)$ not monochromatic implies $\chi(i(b-a)) = 1$;

$s_2(i)$ not monochromatic implies $\chi(ib+a) = 1$;

$s_3(i)$ not monochromatic implies $\chi((i+1)b) = 0$.

Since $b, 2b, \dots, b^2$ are all of color 0 and $a < b$, we have a monochromatic solution given by $(b^2, b(b-a), b^2)$, a contradiction.

Case 3. $2 \le b < a$. Let $n = a^2 + a + 1$. We leave it to the reader in Exercise 9.7 to verify that a valid 2-coloring of $[1, n-1]$ is given by coloring $b[1, a]$ one color and its complement the other color.

For the upper bound, assume, for a contradiction, that there exists $\chi : [1, n] \to \{0, 1\}$ that is valid. Clearly, there exists a maximal k such that, for all i, $1 \le i \le k$, we have $\chi(ib) = 0$. We may rule out $k > a$, since, in this case, $(b^2, b, (a+1)b)$ would be a monochromatic solution.

We now show that $k \ge a-1$. Assume, for a contradiction, that $\chi(kb) = 0$ and $\chi((k+1)b) = 1$ with $k < a-1$. To avoid the triple $((k+1)b, 1, 1+(k+1)a)$ being monochromatic, either $\chi(1) = 0$ or $\chi(1+(k+1)a) = 0$.

First, consider $\chi(1) = 0$. Let $i \in \{1, 2, \dots, k\}$. We must have $\chi(1+ia) = 1$ (so that $(ib, 1, 1+ia)$ is not monochromatic). Next, so that $((k+1)b, 1+ia, 1+(k+i+1)a)$ is not monochromatic, we have $\chi(1+(k+i+1)a) = 0$, provided $k+i \le a$. This implies that $\chi((k+i+1)b) = 1$, so that $((k+i+1)b, 1, 1+(k+i+1)a)$ is not monochromatic. Now, provided $k+i \le a-1$, we have that $\chi((k+2)b) = 1$ implies $\chi(1+(k+i+2)a) = 0$, so that the solution $((k+2)b, 1+ia, 1+(k+i+2)a)$ is not monochromatic. Hence, we have $\chi(1+(k+2)a) = \chi(1+(k+3)a) = 0$ (where $1+(k+3)a \le 1+(a+1)a$

since $k < a-1$). Thus, $(b, 1+(k+2)a, 1+(k+3)a)$ is a monochromatic solution, a contradiction. So, if $\chi(1) = 0$, then $k \ge a-1$.

Next, consider $\chi(1+(k+1)a) = 0$. We may assume $\chi(1) = 1$. So that the solution $(ib, 1+(k-i+1)a, 1+(k+1)a)$ is not monochromatic, we have $\chi(1+ia) = 1$ for $1 \le i \le k$. Now, because $((k+1)b, 1+ia, 1+(k+i+1)a)$ cannot be monochromatic for $i = 0, 1$, we have $\chi(1+(k+1)a) = \chi(1+(k+2)a) = 0$. This gives the monochromatic solution $(b, 1+(k+1)a, 1+(k+2)a)$, a contradiction. Hence, if $\chi(1+(k+1)a) = 0$, then $k \ge a-1$.

Since $k \ge a-1$, from the above two paragraphs we have that $\chi(1+ia) = 1$ and $\chi(ib) = 0$ for $1 \le i \le a-1$. Because $\gcd(a, b) = 1$, $\{ib \pmod{a}\colon 1 \le i \le a-1\}$ is a complete residue system modulo a. Thus, there exists $x \in [1, a-1]$ such that $xb \equiv 1 \pmod{a}$. Since $1 < xb < ab$, we have $xb \in \{1+ia : 1 \le i \le a-1\}$. This is a contradiction, since $\chi(1+ia) = 1$ for all $1 \le i \le a-1$, while $\chi(xb) = 0$. □

Note that as an immediate consequence of Theorem 9.11, we also know the value of $r(ax+by=bz)$ when $\gcd(a, b) \ne 1$. To see this, let $\gcd(a, b) = g$, divide both sides of $ax+by=bz$ by g to obtain $\frac{a}{g}x + \frac{b}{g}y = \frac{b}{g}z$ with $(\frac{a}{g}, \frac{b}{g}) = 1$, and then apply Theorem 9.11. This same analysis may also be applied to the next theorem.

Theorem 9.12. *Let $a, b \ge 1$ with $\gcd(a, b) = 1$. Define $n(a, b)$ to be the least positive integer such that every 2-coloring of $[1, n(a, b)]$ admits a monochromatic solution to $ax + by = (a+b)z$, where x, y, z are distinct. Then $n(a, b) \le 4(a+b)+1$.*

Proof. By means of a computer search, it can be shown that $n(1,2) = 13$, $n(1,3) = 11$, $n(1,4) = 19$, $n(1,5) = 25$, $n(1,6) = 29$, and $n(2,3) = 21$. Furthermore, $n(1,1) = w(3;2) = 9$. These values show that $n(a, b) \le 4(a+b)+1$ for $a+b < 8$. Hence, in the remainder of the proof we assume that $a+b \ge 8$. Let $\mathcal{E}$ represent the equation $ax + by = (a+b)z$.

Assume, for a contradiction, that $\chi : [1, 4(a+b)+1] \to \{red, blue\}$ is a coloring that admits no monochromatic solution to $\mathcal{E}$.

Consider

$$S = \{1 + xa + yb : 0 \le x, y \le 4\} \subseteq [1, 4(a+b)+1].$$

We first show that the elements of S are distinct. Assume, for a contradiction, that there exist $i_1, i_2, j_1, j_2 \in [0,4]$ with $(i_1, j_1) \neq (i_2, j_2)$ such that. Since $(i_1 - i_2)a = (j_2 - j_1)b$ and $\gcd(a,b) = 1$, we have $i_1 - i_2 = kb$ and $j_2 - j_1 = ka$ for some $k \ge 1$. By the restriction on S, $i_1 - i_2$ and $j_2 - j_1$ cannot both equal 4, and thus $i_1 - i_2 + j_2 - j_1 \le 7$, a contradiction since $i_1 - i_2 + j_2 - j_1 \ge a + b \ge 8$.

Rewriting $\mathcal{E}$ as

$$a(x-z) = b(z-y),$$

we see that (x', y', z') is a solution if and only if it has the form $(d + kb, d - ka, d)$ for some $d, k \ge 1$ (provided $d + kb \le 4(a+b)+1$ and $1 \le d - ka$). Hence, for any $k \in \mathbb{Z}^+$, if

$$1 \le 1 + x'a + y'b - ka < 1 + x'a + y'b + kb \le 4(a+b)+1,$$

then

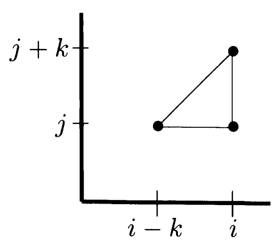
$$\{1 + (x'-k)a + y'b, 1 + x'a + y'b, 1 + x'a + (y'+k)b\}$$

cannot be monochromatic (or else we would have a contradiction).

Before going on with the proof, we introduce some notation: we denote by $\lambda_{(s,t)}$ the integer $1 + sa + tb$. From the previous paragraph we have the following fact.

Fact 1. For $0 \le j \le i \le k$, $k \neq 0$, $\lambda_{(i-k,j)}$, $\lambda_{(i,j)}$, and $\lambda_{(i,j+k)}$ cannot all have the same color.

We call the three integers $\lambda_{(i-k,j)}$, $\lambda_{(i,j)}$, and $\lambda_{(i,j+k)}$ an *isosceles triple* since, associating $\lambda_{(s,t)}$ with the point (s,t), the associated points form an isosceles (right) triangle in the plane:



Applying Fact 1, we have the following two facts.

Fact 2. $\lambda_{(0,0)}$, $\lambda_{(2,2)}$, and $\lambda_{(4,4)}$ cannot all have the same color.

Fact 3. For $i \in \{0,1,2\}$, $\lambda_{(i,i)}$, $\lambda_{(i+1,i+1)}$, and $\lambda_{(i+2,i+2)}$ cannot all have the same color.

Fact 2 holds because otherwise $\lambda_{(0,0)}$, $\lambda_{(2,2)}$, and $\lambda_{(4,4)}$ are all the same color, say red. By Fact 1, this implies that $\lambda_{(2,0)}$, $\lambda_{(4,0)}$ and $\lambda_{(4,2)}$ are all blue, contradicting Fact 1. To prove Fact 3, assume it is false. Then for some $i \in \{0,1,2\}$, the integers $\lambda_{(i,i)}$, $\lambda_{(i+1,i+1)}$, and $\lambda_{(i+2,i+2)}$ all have the same color, say red. By Fact 1, this implies that $\lambda_{(i+1,i)}$, $\lambda_{(i+2,i)}$, and $\lambda_{(i+2,i+1)}$ must all be blue, contradicting Fact 1.

Consider all possible 2-colorings of the set

$$T = \{\lambda_{(x,x)} : 0 \le x \le 4\} \subseteq S.$$

By the pigeonhole principle, one color, say red, must occur at least three times. From Facts 1, 2, and 3 the only possible colorings of

$$(\chi(\lambda_{(0,0)}), \chi(\lambda_{(1,1)}), \chi(\lambda_{(2,2)}), \chi(\lambda_{(3,3)}), \chi(\lambda_{(4,4)}))$$

(using r for red and b for blue) belong to

$$\{(r,r,b,r,b), (b,r,r,b,r), (r,b,b,r,r), (r,r,b,r,r)\}.$$

We illustrate these four colorings graphically as follows, associating $\lambda_{(s,t)}$ with the point (s,t), and using the standard (x,y)-axes.

b	r	r	r
r _	b _	r _	r _
b _ _	r _ _	b _ _	b _ _
r _ _ _	r _ _ _	b _ _ _	r _ _ _
r _ _ _ _	b _ _ _ _	r _ _ _ _	r _ _ _ _
(1)	(2)	(3)	(4)

We finish the proof graphically by determining, for each of the colorings, the colors of most of the remaining elements of T. The steps consist solely of avoiding monochromatic isosceles triples (see Fact 1). By obtaining the colors of enough of the elements, we will arrive at a contradiction in each case.

`Coloring 1.`

$$\begin{array}{r} b\\ r\ \_\\ b\ \_\ \_\\ r\ \_\ \_\ \_\\ r\ \_\ \_\ \_\ \_ \end{array} \longrightarrow \begin{array}{r} b\\ r\ \_\\ b\ \_\ r\\ r\ \_\ b\ \_\\ r\ b\ \_\ b\ \_ \end{array} \longrightarrow \begin{array}{r} b\\ r\ \_\\ b\ \_\ r\\ r\ \_\ b\ \_\\ r\ b\ r\ b\ \_ \end{array} \longrightarrow \begin{array}{r} b\\ r\ \_\\ b\ \_\ r\\ r\ \_\ b\ \_\\ r\ b\ r\ b\ b \end{array} \longrightarrow \begin{array}{r} b\\ r\ r\\ b\ \_\ r\\ r\ \_\ b\ r\\ r\ b\ r\ b\ b \end{array}$$

Regardless of how we color the remaining points, we have a monochromatic isosceles triple, a contradiction.

`Coloring 2.`

$$\begin{array}{r} r\\ b\ \_\\ r\ \_\ \_\\ r\ \_\ \_\ \_\\ b\ \_\ \_\ \_\ \_ \end{array} \longrightarrow \begin{array}{r} r\\ b\ \_\\ r\ \_\ b\\ r\ b\ \_\ b\\ b\ \_\ \_\ r\ \_ \end{array} \longrightarrow \begin{array}{r} r\\ b\ r\\ r\ \_\ b\\ r\ b\ r\ b\\ b\ \_\ \_\ r\ \_ \end{array} \longrightarrow \begin{array}{r} r\\ b\ r\\ r\ \_\ b\\ r\ b\ r\ b\\ b\ \_\ b\ r\ \_ \end{array} \longrightarrow \begin{array}{r} r\\ b\ r\\ r\ \_\ b\\ r\ b\ r\ b\\ b\ x\ b\ r\ r \end{array}$$

Whichever color we assign to x, we have a monochromatic isosceles triple, a contradiction.

`Coloring 3.`

$$\begin{array}{r} r\\ r\ \_\\ b\ \_\ \_\\ b\ \_\ \_\ \_\\ r\ \_\ \_\ \_\ \_ \end{array} \longrightarrow \begin{array}{r} r\\ r\ b\\ b\ \_\ \_\\ b\ r\ \_\ \_\\ r\ \_\ \_\ b\ b \end{array} \longrightarrow \begin{array}{r} r\\ r\ b\\ b\ \_\ \_\\ b\ r\ \_\ r\\ r\ r\ \_\ b\ b \end{array} \longrightarrow \begin{array}{r} r\\ r\ b\\ b\ \_\ \_\\ b\ r\ \_\ r\\ r\ r\ b\ b\ b \end{array} \longrightarrow \begin{array}{r} r\\ r\ b\\ b\ \_\ x\\ b\ r\ r\ r\\ r\ r\ b\ b\ b \end{array}$$

We again obtain a monochromatic isosceles triple for each coloring of x, a contradiction.

`Coloring 4.`

$$\begin{array}{r} r\\ r\ \_\\ b\ \_\ \_\\ r\ \_\ \_\ \_\\ r\ \_\ \_\ \_\ \_ \end{array} \longrightarrow \begin{array}{r} r\\ r\ \_\\ b\ \_\ \_\\ r\ \_\ \_\ x\\ r\ b\ \_\ b\ b \end{array}$$

For either coloring of x we have a monochromatic isosceles triple, a contradiction.

Since we are led to a contradiction using each of the possible colorings, the proof is complete. □

Before moving on to the next equation, we mention a couple of examples that are interesting consequences of Theorem 9.12 and its proof.

Example 9.13. In this book we focus on coloring integers; however, we may also color the set of real numbers by using r-colorings with domain $\mathbb{R}$ and range $\{1, 2, \ldots, r\}$. We can, of course, define an r-coloring on any set, for example the real plane $\mathbb{R}^2$. Using this domain, as an immediate consequence of the proof of Theorem 9.12, we have the following, rather formidable sounding, result: any 2-coloring of $\mathbb{R}^2$ must admit an isosceles right triangle with vertices all of the same color. (In fact, any 2-coloring of $\{(x, y) : x, y \in \mathbb{Z}, 0 \leq y \leq x \leq 4\}$ admits such a triangle, but this may not seem as grand a statement.)

Example 9.14. (A Tic-Tac-Toe Type Triangle) A 2-player game is played on a board B, where $B = \{(x, y) : 0 \leq y \leq x \leq 4\}$. Graphically, B looks like

$$\begin{array}{r} \_\\ \_\ \ \_\\ \_\ \ \_\ \ \_\\ \_\ \ \_\ \ \_\ \ \_\\ \_\ \ \_\ \ \_\ \ \_\ \ \_\ . \end{array}$$

This game is played on the above board, with one player being X and one player being O. The object is to be the first player to create an isosceles right triangle with the same orientation as the board (i.e., the right angle is to the bottom right) with all vertices marked by the player's letter (X or O). As we have seen from the proof of Theorem 9.12, unlike the standard Tic-Tac-Toe game, it is impossible not to have a winner.

The next theorem refines Theorem 9.12 for the cases when $a = 1$ and $a \leq b$.

Theorem 9.15. *Let $b \geq 1$ and define $n(b)$ to be the least positive integer such that for any 2-coloring of $[1, n(b)]$, there is a monochromatic solution to $x + by = (1+b)z$ with x, y, z distinct. Then*

$$n(b) = 4b + 5 \text{ if } 4 \nmid b$$

and

$$n(b) \geq 4b + 3 \text{ if } 4|b.$$

Proof. First note that if $4 \nmid b$, then $n(b) \leq 4b + 5$ is immediate from Theorem 9.12.

To establish the desired lower bounds, we consider three cases. For each case we give a valid 2-coloring of the appropriate interval, and leave it to the reader, in Exercise 9.9, to verify that these colorings are indeed valid.

`Case 1.` $4|b$. Let $b = 4k$. We want to find a valid 2-coloring of $[1, 4(1+b)-2] = [1, 16k+2]$. Let $S = [4k+1, 8k+1] \cup [8k+3, 12k+3]$. Define the coloring α on $[1, 4b+2]$ as follows:

$$\alpha(i) = \begin{cases} 0 & \text{if } i \in S, i \text{ even}, \\ 0 & \text{if } i \notin S, i \text{ odd}, \\ 1 & \text{otherwise}. \end{cases}$$

`Case 2.` b is odd. Let $T = [1, 2b+2]$. Define the coloring β on $[1, 4(1+b)]$ by

$$\beta(i) = \begin{cases} 0 & \text{if } i \in T, i \text{ even}, \\ 0 & \text{if } i \notin T, i \text{ odd}, \\ 1 & \text{otherwise}. \end{cases}$$

`Case 3.` $b \equiv 2 \pmod 4$. Define γ on $[1, 4(1+b)]$ as

$$\gamma(i) = \begin{cases} 0 & \text{if } i \equiv 1 \pmod 4 \text{ or } i \equiv 2 \pmod 4, \\ 1 & \text{if } i \equiv 0 \pmod 4 \text{ or } i \equiv 3 \pmod 4. \end{cases}$$

□

By Theorems 9.2 and 9.4, the Rado numbers given in the last three theorems are guaranteed to exist since the corresponding equations each satisfy the subset condition of Theorem 9.2. We have seen in Theorem 9.8 that some equations that do not satisfy the requirements of Theorem 9.10 (and hence are not regular) are, in fact, r-regular for some $r \geq 2$. In the following three theorems, we consider some homogeneous equations that do not satisfy the subset condition of Theorem 9.2 (which is needed for regularity) that are, nevertheless, 2-regular.

Theorem 9.16. *For $a \geq 1$, $r(ax + ay = z) = a(4a^2 + 1)$.*

Proof. Let $\mathcal{E}$ represent the equation $ax + ay = z$. To show that $r(ax + ay = z) \geq a(4a^2+1)$, we give a 2-coloring of $[1, a(4a^2+1)-1]$ with no monochromatic solution to $\mathcal{E}$. Color $[2a, 4a^2-1]$ one color and its complement a different color. It is left to the reader in Exercise 9.10 to show that no monochromatic solution exists under this coloring.

We complete the proof by showing that any given 2-coloring of $[1, a(4a^2+1)]$ must admit a monochromatic solution to $\mathcal{E}$. Assume, for a contradiction, that $\chi : [1, a(4a^2+1)] \to \{0, 1\}$ admits no monochromatic solution.

We may assume, without loss of generality, that $\chi(1) = 0$. Therefore, $\chi(2a) = 1$. Define $k \geq 2$ to be the maximal integer so that $\chi([1, k-1]) = 0$. Note that we must have $k \leq 2a$, and that $\chi(k) = 1$. This implies that $\chi(2ak) = 0$, since $(x, y, z) = (k, k, 2ak)$ is a solution. Since $\chi(1) = \chi(k-1) = 0$, we have $\chi(ak) = 1$ (to avoid $(1, k-1, ak)$ being of color 0). Since $\chi(ak) = 1$, we have $\chi(2a^2k) = 0$. This, in turn, gives $\chi(2ak - (k-1)) = 1$, so that $(k-1, 2ak-(k-1), 2a^2k)$ is not of color 0. To avoid $(2ak-(k-1), k, a(2ak+1))$ being of color 1, we have $\chi(a(2ak+1)) = 0$. Therefore, $(1, 2ak, a(2ak+1))$ is a monochromatic solution, a contradiction. □

Theorem 9.17. *For $a \geq 1$, $r(ax + ay = 2z) = \frac{a(a^2+1)}{2}$.*

Proof. Let $\mathcal{E}$ represent the equation $ax + ay = 2z$. Let $n(a, b) = r(ax + ay = bz)$ and let $n = n(a, 2)$. From Theorem 9.16, if a is even we have $n = \frac{a(a^2+1)}{2}$, since $n = n\left(\frac{a}{2}, 1\right)$. Furthermore, for $a = 1$ we have the trivial monochromatic solution $x = y = z = 1$. Hence, we assume that $a \geq 3$ is odd.

To show that $n \geq \frac{a(a^2+1)}{2}$, we will give a 2-coloring of the interval $I = \left[1, \frac{a(a^2+1)}{2} - 1\right]$ with no monochromatic solution to $\mathcal{E}$. Color $[a, a^2-1]$ red and its complement in I blue. It is left to the reader in Exercise 9.12 to show that under this coloring there is no monochromatic solution to $\mathcal{E}$.

To finish the proof we show that $n \leq \frac{a(a^2+1)}{2}$. Assume, for a contradiction, that there exists a 2-coloring $\chi : [1, \frac{a(a^2+1)}{2}] \to \{0,1\}$ that admits no monochromatic solution to $\mathcal{E}$. Then neither $(1,1,a)$ nor (a,a,a^2) is monochromatic; therefore 1 and a^2 must have the same color. We may assume that $\chi(1) = \chi(a^2) = 0$ and $\chi(a) = 1$.

Consider the solutions

$$\begin{aligned} s_1(j) &= \left(1, a^2-(a-1)j, \tfrac{a}{2}(a^2-(a-1)j+1)\right), \text{ and} \\ s_2(j) &= \left(1, a^2-(a-1)(j+1), \tfrac{a}{2}(a^2-(a-1)j+1)\right) \end{aligned}$$

for $0 \leq j \leq a-1$.

Denote $s_i(j)$ by $\left(1, s_i^{(2)}(j), s_i^{(3)}(j)\right)$ for $i = 1, 2$ and $0 \leq j \leq a-1$. Since none of these solutions $s_i(j)$ may be monochromatic, we have, for $j = 0, 1, \ldots, a-2$, in order, the following sequence of implications:

$$\begin{aligned} &s_1(j) \text{ and } s_2(j) \text{ not monochromatic implies } \chi\left(s_1^{(3)}(j)\right) = 1, \\ &\chi\left(s_1^{(3)}(j)\right) = 1 \text{ implies } \chi\left(s_2^{(2)}(j)\right) = 0, \\ &\chi\left(s_2^{(2)}(j)\right) = 0 \text{ implies } \chi\left(s_1^{(3)}(j+1)\right) = 1. \end{aligned}$$

Since $\chi\left(s_2^{(2)}(a-1)\right) = \chi\left(a^2-(a-1)^2\right) = 0$, we have that $\left(1, a^2-(a-1)^2, a^2\right)$ is a monochromatic solution, a contradiction. □

The above two theorems are implied by the the following, broader, result, which gives the value of $r(ax+ay=bz)$ for all a and b. We present this without proof.

Theorem 9.18. *Assume* $\gcd(a,b) = 1$ *and let* $r = r(ax+ay=bz)$. *Then*

$$r = \begin{cases} a(4a^2+1) & \text{if } b = 1, \\ \frac{a(a^2+1)}{2} & \text{if } b = 2, \\ 9 & \text{if } b = 3 \text{ and } a = 1, \\ 10 & \text{if } b = 3 \text{ and } a = 2. \end{cases}$$

For $b = 3$ *and* $a \geq 4$,

$$r = \begin{cases} \frac{a(4a^2+2a+3)}{9} & \text{if } a \equiv 1 \pmod 9, \\ \frac{a(4a^2+a+9)}{9} & \text{if } a \equiv 2 \pmod 9, \\ \frac{a(4a^2+2a+9)}{9} & \text{if } a \equiv 4 \pmod 9, \\ \frac{a(4a^2+4a+6)}{9} & \text{if } a \equiv 5 \pmod 9, \\ \frac{a(4a^2+5a+3)}{9} & \text{if } a \equiv 7 \pmod 9, \\ \frac{a(4a^2+a+6)}{9} & \text{if } a \equiv 8 \pmod 9. \end{cases}$$

For $b \geq 4$,

$$r = \begin{cases} r = \binom{b+1}{2} & \text{if } 1 \leq a \leq \frac{b}{4}, \\ r = \lceil \frac{b}{2} \rceil & \text{if } \frac{b}{4} < a < \frac{b}{2}, \\ r = ab & \text{if } \frac{b}{2} < a < b, \\ r = \lceil \frac{a^2}{b} \rceil a & \text{if } b < a. \end{cases}$$

9.3. Generalizations of the Single Equation Theorem

We present two main results in this section, one without proof, both of which generalize Rado's single equation theorem (Theorem 9.2). The way that the first result generalizes Theorem 9.2 is similar to the way in which the Ramsey numbers $R(k_1, k_2, \ldots, k_r)$ generalize $R_r(k)$. Recall, for example (taking $r = 2$), that $R_2(k) = R(k,k)$ is the least positive integer such that every 2-coloring of the edges of the complete graph on $R(k,k)$ vertices admits a monochromatic complete graph on k vertices. Meanwhile, the more general Ramsey number $R(k,\ell)$ denotes the least positive integer such that for every red-blue coloring of the edges on $R(k,\ell)$ vertices, there is either a red complete graph on k vertices or a blue complete graph on ℓ vertices.

The numbers $R(k,k)$ are called *diagonal* Ramsey numbers, and when $k \neq l$, $R(k,\ell)$ is called an *off-diagonal* Ramsey number

Now consider Rado's single equation theorem. The theorem guarantees a monochromatic solution to certain equations. This has a flavor similar to that of the diagonal Ramsey numbers, and we call the analogous Rado numbers *diagonal* Rado numbers. Now, if we have two different homogeneous linear equations, is it true that under any red-blue coloring of $\mathbb{Z}^+$ there must be either a red solution to the first equation or a blue solution to the second? What we are considering here are *off-diagonal* Rado numbers (if they exist). The next theorem proves their existence and is easily derived from the proof of Theorem 9.2.

Theorem 9.19. *Let $r \geq 1$. For $1 \leq j \leq r$, let $n_j \geq 2$ and let $\mathcal{E}_j$ represent the equation $\sum_{i=1}^{n_j} c_i^{(j)} x_i = 0$. If each $\mathcal{E}_j$, $1 \leq j \leq r$, is r-regular, then there is a least positive integer n so that for every r-coloring of $[1,n]$, there exists $t \in \{1,2,\ldots,r\}$ such that $\mathcal{E}_t$ has a solution of color t.*

If a set S of r homogeneous equations satisfies the conditions of Theorem 9.19, we will say that S is r-regular.

Before delving into the proof of this theorem, we look at an example to help solidify the implication of Theorem 9.19.

Example 9.20. Consider the three equations given below:

$$\begin{aligned} -x_1 + 2x_2 + x_3 &= 0, \\ 3x_1 - x_2 + x_3 + 7x_4 &= 0, \\ 2x_1 + 4x_2 - 3x_3 + x_4 + 6x_5 &= 0. \end{aligned}$$

Each of these equations is regular by Rado's theorem, since each has a nonempty subset of coefficients that sums to zero: (using the notation of Theorem 9.19) $c_1^{(1)} + c_3^{(1)} = 0$, $c_2^{(2)} + c_3^{(2)} = 0$, and $c_1^{(3)} + c_3^{(3)} + c_4^{(3)} = 0$. Applying Theorem 9.19, we are guaranteed that any 3-coloring, say red-blue-green, of $\mathbb{Z}^+$ must contain one of the following:

a red solution to $-x_1 + 2x_2 + x_3 = 0$;

a blue solution to $3x_1 - x_2 + x_3 + 7x_4 = 0$;

a green solution to $2x_1 + 4x_2 - 3x_3 + x_4 + 6x_5 = 0$.

We now present a sketch of the proof of Theorem 9.19. The proof is very similar to that of Theorem 9.2 so we leave some justification to the reader as Exercise 9.13.

Proof of Theorem 9.19 (sketch). Let $S = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r\}$. The proof is by induction on r. The case $r = 1$ holds by Theorem 9.2. Let $n = n(\mathcal{S}; r-1)$ be the least positive integer such that for any $(r-1)$-coloring of $[1,n]$ there is a solution to $\mathcal{E}_i$ of color i for some $i \in \{1,2,\ldots,r-1\}$.

Assume that for each j, $1 \leq j \leq r$,

$$c_1^{(j)} + c_2^{(j)} + \cdots + c_{m_j}^{(j)} = 0$$

with $m_j < n_j$ maximal, and that $s_j = c_{m_j+1}^{(j)} + c_{m_j+2}^{(j)} \cdots + c_{n_j}^{(j)}$ is a nonempty, nonzero sum (this needs to be justified).

Let $b_j = \sum_{i=1}^{n_j} |c_i^{(j)}|$. Define $b = \prod_{j=1}^{r} b_j$ and $s = \prod_{j=1}^{r} |s_j|$. We claim that $n(\mathcal{S}; r) \leq b^r w(n+1; r)$. We show that any r-coloring of $[1, b^r w(n+1;r)]$ must admit a solution to $\mathcal{E}_i$ of color i, for some $i \in \{1,2,\ldots,r\}$.

Let $\chi : [1, b^r w(n+1;r)] \to \{0,1,\ldots,r\}$. Let $x_2 = x_3 = \cdots = x_{m_j}$ and $x_{m_j+1} = x_{m_j+2} = \cdots = x_{n_j}$ for each j, $1 \leq j \leq r$. Hence, we can write $\mathcal{E}_j$, for $1 \leq j \leq r$, under these conditions, as

$$c_1^{(j)}(x_1 - x_2) + s_j x_{m_j+1} = 0.$$

By Theorem 4.9, χ admits an $(n+1)$-term monochromatic arithmetic progression with gap a multiple of s. Let $\{a, a+ds, \ldots, a+nds\}$ be one such arithmetic progression of color, say, t. If there exists $j \in \frac{s}{s_t}[1,n]$ such that $\chi\left(jdc_1^{(t)}\right) = t$, then we are done (this needs to be justified). If, on the other hand, for all $j \in \frac{s}{s_t}[1,n]$ we have $\chi\left(jdc_1^{(t)}\right) \neq t$, then $dc_1^{(t)} \frac{s}{s_t}[1,n]$ is $(r-1)$-colored (verify that $dc_1^{(t)} \frac{s}{s_t} n \leq b^r w(n+1;r)$). Let $S' = S - \mathcal{E}_t$. Since we have an $(r-1)$-coloring of $\frac{s}{s_t}[1,n]$ using the colors $1,2,\ldots,t-1,t+1,t+2,\ldots,r$, by the inductive assumption there exists $c \in [1,r] - \{t\}$ such that $\mathcal{E}_c$ has a solution of color c. This completes the proof. $\square$

Example 9.21. Let $S = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r\}$ with $\mathcal{E}_1 = \mathcal{E}_2 = \cdots = \mathcal{E}_r$. Then Theorem 9.19 reduces to Theorem 9.2.

Just as Rado's single equation theorem was strengthened to show the existence of monochromatic solutions consisting of distinct integers, Theorem 9.19 can be strengthened in the same way. This is given by the next theorem, which we state without proof.

Theorem 9.22. *Let $r \geq 1$. Let $S = \{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r\}$, where the $\mathcal{E}_i$'s are defined as in Theorem 9.19. There exist j, $1 \leq j \leq r$, and distinct b_i, $1 \leq i \leq n_j$, of color j, that satisfy $\mathcal{E}_j$, if and only if S is r-regular and for all j, $1 \leq j \leq r$, there exist distinct $y_i^{(j)} \in \mathbb{Z}$, $1 \leq i \leq n_j$, that satisfy $\mathcal{E}_j$.*

As an interesting application of Theorem 9.22, we look at the following example.

Example 9.23. Consider the set of equations $\{x+y = z, x+y = 2z\}$. By Theorem 9.22, there exists a minimal integer n such that any red-blue coloring of $[1, n]$ must contain either a red solution to $x + y = z$ (of distinct integers) or a blue solution to $x + y = 2z$ (of distinct integers). In other words, there must exist either a red Schur triple or a blue 3-term arithmetic progression. In fact, $n = 8$. To see that $n \geq 8$, color $1, 4, 7$ red, and color $2, 3, 5, 6$ blue. The fact that every 2-coloring of $[1, 8]$ admits either a red Schur triple or a blue 3-term arithmetic progression is left to the reader as Exercise 9.4.

We now state, without proof, the full version of Rado's theorem, for which Rado's single equation theorem (Theorem 9.2) is a special case. This is a very powerful theorem, as it describes precisely when we are guaranteed monochromatic solutions to a homogeneous *system* of linear equations. In order to state Rado's full theorem clearly, we make the following definition.

Definition 9.24. Let $C = (\vec{c}_1\ \vec{c}_2\ \ldots\ \vec{c}_n)$ be a $k \times n$ matrix, where $\vec{c}_i \in \mathbb{Z}^k$ for $1 \leq i \leq n$. We say that C satisfies the *columns condition* if we can order the columns $\vec{c}_i$ in such a way that there exist indices $i_0 = 1 < i_1 < i_2 < \cdots < i_t = n$ such that the following two conditions hold for $\vec{s}_j = \sum_{i_{j-1}+1}^{i_j} \vec{c}_i$ $(2 \leq j \leq t)$:

1. $\vec{s}_1 = \vec{0}$.
2. $\vec{s}_j$ can be written as a linear combination of $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_{i_{j-1}}$ for $2 \leq j \leq t$.

To help clarify this rather cumbersome definition, we look at a couple of examples.

Example 9.25. Consider the following matrix:

$$\begin{pmatrix} 1 & 0 & 2 & -3 & 2 \\ 0 & -1 & 2 & -2 & 1 \\ 4 & 2 & -5 & 1 & 6 \end{pmatrix}.$$

To see that this matrix satisfies the columns condition, rearrange the columns to obtain

$$\begin{pmatrix} 1 & 2 & -3 & 0 & 2 \\ 0 & 2 & -2 & -1 & 1 \\ 4 & -5 & 1 & 2 & 6 \end{pmatrix}.$$

Using the notation of Definition 9.24, take $t = 2$, $i_1 = 3$, and $i_2 = 5$. Then $\vec{s}_1 = \vec{0} \in \mathbb{Z}^3$ and $\vec{s}_2 = (2\ 0\ 8)^t = 2\vec{c}_1$ (where t denotes the transpose).

Example 9.26. The following matrix does not satisfy the columns condition:

$$\begin{pmatrix} 1 & 3 & 0 & -3 & 1 \\ -2 & -1 & 2 & 7 & 0 \\ 5 & 3 & -5 & 2 & -3 \\ 1 & -2 & 3 & 4 & -5 \end{pmatrix}.$$

By inspection (check!) we see that there does not exist a set of columns that sums to the zero vector.

The key to comprehending Rado's full theorem is having a clear understanding of the columns condition. We now state Rado's full theorem. We do not include its proof, as it is beyond the scope of this book.

Theorem 9.27 (Rado's Full Theorem). *Let $\mathcal{S}$ be a system of linear homogeneous equations. Write $\mathcal{S}$ as $A\vec{x} = \vec{0}$. $\mathcal{S}$ is regular if and only if A satisfies the columns condition. Furthermore, $\mathcal{S}$ has a monochromatic solution of distinct positive integers if and only if $\mathcal{S}$ is regular and there exist distinct (not necessarily monochromatic) integers that satisfy $\mathcal{S}$.*

Rado's full theorem opens the door to many new Ramsey-type numbers, one of which is presented in the next two examples.

Example 9.28. For $r \geq 1$, there exists $n = n(r)$ such that under any r-coloring of $[1, n]$ we must have both a monochromatic Schur triple and a monochromatic, *of the same color*, 3-term arithmetic progression.

To show the existence of such a monochromatic structure, consider the following system in the variables $x_1, x_2, \ldots, x_6, y$:

$$\begin{aligned} x_1 + x_2 &= x_3, \\ x_4 - x_5 &= y, \\ x_5 - x_6 &= y. \end{aligned}$$

If x_i, $1 \leq i \leq 6$, are all the same color, then $x_1 + x_2 = x_3$ is a monochromatic Schur triple and x_6, x_5, x_4 is a monochromatic 3-term arithmetic progression with gap y.

Writing the above system in matrix form (using the notation of Definition 9.24 and Theorem 9.27) with $\vec{x} = (x_1\ x_2\ \ldots x_6\ y)^t$, we get $C\vec{x} = \vec{0}$, where

$$C = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \end{pmatrix}.$$

Rearranging the rows of C, we have

$$(\vec{c}_1\ \vec{c}_2\ \ldots\ \vec{c}_7) = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 \end{pmatrix}.$$

Since $\sum_{i=1}^{5} \vec{c}_i = \vec{0} \in \mathbb{Z}^3$, $\vec{c}_6 = \vec{c}_1$, and $\vec{c}_7 = \vec{c}_4 + 2\vec{c}_5$, we have satisfied the columns condition. By Rado's full theorem we know that for any finite coloring of $\mathbb{Z}^+$, there is a monochromatic solution to the given system of equations. Hence, we have a monochromatic Schur triple and a monochromatic 3-term arithmetic progression (with gap y) of the same color. Note that, in addition, y has this same color.

Example 9.29. Let n be the least positive integer such that any 2-coloring of $[1, n]$ must contain a Schur triple and a 3-term arithmetic progression, both monochromatic of the same color. We show that $n \leq 16$.

We know that $s(2) = 5$ and $w(3) = 9$, where s and w are the usual Schur and van der Waerden functions, respectively. Clearly, in any red-blue coloring of $[1, 16]$ there exist both a monochromatic Schur triple and a monochromatic 3-term arithmetic progression. Assume, for a contradiction, that the colors of every such pair of triples are different.

Let $a, a + d, a + 2d$ be, say, a red 3-term arithmetic progression. Since $a + a = 2a$, $a + (a + d) = 2a + d$, and $a + (a + 2d) = 2a + 2d$, we must have $2a, 2a + d$, and $2a + 2d$ all be blue in order to avoid a red Schur triple. Note that $2a + 2d \leq 16$, since $w(3) = 9$ implies that $a + 2d \leq 9$ and $a \leq 7$.

Hence, we have that $2a, 2a + d, 2a + 2d$ is a blue 3-term arithmetic progression, a contradiction since we now have a Schur triple and a 3-term arithmetic progression with the same color.

In this section we have seen two generalizations of Rado's single equation theorem: Theorem 9.19 and Rado's full theorem. Rado's full theorem is clearly more powerful, as it completely classifies regular systems. Its strength is also made apparent by comparing Examples 9.23 and 9.29.

9.4. Exercises

9.1 Prove that in the statement of Theorem 9.2 we may take $\mathcal{E} \in \mathbb{Q}[x]$, i.e., the coefficients may be rational.

9.2 Which of the following equations are regular?

a) $x - y = 7z$;

b) $x_1 + 2x_2 + 3x_3 + 4x_4 = 5x_5$;

c) $x_1 - 2x_2 + 4x_3 - 8x_4 + 16x_5 - 32x_6 = 0$;

d) $\frac{1}{3}x_1 - \frac{1}{4}x_2 + 2x_3 - \frac{1}{12}x_5 = 0$.

9.3 Give two different equation(s)/system(s) whose r-regularity proves the existence of monochromatic 3-term arithmetic progressions in any r-coloring of $\mathbb{Z}^+$.

9.4 Show that the minimal integer n such than any red-blue coloring of $[1, n]$ must admit either a red Schur triple, or a blue

3-term arithmetic progression is $n = 8$. The fact that $n \geq 8$ comes from the coloring given in Example 9.23.

9.5 The following system is regular according to Rado's full theorem (Theorem 9.27). Prove this. Also, what monochromatic object is guaranteed to exist?

$$\begin{array}{lllllll} x_1 - x_2 & & & & & = x_7 \\ & x_2 - x_3 & & & & = x_7 \\ & & x_3 - x_4 & & & = x_7 \\ & & & x_4 - x_5 & & = x_7 \\ & & & & x_5 - x_6 & = x_7. \end{array}$$

9.6 Let $b \neq 0$. Let $\mathcal{E}(b)$ represent the equation $x + y - z = b$. Define the 2-coloring $\chi : \left[1, b - \lceil \frac{b}{5} \rceil\right] \to \{0,1\}$ by

$$\chi(x) = \begin{cases} 0 & \text{if } x \in \left[\lceil \frac{b}{5} \rceil + 1, \left\lfloor \frac{b + \lceil \frac{b}{5} \rceil}{2} \right\rfloor\right], \\ 1 & \text{otherwise.} \end{cases}$$

Show that χ is a valid coloring for $\mathcal{E}(b)$.

9.7 Verify the fact that the three colorings used in the proof of Theorem 9.11 are each valid colorings.

9.8 In the proof of Theorem 9.12 it is stated that $n(1,2) = 13$, $n(1,3) = 11$, $n(1,4) = 19$, $n(1,5) = 25$, and $n(2,3) = 21$. Verify this. (A computer may be helpful, but is not necessary.)

9.9 Verify the fact that the three colorings used in the proof of Theorem 9.15 are each valid colorings. Also, what is the standard notation for $n(1)$ in this context?

9.10 Do the following problems concerning the proof of Theorem 9.16.
a) Show that the coloring given for the lower bound is valid.
b) The statement that $(1, 2ak, a(2ak+1)$ is a monochromatic solution, which is given at the end of the proof, is not valid unless $k \leq 2a$. Verify this.
c) What is the standard notation for $n(1)$ in this context?

9.11 Prove that for $a \geq 4$, we have $r(x + y = az) = \binom{a+1}{2}$.

9.12 Verify the fact that the coloring given in Theorem 9.17 is valid. Also, for the upper bound, why must we insist that $a \geq 3$?

9.13 Fill in the details for the proof of Theorem 9.19.

9.14 Deduce the existence of the off-diagonal generalized Schur numbers using one of the theorems in this chapter.

9.15 Find the exact value of the number n of Example 9.29.

9.5. Research Problems

9.1 Find an extension of Rado's full theorem which is analogous to Theorem 9.19's extension of Rado's single equation theorem.
Reference: [**69**]

***9.2** Theorem 9.10 characterizes those nonhomogeneous equations that are not regular. Further characterize these equations by the greatest number, m, of colors for which they are m-regular (i.e., by their degrees of regularity).

9.3 The exact value of $r(ax+by = (a+b)z)$ is known when $a = 1$ and $b \not\equiv 0 \pmod 4$. Determine $r(ax + by = (a + b)z)$ for all other pairs (a, b). Bounds for these remaining pairs are given by Theorems 9.12 and 9.15.
Reference: [**69**]

***9.4** Determine the 2-color Rado numbers for $aw + bx + cy = cz$ for all a, b, and c. Start with $a = b = 1, c > 1$.
References: [**139**], [**140**], [**224**]

9.5 Complete the result analogous to Theorem 9.8 for $b < 0$ (the references listed directly below indicate which cases remain open).
References: [**159**], [**239**]

***9.6** Extend Theorem 9.8 to $r \geq 3$ colors.
Reference: [**239**]

***9.7** Let χ be a given red-blue coloring of $[1, n]$. Define S_χ to be the number of red Schur triples in $[1, n]$ under χ, and V_χ to be the number of blue 3-term arithmetic progressions in

$[1, n]$ under χ. Determine the asymptotic minimum sum of the number of red Schur triples and the number of blue 3-term arithmetic progressions; i.e., find $\min(S_\chi + V_\chi)$, where the minimum is over all red-blue colorings χ of $[1, n]$.
References: [**75**], [**225**], [**248**]

∗**9.8** Let χ be a given red-blue coloring of $[1, n]$. Define S_χ to be the number of monochromatic Schur triples in $[1, n]$ under χ, and V_χ to be the number of monochromatic 3-term arithmetic progressions in $[1, n]$ under χ. Determine the asymptotic minimum sum of the number of monochromatic Schur triples and the number of 3-term arithmetic progressions; i.e., find $\min(S_\chi + V_\chi)$, where the minimum is over all red-blue colorings χ of $[1, n]$.
References: [**75**], [**225**], [**248**]

9.6. References

§**9.1**. Bergelson, Deuber, and Hindman [**29**] extend Rado's theorem to hold over finite fields. The proof of Theorem 9.4 can be found in [**127**, p. 77]. Theorem 9.8 and its proof appear in [**239**]. Rado [**214**] classifies those nonhomogeneous linear systems that are regular. In [**70**], Theorem 9.9 is given. Extensions of Rado's full theorem are given in [**32**] and [**181**]. Generalizations to abelian groups can be found in [**80**] and [**81**].

§**9.2**. Burr and Loo [**69**] present Theorems 9.11, 9.12, and 9.15 with different proofs for 9.11 and 9.12. Theorem 9.16 and Exercises 9.6 and 9.11 may be found in [**70**]. Harborth and Maasberg prove Theorem 9.17 in [**140**] and Theorem 9.18 in [**139**]. Theorem 9.19 is noted, without proof, in [**224**].

§**9.3**. Rado's full theorem's (Theorem 9.27) original proofs are in the series of papers [**215**], [**214**], and [**213**]. Other proofs can be found in [**77**] and [**127**].

Additional References: For a survey with an extensive reference list, see [**77**]. In [**108**], bounds are given for the number of monochromatic solutions in a regular system. Probability results concerning Rado's full theorem can be found in [**226**]. Deuber [**79**] gives a strong

generalization of Rado's theorem. An extension of Deuber's result is in [**178**].

Systems of linear homogeneous inequalities are dealt with in [**242**]. Values and bounds for such inequalities are given by Schaal in [**241**], while in [**240**] he gives values and bounds for linear nonhomogeneous inequalities. Homogeneous systems whose solutions satisfy a linear inequality are presented in [**147**]. Nonlinear homogeneous equations are studied in [**67**] and [**180**].

Chapter 10

Other Topics

The previous chapters were primarily devoted to three classical theorems of Ramsey theory – van der Waerden's theorem, Schur's theorem, and Rado's theorem – and to many generalizations, extensions, and other modifications of these three theorems. There are quite a few interesting topics and problems that are not in the aforementioned category, but which definitely belong to the area of Ramsey theory on the integers. In this chapter we touch upon a few such topics.

10.1. Folkman's Theorem

Folkman's theorem, also known as the Folkman-Rado-Sanders theorem, involves the existence of certain sets of sums that are monochromatic under any finite coloring of the positive integers. We begin with a definition.

Definition 10.1. Let $T \subseteq \mathbb{Z}^+$ be finite. We define

$$\mathcal{S}(T) = \left\{ \sum_{r \in R} r : R \subseteq T, R \neq \emptyset \right\}$$

and call $\mathcal{S}(T)$ the *sumset* of T.

Note that the sumset of a set T consists only of sums of *distinct* elements of T. In particular, if $t \in T$, then $2t \notin \mathcal{S}(T)$.

Example 10.2. Let $T = \{2, 5, 8\}$. Then $\mathcal{S}(T) = \{2, 5, 7, 8, 10, 13, 15\}$. The number $2+2 = 4$, or any sum with the same summand appearing more than once, is not a member of $\mathcal{S}(T)$.

We now state Folkman's theorem.

Theorem 10.3 (Folkman's Theorem). *For all $k, r \geq 1$, there exists a least positive integer $F = F(k; r)$ such that for every r-coloring of $[1, F]$, there is a k-element subset $T \subseteq [1, F]$ such that $\mathcal{S}(T)$ is monochromatic.*

Note that the statement of Theorem 10.3 is the "finite" form of Folkman's theorem. This is equivalent (by means of the compactness principle) to the "infinite" form: if $r \geq 1$, then for every r-coloring of $\mathbb{Z}^+$, there are arbitrarily large sets T such that $\mathcal{S}(T)$ is monochromatic.

Example 10.4. Consider $F(2; r)$. If $n = F(2; r)$, then for each r-coloring of $[1, n]$ there are integers a and b such that the set $\{a, b, a+b\}$ is monochromatic. This may look familiar: it is the statement of Schur's theorem, which says that for any $r \geq 1$, there is an integer $n = s(r)$ such that for every r-coloring of $[1, n]$, there is a monochromatic solution to $x + y = z$.

The astute reader may notice that Folkman's theorem is a special case of Rado's full theorem (Theorem 9.27); it follows by elementary, although somewhat untidy, means. However, Folkman's theorem is of sufficient independent interest to warrant our providing a different proof. We will use the following lemma (which is similar to Definition 2.33) to prove Theorem 10.3.

Lemma 10.5. *For all $k, r \geq 1$, there is an integer $n = n(k; r)$ so that for any r-coloring of $[1, n]$, there exist $x_1 < x_2 < \cdots < x_k \in [1, n]$ with $\sum_{i=1}^{k} x_i \leq n$ such that*

$$S_t = \left\{ \sum_{r \in R} x_r : R \subseteq [1, k], \max_{r \in R} r = t \right\}$$

is monochromatic for $t = 1, 2, \ldots, k$.

Proof. We use induction on k, with $k = 1$ being trivial. Let $k \geq 1$, let r be arbitrary, and assume that $n(k; r)$ exists. We will show that $n(k+1; r) \leq 2w(n(k; r) + 2; r)$, where $w(k; r)$ is the usual van der Waerden function.

Let $m = 2w(n(k; r)+2; r)$, and consider an arbitrary r-coloring of $[1, m]$. By van der Waerden's theorem, using Proposition 2.29, there is a monochromatic arithmetic progression

$$A = \{a + jd : 0 \leq j \leq n(k; r) + 1\} \subseteq \left[\frac{m}{2}, m\right].$$

Now consider the set

$$D = \{d, 2d, \ldots, n(k; r)d\}.$$

Using Proposition 2.30, along with the inductive assumption, there exist $x_1 < x_2 < \cdots < x_k$ in D such that the associated S_t's are monochromatic for $t = 1, 2, \ldots, k$. Our goal is to find an x_{k+1} so that S_{k+1} is also monochromatic.

We will show that we may take $x_{k+1} = a + d$. Since $a > \frac{m}{2}$ and $a + n(k; r)d \leq m$, we see that $n(k; r)d < \frac{m}{2}$. Hence, $a + d > x_k$, since $x_k \in D$. Because $(a + d) + D \subseteq A$, by taking $x_{k+1} = a + d$ we have $x_{k+1} > x_k$ and

$$S_{k+1} \subseteq (a + d) + D \subseteq A.$$

Hence, S_{k+1} is monochromatic, thereby completing the induction. □

We now apply the above lemma to prove Folkman's theorem.

Proof of Theorem 10.3. We show that $F(k; r) \leq n((k-1)r+1; r)$, where $n(k; r)$ is defined as in Lemma 10.5.

Let $x_1 < x_2 < \cdots < x_{(k-1)r+1}$ satisfy Lemma 10.5, with associated sets $S_1, S_2, \ldots, S_{(k-1)r+1}$. By the pigeonhole principle, k of these sets must have the same color. Denote these k sets by $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$.

Consider $T = \{i_1, i_2, \ldots, i_k\}$. Then

$$\mathcal{S}(T) = \left\{ \sum_{r \in R} x_r : R \subseteq T, R \neq \emptyset \right\}$$

is monochromatic, since $\max\{r : r \in R\} \in \{i_1, i_2, \ldots, i_k\}$. □

Now that we have proven the existence of $F(k;r)$ for all $k,r \geq 1$, we refer to $F(k;r)$ as a Folkman number. We also offer, after a definition, an immediate corollary of Folkman's theorem.

Definition 10.6. Let $T \subseteq \mathbb{Z}^+$ be finite. Define

$$\mathcal{P}(T) = \left\{ \prod_{r\in R} r : R \subseteq T, R \neq \emptyset \right\}$$

and call $\mathcal{P}(T)$ the *product-set* of T.

Example 10.7. For $T = \{2,5,8\}$, $\mathcal{P}(T) = \{2,5,8,10,16,40,80\}$.

Corollary 10.8. *For all $k,r \geq 1$, there exists a least positive integer $F^\star = F^\star(k;r)$ such that for every r-coloring of $[1,F^\star]$, there is a k-element subset $T \subseteq [1,F^\star]$ with $\mathcal{P}(T)$ monochromatic.*

Proof. We will show that $F^\star(k;r) \leq 2^{F(k;r)}$. Let $n = 2^{F(k;r)}$ and let χ be an r-coloring of $[1,n]$. Consider $\{2^i : 1 \leq i \leq F(k;r)\}$. Let γ be the r-coloring of $[1,F(k;r)]$ defined by $\gamma(i) = \chi(2^i)$.

Let $T = \{t_1 < t_2 < \cdots < t_k\} \subseteq [1,F(k;r)]$ be a k-element set such that $\mathcal{S}(T) \subseteq [1,F(k;r)]$ is monochromatic under γ. By the definition of γ, this implies that $Q = \{2^{t_1} < 2^{t_2} < \cdots < 2^{t_k}\}$ is monochromatic under χ. Since $\mathcal{S}(T)$ is monochromatic and

$$\prod_{r\in R} 2^{t_r} = 2^{\sum_{r\in R} t_r}$$

for any $R \subseteq T$, we see that $\mathcal{P}(Q) \subseteq [1,n]$ is monochromatic. □

With regard to bounds on the Folkman numbers, we see from the proofs of Theorem 10.3 and Lemma 10.5 that $F(k;r)$ is bounded from above by $2w(t;r)$ for some suitably large t, where w is the usual van der Waerden function. As such, Gower's bound for the van der Waerden numbers (Remark 2.22) also provides an upper bound for the Folkman numbers. For a lower bound, we turn to a result of Erdős and Spencer, which we offer without proof.

Theorem 10.9. $F(k;2) > 2^{ck^2/\log k}$ *for some fixed constant $c > 0$.*

A question you may have considered, based on the above results, is: for any $k,r \geq 1$, does every r-coloring of $\mathbb{Z}^+$ admit a k-element subset $T \subseteq \mathbb{Z}^+$, such that $\mathcal{S}(T) \cup \mathcal{P}(T)$ is monochromatic? The answer to this question is unknown, even for $k = 2$.

10.2. Doublefree Sets

Taking $k = 2$ in Folkman's theorem, we have a "strengthened" Schur's theorem, which says that, under any given r-coloring of $\mathbb{Z}^+$, there exist $x \neq y$ such that $\{x,y,x+y\}$ is monochromatic. We call this a strengthened Schur's theorem since we are requiring $x \neq y$, something that is not required in Schur's theorem. So, can we go the other way and require $x = y$? In other words, does every r-coloring of $\mathbb{Z}^+$ admit a monochromatic set of the form $\{x,2x\}$? The answer is no. Consider the 2-coloring χ of $\mathbb{Z}^+$ defined as follows: for $i = 2^j q$, where q is odd, let $\chi(i) = 0$ if j is even, and $\chi(i) = 1$ if j is odd. After a moment of thought, we see that such a coloring is indeed a well-defined coloring of $\mathbb{Z}^+$ with no monochromatic set of the form $\{x,2x\}$.

Definition 10.10. A subset $T \subseteq [1,n]$ is called *doublefree* if T contains no set of the form $\{x,2x\}$.

Since colorings of $[1,n]$ are just partitions of $[1,n]$ into subsets, we consider the question: what is the size of the largest doublefree subset of $[1,n]$? The following result answers this question.

Theorem 10.11. *The largest size of a doublefree subset of $[1,n]$ is $|T|$, where*

$$T = \{u4^i : u \text{ odd and } u4^i \leq n\}.$$

Proof. Each positive integer can be expressed uniquely in the form $u2^i$, where u is odd and $i \geq 0$. For each fixed odd positive integer $u \leq n$, let i_u denote the largest integer such that $u2^{i_u} \leq n$.

Consider $T_u = \{u2^i : 0 \leq i \leq i_u\}$. Assume T is a double-free subset of $[1,n]$. Then T cannot contain a pair of elements $u2^{i-1}$ and $u2^i$, with u odd and $i \geq 1$. Therefore, for each fixed odd positive integer u, a maximum-sized subset of T_u that is contained in T is $A_u = \{u2^0, u2^2, u2^4, \ldots, u2^m\}$, where

$$m = \begin{cases} i_u & \text{if } 4|i_u, \\ i_u - 1 & \text{if } 4 \nmid i_u. \end{cases}$$

Therefore, the size of T is no larger than the size of

$$S = \{u2^{2k} : u \text{ is odd and } u2^{2k} \leq n\} = \{u^{4k} : u \text{ is odd and } u4^k \leq n\}.$$

Since S is doublefree, we are done. □

10.3. Diffsequences

In Chapter 4 we investigated the Ramsey properties of certain subfamilies of the family of arithmetic progressions, where the common gap between consecutive integers is restricted to a prescribed set. In particular, recall that if $D \subseteq \mathbb{Z}^+$, then A_D denotes the family of all arithmetic progressions with gaps belonging to D. We remind the reader also that a set D is called *large* if for all $k, r \geq 1$, there is a least positive integer $R(A_D, k; r)$ such that every r-coloring of $\mathbb{Z}^+$ yields a k-term member of A_D (when this condition is satisfied for a fixed r, D is said to be r-large). In this section we consider a notion related to largeness. Yet, the topic of this section could be considered more like the families we encountered in Chapter 3, where we "loosened" the requirements in the sense that the gaps were allowed some "slack." How can these two contrasting concepts be somehow blended? Actually, it is quite simple: we restrict the gaps to a prescribed set D, but we throw out the requirement that the terms of the sequence form an arithmetic progression.

We begin with some terminology.

Definition 10.12. Let $D \subseteq \mathbb{Z}^+$. A sequence of positive integers $x_1 < x_2 < \cdots < x_k$ is called a k-term *D-diffsequence* if $x_i - x_{i-1} \in D$ for $i = 2, 3, \ldots, k$.

Example 10.13. Let $D = \{1, 5\}$ and let $E = 2\mathbb{Z}^+$. The sequence of integers $3, 8, 9, 14, 19$ is a 5-term D-diffsequence, and the sequence $1, 2, 7, 8$ is a 4-term D-diffsequence. A set is an E-diffsequence if and only if it consists entirely of even numbers or entirely of odd numbers. Thus, for example, $3, 7, 21, 25, 31, 41$ is an E-diffsequence.

Definition 10.14. Let $r \geq 1$. A set of positive integers D is called *r-accessible* if for every $k \geq 1$, there exists a least positive integer $\Delta = \Delta(D, k; r)$ such that whenever $[1, \Delta]$ is r-colored, there is a monochromatic k-term D-diffsequence. If D is r-accessible for all positive integers r, we say that D is *accessible*.

Note that by the compactness principle, the following is an alternate definition for a set D being r-accessible: whenever $\mathbb{Z}^+$ is r-colored, there are arbitrarily long monochromatic D-diffsequences.

Let us consider some examples.

Example 10.15. Consider $D = \{10n : n \geq 1\}$. We know from Theorem 4.9 (or Example 2.2) that for every finite coloring of the positive integers, there are arbitrarily long monochromatic arithmetic progressions whose gaps belong to D. Certainly, if we remove the requirement of being an arithmetic progression, and instead require merely that the difference between consecutive terms of the progression be a multiple of 10, the corresponding Ramsey property will still hold; i.e., D is accessible.

Example 10.16. Let D be the set of odd positive integers. Coloring $\mathbb{Z}^+$ as $101010\ldots$ shows that D is not 2-accessible (in fact, this coloring even avoids monochromatic 2-term D-diffsequences).

We remark that, as made evident in Example 10.15, if a set is r-large, then it must also be r-accessible.

We start our investigation of the Ramsey theory of diffsequences with a very useful lemma. Recall that, for S a set and $c \in \mathbb{R}$, $S + c$ denotes the set $\{s + c : s \in S\}$.

Lemma 10.17. *Let $c \geq 0$ and $r \geq 2$, and let D be a set of positive integers. If every $(r-1)$-coloring of D yields arbitrarily long monochromatic $(D+c)$-diffsequences, then $D + c$ is r-accessible.*

Proof. Let $D = \{d_i : i = 1, 2, \ldots\}$ and assume every $(r-1)$-coloring of D admits arbitrarily long monochromatic $(D + c)$-diffsequences. Let χ be an r-coloring of $\mathbb{Z}^+$. By induction on k, we show that, under χ, there are k-term monochromatic $(D + c)$-diffsequences for all k. Since there are obviously 1-term sequences, assume $k \geq 1$ and that under χ there is a k-term monochromatic $(D + c)$-diffsequence $X = \{x_1, x_2, \ldots, x_k\}$. We may assume X has color red. Consider the set $A = \{x_k + d_i + c : d_i \in D\}$. If some member of A is colored red, then we have a red $(k+1)$-term $(D+c)$-diffsequence. Otherwise we have an $(r-1)$-coloring of A and therefore, by the hypothesis, A must contain arbitrarily long monochromatic $(D+c)$-diffsequences. $\square$

Remark 10.18. The converse of Lemma 10.17 is false. As one example, let S be the set of odd positive integers, and let $D = S \cup \{2\}$. Let χ be the 2-coloring of D defined by $\chi(x) = 1$ if $x \equiv 1 \pmod 4$ or $x = 2$, and $\chi(x) = 0$ if $x \equiv 3 \pmod 4$. Then χ does not yield arbitrarily long monochromatic D-diffsequences (there are none of length four). On the other hand, Theorem 10.24 below tells us that D is 3-accessible (and, in fact, $\Delta(D, k; 3) \leq 6k^2 - 13k + 6$).

We now present two additional lemmas, the proofs of which are left to the reader as Exercises 10.3 and 10.4.

Lemma 10.19. *Let $m \geq 2$ and $i \geq 1$, and assume* $\gcd(i, m) = 1$. *Let $D = \{x \in \mathbb{Z}^+ : x \equiv i \pmod m\}$. Then D is not 2-accessible.*

Lemma 10.20. *If D is not r-accessible and E is not s-accessible, then $D \cup E$ is not rs-accessible.*

We next investigate the accessibility of some specific sets.

Theorem 10.21. *Let $D = \{2^i : i = 1, 2, \ldots\}$. Then*

$$8(k-3) + 1 \leq \Delta(D, k; 2) \leq 2^k - 1.$$

Proof. We start with the upper bound. Let $\alpha : [1, 2^k - 1] \to \{0, 1\}$. We show that under α there must be a monochromatic k-term D-diffsequence. We do this by induction on k. Obviously, it holds for $k = 1$. Now assume $k \geq 2$, and that $\Delta(D, k-1; 2) \leq 2^{k-1} - 1$. Let $X = \{x_1, x_2, \ldots, x_{k-1}\}$ be a monochromatic D-diffsequence, say of color 0, that is contained in $[1, 2^{k-1} - 1]$. Consider the set

$$A = \{x_{k-1} + 2^i : i = 0, 1, \ldots, k-1\}.$$

Note that $A \subseteq [1, 2^k - 1]$. If there exists $y \in A$ of color 0, then $X \cup \{y\}$ is a monochromatic k-term D-diffsequence. If, on the other hand, no such y exists, then A is a monochromatic k-term D-diffsequence.

For the lower bound, first note that by direct calculation we find that $\Delta(D, 3; 2) = 7$ and $\Delta(D, 4; 2) = 11$ (see Table 10.1 at the end of this section). To complete the proof we show by induction on k

that, for $k \geq 5$, the 2-coloring χ_k of $[1, 8(k-3)]$ that is represented by $(10010110)^{k-3}$ avoids monochromatic k-term D-diffsequences. It is easy to check that this statement is satisfied by $k = 5$. So now assume $k \geq 5$, that χ_k avoids k-term D-diffsequences, and consider χ_{k+1}.

Let $X = \{x_1, x_2, \ldots, x_m\}$ be a maximal length monochromatic D-diffsequence under χ_{k+1}. We wish to show that $m \leq k$. Assume, by way of contradiction, that $m \geq k+1$. Then x_{m-1} and x_m both belong to $[8(k-3)+1, 8(k-2)]$, or else the inductive assumption would be contradicted. We have the following two cases.

`Case 1.` $\chi_{k+1}(X) = 1$. We consider 2 subcases.

`Subcase i.` $x_{m-2} \in [8(k-3)+1, 8(k-2)]$. We have $x_{m-2} = 8k-20$, $x_{m-1} = 8k-18$, and $x_m = 8k-17$. By the structure of χ_k, we see that $x_{m-3} \equiv 4 \pmod 8$. Hence, there exists, under χ_k, a monochromatic D-diffsequence of length $m-1$, contradicting our assumption about χ_k.

`Subcase ii.` $x_{m-2} \notin [8(k-3)+1, 8(k-2)]$. Then $x_{m-1} = 8k-18$ and $x_m = 8k-17$. By the structure of χ_k, this implies $x_{m-2} \equiv 6 \pmod 8$. Then there is an $(m-1)$-term monochromatic D-diffsequence under χ_k, a contradiction.

`Case 2.` $\chi_{k+1}(X) = 0$. We again consider 2 subcases.

`Subcase i.` $x_{m-2} \in [8(k-3)+1, 8(k-2)]$. We have $x_{m-2} = 8k-22$, $x_{m-1} = 8k-21$, and $x_m = 8k-19$. Then either $x_{m-3} = 8(k-3)$ or $x_{m-3} \equiv 2 \pmod 8$. If $x_{m-3} = 8(k-3)$, then $m-3 \leq k-3$, because there can be only one term of a D-diffsequence per 10010110-string, a contradiction. If $x_{m-3} \equiv 2 \pmod 8$, then there is an $(m-1)$-term D-diffsequence of color 0 under χ_k, a contradiction.

`Subcase ii.` $x_{m-2} \notin [8(k-3)+1, 8(k-2)]$. Then $x_{m-1} = 8k-21$ and $x_m = 8k-19$, and hence $x_{m-2} \equiv 3 \pmod 8$. This is not possible, since there would then be a monochromatic $(m-1)$-term D-diffsequence under χ_k. $\square$

Before continuing, we need one more definition.

Definition 10.22. If $D \subset \mathbb{Z}^+$ is not accessible, the *degree of accessibility* of D is the largest value of r such that D is r-accessible.

We denote the degree of accessibility of a set D by $doa(D)$.

As a corollary to Theorem 10.21, we have the following result concerning the degree of accessibility of the set of powers of 2. The proof is left to the reader as Exercise 10.5.

Corollary 10.23. *Let $D = \{2^i : i = 1, 2, \dots\}$. Then $doa(D) = 2$.*

As mentioned in Remark 10.18, if D consists of the set of odd positive integers along with 2, then $doa(D) \geq 3$. The following theorem tells us more.

Theorem 10.24. *Let $D = \{2i + 1 : i \geq 0\} \cup \{2\}$. Then $doa(D) = 3$. Furthermore,*

$$\Delta(D, k; 2) = \begin{cases} 3k - 4 & \text{if } k \text{ is odd,} \\ 3k - 3 & \text{if } k \text{ is even.} \end{cases} \tag{10.1}$$

Proof. We first show that $doa(D) \geq 3$. Assume, for a contradiction, that $\gamma : \mathbb{Z}^+ \to \{0, 1, 2\}$ is a 3-coloring with no arbitrarily long monochromatic D-diffsequence. Let $s_1 < s_2 < \cdots < s_m$ be a monochromatic D-diffsequence of maximal length. We may assume this diffsequence has color 2. Then

$$\{s_m + 2\} \cup \{s_m + j : j \in \mathbb{Z}^+, j \text{ odd}\}$$

must be void of color 2. We may assume, without loss of generality, that $\gamma(s_m + 2) = 0$. Since m is maximal, at most $m - 1$ elements of $\{s_m + j : j \in \mathbb{Z}^+ \text{ odd}\}$ may have color 0. Let $\gamma(s_m + n) = 0$ with n maximal. Then $\{s_m + j : j \in \mathbb{Z}^+ \text{ odd}, j > n\}$ must be of color 1, a contradiction since this is a monochromatic D-diffsequence of infinite length.

The fact that $doa(D) < 4$ follows from Lemma 10.20. To see this, note that by Example 10.16 the set of odd positive integers is not 2-accessible, and that the 2-coloring of $\mathbb{Z}^+$ represented by $001100110011\dots$ shows that $\{2\}$ is not 2-accessible. Thus, by Lemma 10.20, $doa(D) \leq 3$, and hence $doa(D) = 3$.

Let $f(k)$ be the function on the right side of (10.1). We next prove that $f(k)$ is an upper bound for $\Delta(D, k; 2)$. First, by direct computation, we find that this is true for $k = 2$ and $k = 3$. To prove that $f(k)$ serves as an upper bound if $k \geq 4$, it is sufficient to show that for every $\chi : [1, f(k)] \to \{0, 1\}$, there exist D-diffsequences $S = \{s_1, s_2, \dots, s_{k_1}\}$ and $T = \{t_1, t_2, \dots, t_{k_2}\}$ such that $\chi(S) = 0$, $\chi(T) = 1$, and $k_1 + k_2 \geq 2k - 1$. We leave it as Exercise 10.6 to show that this condition holds for $k = 4$ and $k = 5$. To show it holds for all k, we proceed by induction on k, showing that its truth for $k + 2$ follows from its truth for k.

Assume that $k \geq 4$, and that every 2-coloring of $[1, f(k)]$ admits monochromatic sequences S and T as described above. Let $\chi : [1, f(k + 2)] \to \{0, 1\}$ be an arbitrary 2-coloring. To complete the proof we show that there exist a k_1'-term D-diffsequence of color 0 and a k_2'-term D-diffsequence of color 1 such that

$$k_1' + k_2' \geq 2k + 3. \tag{10.2}$$

We may assume, without loss of generality, that $k_1 \geq k_2$. Let

$$U = \{s_{k_1} + 1, s_{k_1} + 2, \dots, s_{k_1} + 6\}.$$

We consider three cases.

`Case 1.` There exist at least four elements of U that have color 0. It is easy to see that by appending these four elements to S, we have a monochromatic D-diffsequence, and hence (10.2) holds.

`Case 2.` Exactly three elements of U have color 0. Then there exist two of these three elements, a and b, such that $S \cup \{a, b\}$ forms a $(k_1 + 2)$-term D-diffsequence. Likewise there exist two elements, c and d, of T, with color 1 and such that $T \cup \{c, d\}$ forms a $(k_2 + 2)$-term T-diffsequence. This implies that (10.2) holds for this case.

`Case 3.` At most two elements of U have color 0. Then we may extend T to a D-diffsequence, monochromatic of color 1, having length $k_2' \geq k_2 + 4$. Again (10.2) holds.

To complete the proof, we show that $\Delta(D, k; 2) \geq f(k)$ by exhibiting a specific 2-coloring of $[1, f(k) - 1]$ that avoids monochromatic k-term D-diffsequences. Starting with the case in which k is even, let

$$\chi_k : [1, 3k - 4] \to \{0, 1\}$$

be the 2-coloring represented by $1(000111)^{\frac{k-2}{2}}0$. By symmetry it suffices to show that, under χ_k, there is no k-term D-diffsequence with color 1. We show this by induction on j, where $k = 2j$. Obviously the coloring 10 avoids 2-term monochromatic D-diffsequences, and the coloring 10001110 avoids 4-term monochromatic D-diffsequences, and hence the result holds for $j = 1$ and $j = 2$.

Now assume $j \geq 2$, and that χ_k does not yield any k-term monochromatic D-diffsequences with color 1. Note that χ_{k+2} may be represented by $\chi_k001110$. Let A be a monochromatic D-diffsequence, under χ_k, of color 1, and having maximal length. Obviously, at least one of $3k-7, 3k-6$ belongs to A, which implies that $3k-5$ also belongs to A. Therefore, at most two members of $\{3k-1, 3k, 3k+1\}$ may be tacked on to A to form a monochromatic D-diffsequence. Thus, under the coloring χ_{k+2}, there is no $(k+2)$-term D-diffsequence with color 1. This completes the proof for k even.

Next, consider k odd. Let λ_k be the coloring represented by $11(000111)^{\frac{k-3}{2}}00$. The proof is completed in a straightforward manner, similar to the even case, by induction on $\ell = \frac{k-1}{2}$, by showing that the longest D-diffsequence with color 1 cannot have length greater than $k-1$. We leave the details to the reader as Exercise 10.7. □

One very intriguing area of research that remains rather open is the relationship between accessibility and largeness. Clearly, no matter how we decide to do the counting, for any given (nontrivial) k and D, there are a lot more k-term D-diffsequences than there are k-term members of A_D (arithmetic progressions with gaps in D). Yet, it is currently unknown whether there exist any accessible sets that are not large. Perhaps settling this question is not a very difficult problem.

There are, however, some fundamental differences between accessibility and largeness that are known. For example, currently there are no known sets that are r-large for some $r \geq 2$, but which are not large. In fact, it has been conjectured that any set that is 2-large is also large (the conjecture remains open). In contrast to this, we have the following theorem concerning r-accessibility.

Theorem 10.25. *For each $r \geq 1$, there exists a set $D \subseteq \mathbb{Z}^+$ such that $doa(D) = r$.*

As stated, it may seem a daunting task to prove this theorem, especially given the rather complete lack of such information about r-large sets. However, Theorem 10.25 is an immediate corollary of the following result, which is, itself, rather easy to prove.

Theorem 10.26. *For $m \geq 2$, let V_m be the set of all positive integers not divisible by m. Then $doa(V_m) = m - 1$.*

Proof. We first show that $doa(V_m) \leq m-1$ by proving the following more general result: if $r \in \mathbb{Z}^+$ and D contains no multiples of r, then D is not r-accessible. To see this, consider $\chi : \mathbb{Z}^+ \to \{0, 1, \ldots, r-1\}$ defined by $\chi(x) = i$ if $x \equiv i \pmod r$. It is obvious that χ avoids 2-term monochromatic D-diffsequences. Thus, since (by definition) V_m contains no multiple of m, we conclude that $doa(V_m) \leq m - 1$.

Since V_2 is the set of odd positive integers which, as we have already seen is not 2-accessible, we need only consider $m \geq 3$. Let χ be any $(m-2)$-coloring of V_m. By the pigeonhole principle, some color must yield an infinite number of elements from each of at least two of the congruence classes $1 \pmod m, 2 \pmod m, \ldots, (m-1) \pmod m$. Thus, some color admits arbitrarily long V_m-diffsequences. By Lemma 10.17, V_m is $(m-1)$-accessible, and the proof is complete. □

As mentioned above, the question as to whether every accessible set is also a large set is still open. If there do exist any accessible sets that are not large, then the following theorem seems like a good place to start looking.

Theorem 10.27. *Let $T \subseteq \mathbb{Z}^+$ be infinite. Then*

$$T \ominus T = \{t - s : s < t \text{ and } s, t \in T\}$$

is accessible.

Proof. Let $r \geq 1$, and consider any r-coloring of $T \ominus T$. Fix $s \in T$. Let $\{t_1 < t_2 < \cdots\} = \{t \in T : t > s\}$ and define

$$A = \{t_i - s : i = 1, 2, \ldots\}.$$

Obviously, there exists a monochromatic infinite $B \subseteq A$. Since B is a $(T \ominus T)$-diffsequence, by Lemma 10.17, $T \ominus T$ is $(r+1)$-accessible. Since r is arbitrary, $T \ominus T$ is accessible. □

From the above theorem, we see that a set can be very "sparse" and still be accessible. For example, the set $\{t! - s! : 1 \leq s < t\}$ is accessible. It would be desirable to know if it is large.

The last stop on our tour of accessibility concerns the set of primes, which we will denote by P. We know that P is not a large set; in fact, it is not even 2-large, because a necessary condition for a 2-large set is that it contain a multiple of every positive integer (P contains no multiple of 4, for example). An open question is whether any translation of P (i.e., a set $c + P$ where $c \in \mathbb{Z}^+$) is large, or perhaps r-large for some $r \geq 2$. We may ask analogous questions in the context of accessibility; that is, what can we say, if anything, about the degree of accessibility of $P + c$ for $c \geq 0$.

First of all, what about P itself? Is it 2-accessible? As we explained in the previous paragraph, P is not 2-large; but the reason we gave does not apply to 2-accessibility (for example, the set of powers of 2 is 2-accessible, but has no multiple of 3). On the other hand, we do know, by Theorem 10.26, that P is not 4-accessible, since it is, in particular, a subset of V_4. In fact, we do not know whether P is 2-accessible. By the next theorem, however, we do know for sure that it is not 3-accessible.

Theorem 10.28. *Let P be the set of primes. Then $doa(P) \leq 2$.*

Proof. It suffices to find a 3-coloring of the positive integers that does not yield arbitrarily long monochromatic P-diffsequences. Consider the coloring $\chi : \mathbb{Z}^+ \to \{0, 1, 2\}$ defined by

$$\chi(i) = \begin{cases} 0 & \text{if } 9|i, \\ 1 & \text{if } i \text{ is even and } 9 \nmid i, \\ 2 & \text{if } i \text{ is odd and } 9 \nmid i. \end{cases}$$

It is clear that there is not even a 2-term P-diffsequence of color 0. Consider the elements of color 1. For any sequence $a_1 < a_2 < \cdots < a_9$

of color 1, there must be some i, $2 \leq i \leq 9$, such that $a_i - a_{i-1}$ exceeds 2 (and is even). Hence there is no 9-term P-diffsequence of color 1. Similarly, there is no 9-term P-diffsequence of color 0. □

If we consider $P + c$, with $c \geq 2$ even, then it is not hard to show that $doa(P + c) \leq 3$. However, we do not know if any such even translations are 2-accessible, let alone 3-accessible.

Based on what we have presented above, we could say that we do not have any positive results about the accessibility of even translations of P (we still do not know if there exists an even translation that is 2-accessible). We do, however, have a nice positive result for odd translations, presented below as Theorem 10.30. Before stating it, we give a lemma that, when used with Lemma 10.17, gives us Theorem 10.30.

The next lemma we present says that for any odd positive integer c, there exist arbitrarily $(P + c)$-diffsequences consisting of only primes.

Lemma 10.29. *Let $c \in \mathbb{Z}^+$ be odd. Then for any $k \geq 2$, there exist $p_1, p_2, \ldots, p_k \in P$ such that $p_i - p_{i-1} \in P + c$ for all i, $2 \leq i \leq k$.*

We do not include the proof of Lemma 10.29, which is beyond the scope of the book. Using Lemma 10.29, the following result can be shown. We leave the proof to the reader as Exercise 10.12.

Theorem 10.30. *Let P be the set of primes. If c is an odd positive integer, then $P + c$ is 2-accessible.*

There are many interesting questions left unanswered about accessibility. We present several of these in Section 10.8.

We end this section with a table of values of $\Delta(D, k; 2)$ for several choices of D and small k. The symbols T, F, P, and V_n denote $\{2^i : i \geq 0\}$, the set of Fibonacci numbers, the set of primes, and the set of positive integers not divisible by n, respectively.

$D \setminus k$	2	3	4	5	6	7	8	9	10
T	3	7	11	17	25	35	51	?	?
F	3	5	9	11	15	19	21	?	?
P	5	9	13	21	25	33	37	42	49
$P+1$	7	13	21	27	35	?	?	?	?
$P+2$	9	17	25	33	?	?	?	?	?
$P+3$	11	21	31	42	?	?	?	?	?
$P+4$	13	25	37	?	?	?	?	?	?
$P+5$	15	29	?	?	?	?	?	?	?
$P+6$	17	33	?	?	?	?	?	?	?
$P+7$	19	37	?	?	?	?	?	?	?
V_5	3	5	7	11	13	15	19	?	?
V_6	3	5	7	9	13	15	17	?	?

Table 10.1: Values of $\Delta(D, k)$

10.4. Brown's Lemma

The main result of this short section could have been included in the last section, since it is so closely related to diffsequences. However, the result is of sufficient interest to warrant its own section. We will formalize this result after presenting a definition.

Definition 10.31. A set $S \subseteq \mathbb{Z}^+$ is called *piecewise syndetic* if there exists $d \in \mathbb{Z}^+$ such that S contains arbitrarily long $\{1, 2, \ldots, d\}$-diffsequences.

Theorem 10.32 (Brown's Lemma). *Let $r \geq 1$. Any r-coloring of $\mathbb{Z}^+$ admits a monochromatic piecewise syndetic set.*

Proof. We induct on the number of colors. For $r = 1$, the result is trivial since $\mathbb{Z}^+$ is clearly piecewise syndetic. Assume the result holds for $r \geq 1$ colors and let $\chi : \mathbb{Z}^+ \to \{0, 1, \ldots, r\}$ be an arbitrary $(r+1)$-coloring.

If the color 0 occurs only finitely many times, then there exists n such that $\{n, n+1, \ldots\}$ is r-colored. By Proposition 2.30 and the inductive hypothesis, we are done. Hence, we assume that the color 0 occurs infinitely often. Let $R = \{r_1 < r_2 < \ldots\}$ be the set of integers with color 0.

If R is piecewise syndetic we are done, so we assume that R is not piecewise syndetic. Hence, the differences between consecutive elements of R are not bounded. Let $r_2 - r_1 = k_0$. Then there exists an $i_1 \geq 2$ such that $r_{i_1+1} - r_{i_1} = k_1 > k_0$, so that the interval $[r_{i_1} + 1, r_{i_1+1} - 1]$ is void of color 0. Similarly, we may choose $i_2 \geq i_1$ such that $r_{i_2+1} - r_{i_2} = k_2 > k_1$. Repeating this process, for each $j \geq 2$, denote by I_j the interval $[r_{i_j} + 1, r_{i_j+1} - 1]$, where i_j is chosen so that $r_{i_j+1} - r_{i_j} > r_{i_{j-1}+1} - r_{i_{j-1}}$ and $i_j \geq i_{j-1}$. Hence, the interval I_j is void of color 0 and $|I_j| > |I_{j-1}|$ for $j \geq 2$.

We now define γ, an r-coloring of $\mathbb{Z}^+$, as follows. Denote by $I_j(k)$ the k^{th} smallest element of I_j. Within $\chi(I_1(1)), \chi(I_2(1)), \chi(I_3(1)), \ldots$ some color (not 0, since the I_j's are void of the color 0) must occur an infinite number of times. Call this color c_1 and let $\gamma(1) = c_1$. Let $\mathcal{T}_1$ be the set consisting of those intervals I_j such that $\chi(I_j(1)) = c_1$. Within $\{\chi(I_j(2)) : I_j \in \mathcal{T}_1\}$ there must be some color c_2 that occurs an infinite number of times. Let $\gamma(2) = c_2$ and let $\mathcal{T}_2$ be the set of those intervals $I_j \in \mathcal{T}_1$ such that $\chi(I_j(2)) = c_2$. We continue in this way to find, for each $i \geq 2$, some color c_i such that

$$\mathcal{T}_i = \{I_j \in \mathcal{T}_{i-1} : \chi(I_j(i)) = c_i\}$$

is infinite, and we define $\gamma(i) = c_i$. The coloring $\gamma : \mathbb{Z}^+ \to \{1, 2, \ldots, r\}$ has the property that for any $n \geq 1$, $\mathcal{T}_n$ is the set of intervals I_j such that $\chi(I_j(i)) = \gamma(i)$ for $i = 1, 2, \ldots, n$.

By the inductive assumption, γ yields a monochromatic piecewise syndetic set, say X. Thus, X contains arbitrarily long $\{1, 2, \ldots, d\}$-diffsequences for some $d \in \mathbb{Z}^+$. Let $\{a_1 < a_2 < \cdots < a_n\} \subseteq X$ be one such diffsequence. Let $I \in \mathcal{T}_{a_n}$ and consider

$$S = \{I(a_1) < I(a_2) < \cdots < I(a_n)\}.$$

By the definition of γ, S is monochromatic. By construction, S is piecewise syndetic. Since n can be arbitrarily large, we have found an arbitrary large monochromatic (under χ) piecewise syndetic set. □

There is also a "finite form" of Brown's lemma. The fact that this is equivalent to Theorem 10.32 is left to the reader as Exercise 10.13. To state it, we define, for a finite set $A = \{a_1 < a_2 < \cdots < a_n\} \subseteq \mathbb{Z}^+$, the *gap size* of A to be

$$gs(A) = \max\{a_{j+1} - a_j : 1 \leq j \leq n-1\}.$$

If $|A| = 1$, we set $gs(A) = 1$.

Theorem 10.33. *Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be any function, and let $r \geq 1$. Then there exists a least positive integer $B(f;r)$ such that for every r-coloring of $[1, B(f,r)]$, there exists a monochromatic set A with $|A| > f(gs(A))$.*

Proof. Assume, without loss of generality, that f is nondecreasing. For simplicity of notation, denote $B(f,r)$ by $B(r)$. We use induction on r.

Clearly $B(1) = f(1)+1$, since if the interval $[1, f(1)+1]$ is colored with one color, the interval itself constitutes a monochromatic set A with $gs(A) = 1$, so that $|A| > f(gs(A)) = f(1)$.

Now, let $r \geq 2$ and assume that $B(r-1)$ exists. Let

$$m = rf(B(r-1)) + 1. \tag{10.3}$$

We will show that $B(r) \leq m$. Assume, for a contradiction, that there is an r-coloring χ of $[1, m]$ such that for every monochromatic set A we have $|A| \leq f(gs(A))$. Let $C_i = \{j : \chi(j) = i\}$, $1 \leq i \leq r$. Then $|C_i| \leq f(gs(C_i))$.

Also, $gs(C_i) \leq B(r-1)$ for each i, $1 \leq i \leq r$; otherwise, for some $a \geq 1$, the set $\{a+1, a+2, \ldots, a+B(r-1)\} \subseteq [1, m]$ would have only $r-1$ colors (it would be void of color i). By Proposition 2.30 and the inductive assumption, this would give a monochromatic set T with $|T| > f(gs(T))$, contradicting our assumption about χ.

Since f is nondecreasing, $f(gs(C_i)) \leq f(B(r-1))$. Hence,

$$|C_i| \leq f(gs(C_i)) \leq f(B(r-1))$$

for all i, $1 \leq i \leq r$. Since $[1, m] = C_1 \cup C_2 \cup \cdots \cup C_r$, we get $m \leq rf(B(r-1))$, contradicting (10.3). Thus, any r-coloring of $[1, m]$ satisfies the conditions of the theorem. This proves the existence of $B(r)$, since $B(r) \leq m = rf(B(r-1)) + 1$. □

We may rephrase Brown's lemma (Theorem 10.32) as follows: *for any r-coloring of $\mathbb{Z}^+$, there exists $d \geq 1$ such that for any $n \geq 2$, there exists a monochromatic set $A = \{a_1 < a_2 < \cdots < a_n\}$, where $a_{j+1} - a_j \leq d$ for $1 \leq j \leq n-1$.* Comparing this statement to the statements given in Theorem 2.5, we see that Brown's lemma is very reminiscent of van der Waerden's theorem. However, it is known that Brown's lemma neither implies, nor is implied by, van der Waerden's theorem.

10.5. Patterns in Colorings

Most of this book has dealt with the presence of specific monochromatic structures under finite colorings of the integers. In this section, rather than being concerned with finding structures, we will investigate colorings themselves.

For convenience, we will be working with colorings of $\mathbb{Z}^+$ or $[1, n]$ ($n \in \mathbb{Z}^+$); however, the integers are used only as placeholders for the colors.

Definition 10.34. Let $n, r \geq 1$ and let χ be an r-coloring of $[1, n]$ or $\mathbb{Z}^+$. We call χ *squarefree* if we cannot write $\chi = xzzy$ where each of x, y, z is an r-coloring (written as a string of colors) and z is nonempty. We call s *cubefree* if we cannot write $s = xzzzy$ where x, y, z are r-colorings and z is nonempty.

Note that Definition 10.34 does not say that x or y must be nonempty.

Example 10.35. There are no 2-colorings of $[1, 4]$ that are squarefree. To see this, assume $\chi : [1, 4] \to \{0, 1\}$ is a squarefree coloring. Without loss of generality, we may assume that $\chi(1) = 0$. Since χ is squarefree, $\chi(2)$ must be 1. Likewise, $\chi(3) = 0$, and $\chi(4) = 1$. Now we have the coloring zz, where $z = 01$, a contradiction.

Since it is quite easy to show that there is no squarefree 2-coloring of an interval of length more than three (by Example 10.35), the next result might seem somewhat remarkable.

Theorem 10.36. *Let $r \geq 3$. There exists $\gamma : \mathbb{Z}^+ \to \{0, 1, \ldots, r-1\}$ such that γ is squarefree.*

Concerning cubefree colorings, we have the following result.

Theorem 10.37. *Let $r \geq 2$. There exists $\gamma : \mathbb{Z}^+ \to \{0, 1, \ldots, r-1\}$ such that γ is cubefree.*

We next provide the colorings that are used to establish Theorems 10.36 and 10.37. The proofs that these colorings yield the desired conclusions are left to the reader as Exercise 10.14.

Definition 10.38. Let c be a string of colors 0 and 1. Let f act on c from left to right by

$$f \cdot 0 = 01 \text{ and } f \cdot 1 = 10.$$

The *Thue-Morse sequence*, also known as the Prouhet-Thue-Morse sequence, is

$$\lim_{n \to \infty} f^n \cdot 0 = 01101001100101101 \ldots.$$

To understand the Thue-Morse sequence, consider the first few iterations of f acting on 0. We have

$$0 \xrightarrow{f} 01 \xrightarrow{f} 0110 \xrightarrow{f} 01101001 \xrightarrow{f} 0110100110010110 = f^4 \cdot 0.$$

We see that the action of f appears to be such that $f^{n-1} \cdot 0$ is a leftmost substring of $f^n \cdot 0$. This is indeed true, but will not be proven here.

Using the Thue-Morse sequence, we can derive a related sequence.

Definition 10.39. Let $\{t_1, t_2, t_3, \ldots\}$ be the Thue-Morse sequence. Let $s = \{s_1, s_2, s_3, \ldots\}$ be the sequence defined by $s_i = t_{i-1}t_i$ (concatenation, not multiplication) so that

$$S = \{01, 11, 10, 01, 10, 00, 01, 11, 10, \ldots\}.$$

Define M to be the sequence obtained from S by replacing every 01 by 0, every 10 by 1, and every 00 and every 11 by 2 to get

$$M = \{0, 2, 1, 0, 1, 2, 0, 2, 1, \ldots\}.$$

We call M the *Morse-Hedlund sequence*.

We leave it to the reader as Exercise 10.14 to prove that the Thue-Morse sequence is an example of a cubefree 2-coloring of $\mathbb{Z}^+$, and that the Morse-Hedlund sequence is an example of a squarefree 3-coloring of $\mathbb{Z}^+$.

Some natural questions we may now ask are:

1. For $n \geq 1$, how many squarefree 3-colorings of length n are there?
2. For $n \geq 1$, how many cubefree 2-colorings of length n are there?

In investigating these questions we will use the following notation.

Notation. For $n \in \mathbb{Z}^+$, let $sq(n)$ denote the number of squarefree 3-colorings of length n, and let $c(n)$ denote the number of cubefree 2-colorings of length n.

For small n, the exact values of $sq(n)$ and $c(n)$ are known. We give the first 10 values of each:

n	$sq(n)$	$c(n)$
1	1	1
2	3	2
3	6	4
4	12	6
5	18	10
6	30	16
7	42	24
8	60	36
9	78	56
10	108	80

Table 10.2: Number of squarefree and cubefree colorings

Although no exact formula for $sq(n)$ or $c(n)$ is known, the following result, which we present without proof, gives upper and lower bounds for these functions.

Theorem 10.40. *Let $n \geq 1$. There exist constants $a, b > 0$ such that*

$$(1.118)^n < 110^{\frac{n}{42}} \leq sq(n) < a(1.3021)^n,$$
$$2 \cdot (1.080)^n < 2^{\frac{n}{9}+1} \leq c(n) < b(1.4576)^n.$$

10.6. Zero-sums

This section seems a fitting way to conclude this book, insomuch as it involves results obtained from the pigeonhole principal, the most basic principle of Ramsey theory. We begin with a definition.

Definition 10.41. Let $k, r \geq 1$ and let $\chi : \mathbb{Z}^+ \to \{0, 1, \ldots, r-1\}$ be an r-coloring. For $x_1, x_2, \ldots, x_k \in \mathbb{Z}^+$ we say that $(x_1, x_2, \ldots, x_k)$ is a *zero-sum sequence* if $\sum_{i=1}^{k} \chi(x_i) \equiv 0 \pmod{r}$.

It is convenient to use the language "S is zero-sum" when we want to say that S is a zero-sum sequence.

Before stating perhaps the most well-known zero-sum theorem, we start with two nice applications of the pigeonhole principle.

Theorem 10.42. *Let $r \geq 1$. Let $\chi : [1, r] \to \{0, 1, \ldots, r-1\}$ be an r-coloring. There exist $i, j \in [1, r]$ with $i + j \leq r$ such that*

$$(i, i+1, i+2, \ldots, i+j)$$

is zero-sum.

Proof. Assume, for a contradiction, that no such i and j exist. For $k = 1, 2, \ldots, r$ define

$$s_k = \sum_{i=1}^{k} \chi(i) \pmod{r}.$$

By assumption, $s_k \neq 0$ for $k = 1, 2, \ldots, r$. By the pigeonhole principle, there exist x, y, $1 \leq x < y \leq r$, such that $s_x = s_y$. Thus,

$$\sum_{i=1}^{x} \chi(i) \equiv \sum_{i=1}^{y} \chi(i) \pmod{r}.$$

This yields

$$\sum_{i=x+1}^{y} \chi(i) \equiv 0 \pmod{r},$$

and we are done by taking $i = x + 1$ and $j = y - x - 1$. $\square$

Lemma 10.43. *Let $r \geq 2$, and let $\chi : \mathbb{Z}^+ \to [0, r-1]$ be an r-coloring. Let $A \subseteq \mathbb{Z}^+$ be such that $|\{\chi(a) : a \in A\}| = k > 1 + \frac{r}{2}$ and write $\{\chi(a) : a \in A\} = \{c_1, c_2, \ldots, c_k\}$. Then, for any $c \in [0, r-1]$, there exist i, j, $1 \leq i < j \leq k$, such that $c \equiv (c_i + c_j) \pmod{r}$.*

Proof. Let χ and A be as in the statement, and let $c \in [0, r-1]$ be given. Let

$$B = \{(c - c_1) \pmod{r}, (c - c_2) \pmod{r}, \ldots, (c - c_k) \pmod{r}\}$$

(so that $|B| = k$). Since $|A \cap B| = |A| + |B| - |A \cup B|$ and $|A| + |B| > 2 + r$, while $|A \cup B| \leq r$, we see that A and B must have at least 3 elements in common. Say $x, y, z \in A \cap B$. Thus,

$$x \equiv (c - x') \pmod{r}, \quad y \equiv (c - y') \pmod{r}, \quad z \equiv (c - z') \pmod{r}$$

for some $x', y', z' \in A$. If we have $x \neq x'$, $y \neq y'$, or $z \neq z'$, then we are done because we have two distinct elements of A summing (modulo r) to c. Hence, assume $x = x', y = y'$, and $z = z'$. Since $2x \equiv 2y \equiv 2z \equiv c \pmod{r}$ we have

$$2(x - y) \equiv 0 \pmod{r}, \quad 2(x - z) \equiv 0 \pmod{r}, \quad 2(y - z) \equiv 0 \pmod{r},$$

i.e., there are three solutions to $2t \equiv 0 \pmod{r}$. Since we may have at most two distinct solutions to $2t \equiv 0 \pmod{r}$ (check!), one of the following must hold:

(i) $x - z \equiv (y - z) \pmod{r}$;

(ii) $x - y \equiv (x - z) \pmod{r}$;

(iii) $x - y \equiv (y - z) \pmod{r}$.

If (i) holds, then $x = y$, contradicting the fact that x and y are distinct. If (ii) holds, then $y = z$, again a contradiction. If (iii) holds, we have $x + z \equiv 2y \equiv c \pmod{r}$, which implies that $x = z$ since $2x = x + x \equiv c \pmod{r}$, contradicting the fact that x and z are distinct. $\square$

We now state and prove the main theorem of this section.

Theorem 10.44 (Erdős-Ginzburg-Ziv Theorem)**.** *Let $r \geq 2$. Let S be a set of $2r-1$ elements. For any r-coloring $\chi : S \to \{0, 1, \ldots, r-1\}$, there exist distinct $t_1, t_2, \ldots, t_r \in S$ such that $(t_1, t_2, \ldots, t_r)$ is zero-sum.*

Proof. Let χ be a given r-coloring. We use induction on the number of (not necessarily distinct) prime factors of r. We start with the base case: r is prime. Let $0 \leq c_1 \leq c_2 \leq \cdots \leq c_{2r-1}$ be the colors used by χ (all of the colors are not necessarily used). If $c_i = c_{i+r-1}$ for some $i \leq r-1$, then we are done since $\sum_{j=i}^{i+r-1} c_j \equiv 0 \pmod{r}$. Hence, assume $C_i = \{c_i, c_{i+r-1}\}$, $i = 1, 2, \ldots, r-1$, are 2-element sets.

Define, for $1 \le k \le r-1$,

$$X_k = \left\{ \sum_{i=1}^{k} x_i \,(\mathrm{mod}\, r) : x_i \in C_i, i = 1,2,\ldots,k \right\}.$$

We will show that $|X_k| \ge k+1$ by means of induction on k. For $k=1$ the result is trivial. Let $k < r-1$ and assume $|X_k| \ge k+1$. We will show that $|X_{k+1}| \ge k+2$. Clearly, the set

$$Y = \{(s_i + c_{k+1}) \,(\mathrm{mod}\, r) : s_i \in X_k, i = 1,2,\ldots,|X_k|\}$$

consists of $|X_k|$ distinct elements. We may assume that $|X_k| = k+1$, or else $|Y| \ge k+2$ and we are done since $Y \subseteq X_{k+1}$.

Hence, it remains to show that there exists $s_j \in X_k$ such that $(s_j + c_{r+k}) \,(\mathrm{mod}\, r) \notin Y$. Assume, for a contradiction, that no such s_j exists, i.e., for each s_j, $j = 1,2,\ldots,k+1$, there exists $s'_j \in X_k$ such that

$$s'_j + c_{k+1} \equiv (s_j + c_{r+k}) \,(\mathrm{mod}\, r). \tag{10.4}$$

Let $d = c_{k+1} - c_{r+k}$. Then $d \neq 0$, and hence we can rewrite (10.4) as

$$s'_j \equiv (s_j - d) \,(\mathrm{mod}\, r). \tag{10.5}$$

Since $|X_k| = k+1$, there exist $t_1, t_2, \ldots, t_{r-k-1}$, distinct residues modulo r, that are not members of X_k. Note that $k < r-1$ gives us $r-k-1 \ge 1$. Since r is prime, there exists $s \in X_k$ such that $s - d \equiv t_i \,(\mathrm{mod}\, r)$ for some i, $1 \le i \le r-k-1$, contradicting the existence of an $s'_j \in X_k$ satisfying (10.5). This completes the proof that $|X_k| \ge k+1$ for $1 \le k \le r-1$.

Let $X = X_{r-1}$. Hence, $|X| = r$ (since there are only r residues modulo r). Consider c_{2r-1}, which is not a member of any C_i. Since X contains all residues modulo r, there exist $x_i \in C_i$, $1 \le i \le r-1$, such that

$$\sum_{i=1}^{r-1} x_i \equiv -c_{2r-1} \,(\mathrm{mod}\, r).$$

Hence,

$$x_1 + x_2 + \cdots + x_{r-1} + c_{2r-1} \equiv 0 \,(\mathrm{mod}\, r),$$

and we have r elements that satisfy the conclusion of the theorem. This completes the base case.

Now, let $r = pm$ with p a prime and $m \neq 1$. Clearly, the number of prime factors of m is less than the number of prime factors of r. Taking all colors modulo p, there exists a p-element subset $T_1 \subseteq S$ such that $\sum_{t \in T_1} \chi(t) \equiv 0 \,(\mathrm{mod}\, p)$. Consider $S_1 = S - T_1$, so that $|S_1| = 2(m-1)p - 1$. Again, taking all colors modulo p, there exists a p-element subset $T_2 \subseteq S_1$, such that $\sum_{t \in T_2} \chi(t) \equiv 0 \,(\mathrm{mod}\, p)$. Let $S_2 = S_1 - T_2$. Continuing to view all colors modulo p, for each i, $2 \le i \le 2m-1$, there exists a p-element subset $T_i \subseteq S_{i-1}$ where $S_{i-1} = S_{i-2} - T_{i-1}$ (taking $S_0 = S$). Hence, there exist $2m-1$ pairwise disjoint subsets of S, say, $T_1, T_2, \ldots, T_{2m-1}$, with each T_i satisfying $\sum_{t \in T_i} \chi(t) \equiv 0 \,(\mathrm{mod}\, p)$.

For $i = 1,2,\ldots,2m-1$, let $k_i p = \sum_{t \in T_i} \chi(t)$ and consider the set $S' = \{k_i : i = 1,2,\ldots,2m-1\}$. By the induction hypothesis, S' contains $k_{i_1}, k_{i_2}, \ldots, k_{i_m}$ such that $\sum_{j=1}^{m} k_{i_j} \equiv 0 \,(\mathrm{mod}\, m)$. Hence, $T = T_{i_1} \cup T_{i_2} \cup \cdots \cup T_{i_m}$ is a subset of S with pm elements such that $\sum_{t \in T} \chi(t) \equiv 0 \,(\mathrm{mod}\, pm)$, which completes the proof. $\square$

As can be seen from the above proof, the fact that the set S has $2r-1$ elements is crucial. To see that we cannot r-color a set with only $2r-2$ elements and expect the same result to hold, consider the following example.

Example 10.45. For all $r \ge 2$, there exists $\chi : [1, 2r-2] \to [0, r-1]$ such that for any distinct $t_1, t_2, \ldots, t_r \in [1, 2r-2]$, $(t_1, t_2, \ldots, t_r)$ is not zero-sum. To show this, let χ to be the coloring defined by $\chi(i) = 0$ for $i = 1,2,\ldots,r-1$, and $\chi(i) = 1$ for $i = r, r+1, \ldots, 2r-2$ (χ uses only 2 of the possible r colors). Since $\sum_{i=1}^{r} \chi(x_i) \le r-1$ and is positive, for any distinct $t_1, t_2, \ldots, t_r \in [1, 2r-2]$, we see that $(t_1, t_2, \ldots, t_r)$ is not zero-sum.

In Example 10.45, we used only 2 colors. In fact, it is known that for any r-coloring $\chi : S \to \{0, 1, \ldots, r-1\}$, where S is a set of $2r-2$ elements, if there do not exist $t_1, t_2, \ldots, t_r \in S$ such that $(t_1, t_2, \ldots, t_r)$ is zero-sum, then, in fact, it must be the case that $\chi : S \to \{x, y\}$, where $r-1$ elements of S have color x, and the remaining $r-1$ elements of S have color y. Hence, if χ is an r-coloring that uses at least three colors, then we may use $|S| = 2r-2$ in Theorem 10.44.

Based on the previous example, we refine Theorem 10.44 by means of the following definition and subsequent results.

Definition 10.46. For $1 \le k \le r$, define $g = g(r,k)$ to be the least integer such that for all S with $|S| = g$ and all $\chi : S \to \{0,1,\ldots,r-1\}$, whenever $|\chi(S)| = k$ (i.e., the range of χ has size k), there exist $t_1, t_2, \ldots, t_r \in S$ such that $(t_1, t_2, \ldots, t_r)$ is zero-sum.

Note that Theorem 10.44 implies that $g(r,k)$ is well-defined, since $g(r,k) \le 2r-1$.

From the discussion after Example 10.45, we see that the following result is true.

Proposition 10.47. *Let $2 < k \le r$. Then $g(r,2) = 2r-1$ and $g(r,k) \le 2r-2$.*

Investigating $g(r,k)$ further, we have the following two theorems.

Theorem 10.48. *For $r \ge 1$,*

$$g(r,r) = \begin{cases} r & \text{if } r \text{ is odd,} \\ r+1 & \text{if } r \text{ is even.} \end{cases}$$

Proof. By definition, $g(r,r) \ge r$ for all r. For an upper bound, let $|S| = r$ and let χ be a coloring of S such that $\chi(S) = \{0,1,\ldots,r-1\}$.

First consider r odd. By definition, all colors are used by χ, so that

$$\sum_{s \in S} \chi(s) = \sum_{i=0}^{r-1} i = \frac{r(r-1)}{2}. \tag{10.6}$$

Since r is odd we see that $\frac{r-1}{2} \in \mathbb{Z}^+$, and hence the elements of S are zero-sum.

Now let r be even, so that $\frac{r}{2} \in \mathbb{Z}^+$. By (10.6), since $r \ge 2$, we have $\gcd(r, r-1) = 1$, so that $\frac{r}{2} \not\equiv 0 \pmod{r}$. Hence, $g(r,r) \ge r+1$.

Now let $T = \{t_1, t_2, \ldots, t_{r+1}\}$ and $\gamma : T \to \{0,1,\ldots,r-1\}$, with $|\gamma(S)| = r$. We may assume that $\gamma(t_{r+1})$ is the sole duplicate color.

Hence, for r even,

$$\sum_{i=1}^{r} \gamma(t_i) = \sum_{i=0}^{r-1} i \equiv s \pmod{r}$$

for some $s \ne 0$. Let $d = s + \gamma(t_{r+1})$. Then there exists $j \in \{1,2,\ldots,r\}$ such that $\gamma(t_j) = d$. Thus,

$$\sum_{\substack{i=1 \\ i \ne j}}^{r} \gamma(t_i) \equiv (s-d) \pmod{r},$$

which gives us

$$\gamma(t_{r+1}) + \sum_{\substack{i=1 \\ i \ne j}}^{r} \gamma(t_i) \equiv 0 \pmod{r},$$

and hence $(t_1, t_2, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{r+1})$ is zero-sum. Since γ is arbitrary, this implies that $g(r,r) \le r+1$. Hence, we have $g(r,r) = r+1$ for r odd, thereby completing the proof. □

Theorem 10.49. *If $r \ge 5$ and $1 + \frac{r}{2} < k \le r-1$, then $g(r,k) = r+2$.*

Proof. To show that $g(r,k) \le r+2$, let $\chi : [1, r+2] \to \{0,1,\ldots,r-1\}$ be a coloring that uses exactly k colors. Let $s = \sum_{i=1}^{r+2} \chi(i) \pmod{r}$. By Lemma 10.43, there exist $x, y \in [1, r+2]$, with $x < y$, such that $s \equiv (x+y) \pmod{r}$. Hence,

$$(1, 2, \ldots, x-1, x+1, x+2, \ldots, y-1, y+1, y+2, \ldots, r+2)$$

is zero-sum. This proves that $g(r,k) \le r+2$.

To prove that $g(r,k) \ge r+2$ we present a valid 2-coloring of $[1, r+1]$. We leave to the reader, as Exercise 10.15, the existence of $x \in \{2, 3, \ldots, k\}$ such that

$$2x \equiv \left(\binom{k}{2} + 1\right) \pmod{r}.$$

Let $\chi : [1, r+1] \to \{0, 1, \ldots, x-1, x+1, \ldots, k\}$ be the k-coloring represented by

$$\underbrace{0 \ldots 0}_{r-k+1}\, 1\, 1\, 2\, 3 \ldots (x-1)\,(x+1)\,(x+2) \ldots k.$$

Let

$$s = 1 + \sum_{\substack{i=1 \\ i \neq x}}^{k} i,$$

so that

$$s = \binom{k}{2} - x + 1 \equiv x \pmod{r}.$$

For any r elements in $[1, r+1]$, the sum of their colors, under χ, is $(s-t) \pmod{r}$ for some $t \in \{0, 1, \ldots, x-1, x+1, x+2, \ldots, k\}$. By our choice of x, we have $s - t \equiv (x - t) \not\equiv 0 \pmod{r}$. Hence, there is no collection of r elements which is zero-sum under χ. This proves that $g(r, k) \geq r + 2$, and completes the proof. □

10.7. Exercises

10.1 Find $F(3; 2)$.

10.2 Let $\widehat{F}$ be the least positive integer such that for any 2-coloring of $[1, \widehat{F}]$ there exist $x, y \in [1, \widehat{F}]$, not necessarily distinct, such that $\{x, y, x+y, xy\}$ is monochromatic. Show that $\widehat{F} = 39$.

10.3 Prove Lemma 10.19.

10.4 Prove Lemma 10.20. (Hint: see Theorem 4.32.)

10.5 Prove Corollary 10.23 by showing that D is not 3-accessible.

10.6 Using the notation of the proof of Theorem 10.24, show that for $k = 4$ and $k = 5$, for every 2-coloring of $[1, f(k)]$, there exist S and T as described in the proof.

10.7 Complete the proof of Theorem 10.24.

10.8 Prove that $a = 2$ is the only value of a, $a \geq 2$, for which $\{a^i : i = 0, 1, 2, \ldots\}$ is 2-accessible.

10.9 Let $F = \{F_0, F_1, \ldots\}$ be the set of Fibonacci numbers initialized by $F_0 = F_1 = 1$. Show that $\Delta(F, k; 2) \leq F_{k+3} - 2$.

10.10 Let $k \geq 2$. Show that $\Delta(S_3, k; 2) = 4k - 5$ and

$$\Delta(S_4, k; 2) = \begin{cases} 3k - 4 & \text{if } k \text{ is odd,} \\ 3k - 3 & \text{if } k \text{ is even.} \end{cases}$$

10.11 Let $V_{m,n}$ be the set of positive integers divisible by neither m nor n. Do parts (a) and (b) to show that $\Delta(V_{3,4}, k; 2) = 7k - 12$ for $k \geq 3$.

a) Show that $\Delta(V_{3,4}, k; 2) \geq 7k - 12$ for $k \geq 3$ by proving:

i) for k even, the coloring $1(10011000110011)^{\frac{k-2}{2}}$ avoids monochromatic k-term $V_{3,4}$-diffsequences, and

ii) for k odd, the coloring $1(10011000110011)^{\frac{k-3}{2}}(1001100)$ avoids monochromatic k-term $V_{3,4}$-diffsequences.

b) Show that $\Delta(V_{3,4}, k; 2) \leq 7k - 12$ for $k \geq 3$. (Hint: show that every 2-coloring of $[1, 7k-12]$ has monochromatic $V_{3,4}$-diffsequences $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, x_2, \ldots, y_n\}$, of different colors, such that $m + n \geq 2k - 1$. Then consider $x_m - y_n \equiv i \pmod{12}$ for $i = 0, 1, \ldots, 11$.)

10.12 Prove Theorem 10.30.

10.13 Prove that Theorems 10.32 and 10.33 are equivalent.

10.14 Prove that the Thue-Morse sequence is cubefree and that the Morse-Hedlund sequence is squarefree.

10.15 Finish the proof of Theorem 10.49 by proving the existence of x. Also, where is $r \geq 5$ needed?

10.8. Research Problems

***10.1** Prove or disprove: For all $r \geq 1$, any r-coloring of $\mathbb{Z}^+$ must admit a monochromatic set of the form $\{x, y, x+y, xy\}$.
References: [**212**], [**232**]

10.2 Define $f(n)$ to be the least positive integer such that for every 2-coloring of $[1, f(n)]$ there is an n-element set with no pair of elements differing by exactly 2. For example, the subset $\{1, 2, 5\}$ of $[1, 5]$ shows that $f(3) \leq 5$. On the other hand, it is impossible to find a 3-element subset of $[1, 4]$ for which no pair differs by exactly 2. Hence $f(3) \geq 5$. Do a study of $f(n)$. Try the same problem when the difference 2 is replaced by some other constant. Try more than two colors.
References: [**156**], [**157**], [**158**]

***10.3** Determine if P, the set of primes, is 2-accessible. More generally, does there exist an even number e such that $P + e$

is 2-accessible? If so, what is the order of magnitude of $\Delta(P+e,k;2)$?
Reference: **[174]**

***10.4** Let c be an odd positive integer. What is $doa(P+c)$? In particular, is $P+c$ 3-accessible? Run a computer program to calculate various values of $\Delta(P+c,k;2)$ (some are given in Table 10.1) and $\Delta(P+c,k;3)$.
Reference: **[174]**

10.5 Let $S=\{2\}\cup\{2i+1:i\geq 1\}$. Study the function $\Delta(S,k;2)$.
Reference: **[174]**

10.6 Determine the true order of magnitude of $\Delta(F,k;2)$, where F is the set of Fibonacci numbers.
Reference: **[174]**

10.7 Let $T=\{2^i:i\geq 0\}$. Determine the true order of magnitude of $\Delta(T,k;2)$ (the present authors have conjectured that $\Delta(T,k;2)=8k-23$, the lower bound given by Theorem 10.21).
References: **[174]**

10.8 Extend the results on $\Delta(T,k;2)$ (see Research Problem 10.7) and $\Delta(F,k;2)$ (see Research Problem 10.6) to more than 2 colors.
Reference: **[174]**

10.9 Does there exist an accessible set that is not large (see Definition 4.28)?
References: **[62]**, **[174]**

10.10 Find a formula for $f(V_m,k;2)$ for $m\geq 5$.
Reference: **[174]**

10.11 Find a formula for $f(V_{m,n},k;2)$ for $3\leq m<n$. (See Exercise 10.11 for the definition of $V_{m,n}$.)
Reference: **[174]**

10.12 Improve the bounds found in Theorem 10.40.
References: **[50]**, **[83]**, **[201]**, **[262]**

10.13 Find $g(r,k)$ for $5\leq k\leq 1+\frac{n}{2}$ (see Definition 10.46).
References: **[11]**, **[37]**, **[40]**, **[48]**, **[49]**, **[117]**

10.9. References

§10.1. The original proof of Folkman's theorem is unpublished, but reproduced in **[127]**. The proof presented in this section is essentially this proof. It was also proved by Rado **[212]** and, independently, by Sanders **[232]**. The proof of Theorem 10.9 can be found in **[97]**. A proof based only on Ramsey's theorem (and not van der Waerden's theorem) can be found in **[197]**. A powerful generalization of Folkman's theorem is due to Hindman **[143]** (it is known as Hindman's theorem). A simpler proof of Hindman's theorem was given by Baumgartner **[19]**. Related work is found in **[144]** and **[145]**; also see **[146]** for an excellent survey, and **[253]**.

§10.2. Extensions of doublefree sets can be found in **[98]**.

§10.3. All results presented concerning diffsequences are from **[174]**, which contains further work on the subject. The conjecture that the set of large sets is equal to the set of accessible sets was posed by Tom Brown.

§10.4. Brown's lemma is due to Tom Brown and is taken from **[52]**. We have denoted the numbers in Theorem 10.33 by $B(f;r)$ in honor of his contributions to the field of Ramsey theory on the integers. A generalization of Brown's lemma is investigated in **[54]**.

§10.5. The Thue-Morse sequence was defined originally by Prouhet **[208]**, and rediscovered by Thue (see **[268]**) with additional work by Morse **[193]**. For a good expository article about Thue, see **[35]**. The Morse-Hedlund sequence is from **[194]**. Regarding Theorem 10.40, the lower bound for $sq(n)$ is from **[262]**, the upper bound for $sq(n)$ is from **[201]**, the lower bound for $c(n)$ is from **[50]**, and the upper bound for $c(n)$ is from **[83]**. Good general references for this section are **[13]**, **[182]**, and **[183]**.

§10.6. There are at least five different proofs of the Erdős-Ginzburg-Ziv theorem. The one given here follows the original **[91]**, which can be found in **[14]**, along with four other proofs of the Erdős-Ginzburg-Ziv theorem. Some generalizations of the Erdős-Ginzburg-Ziv theorem are discussed in **[38]**. The definition of $g(n,k)$ is given in **[40]**. Theorem 10.42 can be found in **[11]**. The proof of Lemma 10.43 is from **[117]**, but was first proved in **[48]**. In **[49]**, Brakemeier also considers $g(n, k)$ for n prime. Proposition 10.47 is from [**37**]. The cases $k = 3, 4$ of $g(n, k)$ are studied in [**40**]. Theorem 10.49 was first proved in [**49**] and then independently in [**117**].

Additional References: There are many other Ramsey-type problems that have not been discussed in this book, but which are certainly of interest. Abbott, Liu, and Riddell [**10**] considered a function much like the Erdős and Turán function (see Section 2.5), but where one tries to avoid arithmetic progressions in a set of n real numbers, rather than in $[1, n]$. Let $m(n)$ represent the largest integer m such that there exists a k-term sequence in $[1, m]$ with the property that no member of the sequence is equal to the mean of the other members. Bounds on $m(n)$ are given in [**1**], [**2**], [**47**], [**96**], and [**100**].

Notation

Notation	Description	Page
$\lceil \cdot \rceil$	Ceiling function	10
$\lfloor \cdot \rfloor$	Floor function	10
$\oplus$	Modular addition	107
$[a, b]$	$\{a, a+1, \ldots, b\}$	9
$A - B$	$\{x \in A : x \notin B\}$	9
A_D	Family of arithmetic progressions with gaps in D	104
AP	Family of arithmetic progressions	14
$AP_{a(m)}$	Family of arithmetic progressions with gaps congruent to a (mod m)	163
$AP^*_{a(m)}$	$AP_{a(m)} \cup A_{\{m\}}$	168
$AP_{(m)}$	Set of arithmetic progressions (mod m)	164
AUG_b	Family of augmented progressions with tail b	152
$B(f; r)$	Brown number	278
cul_j	Culprit of color j	32
$\Delta_{\{a,b,c\}}$	Triangle on vertices a, b, c	205
$\Delta(D, k; r)$	Diffsequence Ramsey-type number	266
doa	Degree of accessibility	270
dor	Degree of regularity	138
dor_k	Degree of regularity for $T_{k-1}(a)$	145
$DW(k)$	Descending wave 2-color Ramsey-type number	70
$F(k; r)$	Folkman number	262
$\Gamma_m(k)$	Least s guaranteeing k-term arithmetic progressions in all s-term $[1, m]$-gap sequences	73
$GQ_\delta(k)$	2-color Ramsey-type function for generalized quasi-progressions	67

Notation	Description	Page
$g(r,k)$	Least integer such that for all S with $\lvert S\rvert = g$ and all $\chi : S \to \{0,\ldots,r-1\}$, whenever $\lvert\chi(S)\rvert = k$ there exist $t_1,t_2,\ldots,t_r \in S$ with $(t_1,t_2,\ldots,t_r)$ zero-sum	286
gs	Gap size	278
(k,n,d)-progression	k-term quasi-progression with diameter n and low-difference d	57
$H(s_1,\ldots,s_k)$	2-color Ramsey-type function for homothetic copies of $\{1,1+s_1,\ldots,1+s_1+\cdots+s_k\}$	147
$\lambda(c,k;r)$	Special r-coloring of $[1,cr(k-1)^2]$ that avoids monochromatic k-term c-a.p.'s	112
$\mathcal{L}(t)$	Equation $x_1+\cdots+x_{t-1}=x_t$	212
$\mu(k)$	$\min\{\lvert\mathcal{E}\rvert : \Gamma=(V,\mathcal{E})$ is a hypergraph not satisfying Property B and $\lvert E\rvert = k$ for all $E\in\mathcal{E}\}$	36
$M_\chi(n)$	Number of monochromatic Schur triples in $[1,n]$ under χ	205
$\nu(k)$	Erdős and Turán function	41
$\Omega_m(k)$	Least n so that every $\{x_1,\ldots,x_n\}$ with $x_i\in[(i-1)m, im-1]$ contains a k-term arithmetic progression	182
P_n	Family of sequences generated by iteration of a polynomial of degree n	81
$P_{n,k}$	Family of k-term members of P_n	81
$Q_n(k)$	2-color Ramsey-type function for quasi-progressions	57
$R(AP^*_{a(m)},k,l;r)$	Generalization of $R(AP^*_{a(m)},k;r)$	168
$r(\mathcal{E};s)$	Rado number for equation $\mathcal{E}$	230
$R(\mathcal{F},k;r)$	Ramsey-type number for family $\mathcal{F}$	14
$R(k_1,\ldots,k_r)$	r-color (classical) Ramsey number	8
$R_r(k)$	$R(\underbrace{k,\ldots,k}_{r})$	8
$RR(S;r)$	Reverse r-regular number	185
$S(k_1,\ldots,k_r)$	Generalized Schur number	212
$\hat{S}(k_1,\ldots,k_r)$	Strict generalized Schur number	218
S_n	Family of sequences generated by iteration of a polynomial of degree at most n	82
$S_{n,k}$	Family of k-term members of S_n	82
$SP_m(k)$	Ramsey-type number for semi-progressions	72
$s(r)$	Schur number	201
$\hat{s}(r)$	Strict Schur number	217
$T_{a,b}$	Set of (a,b)-triples	136
$T(a,b;r)$	Ramsey-type number for (a,b)-triples	136

Notation	Description	Page
$T(a_1,\ldots,a_{k-1})$	Ramsey-type function for generalization of (a,b)-triples	145
$\Theta(n)$	Set of permutations of $[1,n]$ with no 3-term arithmetic progression	194
$\theta(n)$	$\lvert\Theta(n)\rvert$	193
V_m	$\{x\in\mathbb{Z}^+ : m\nmid x\}$	273
$V_{m,n}$	$\{x\in\mathbb{Z}^+ : m\nmid x, n\nmid x\}$	289
$w(k)$	$w(k;2)$	25
$w(k;r)$	van der Waerden number	11
$w(k_1,\ldots,k_r;r)$	Mixed van der Waerden number	33
$w'(c,k;r)$	Ramsey-type number for arithmetic progressions with gaps at least c	111
$w'(f(x),k;r)$	Ramsey-type number for f-a.p.'s	116
$w^*(k,j)$	Ramsey-type number for arithmetic progressions with color discrepancy at least j	187
$\mathbb{Z}^+$	Positive integers	9

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Errata for Ramsey Theory on the Integers by Bruce M. Landman and Aaron Robertson

Date: June 23, 2004

FRONT MATTER

p. xi, Table 4.1: change "w" to "w'"

CHAPTER 1

p. 12, line -11: change "We" to "The"

p. 12, line -4: insert "monochromatic" before "integers"

p. 15, line -5: change "F" to "$\mathcal{F}$"

p. 16, line 2: change "exists" to "exist"

p. 17, line 17: change "$M(k;1)$" to "$M(k+1;1)$"

p. 20, line -3: change "contain" to "contains"

CHAPTER 2

p. 24, line -2: change "exist" to "exists"

p. 29, line -15: change "$w(3,4)$" to "$w(3;4)$"

p. 34, Table 2.1: change "$w(5,3,3;3) = 77$" to "$w(5,3,3;3) = 80$"

p. 37, line 3: change "$k \geq 1$" to "$k \geq 2$"

p. 42, line -2: change "$\frac{m}{r}$" to "$\frac{m-1}{r}$"

p. 48, lines 5–8: replace the definitions of the x_i''s by: $x_0' = d; x_i' = x_{i-1}$ for $1 \leq i \leq t+1$.

p. 49, exercise 1: Should read: "Show that within $[1,n]$ there are $\frac{n^2}{2(k-1)}(1+o(1))$ k-term arithmetic progressions."

p. 50, exercise 6: change "$w(k,2;r)$" to "$w(k,2;2)$"

p. 50, exercise 8: in (b) change "$\sum_{i=1}^{\frac{n^2}{4}+O(n)} 2^{1-k} = \frac{n^2+O(n)}{2^{k+1}}$" to "$\sum_{i=1}^{\frac{n^2}{2(k-1)}(1+o(1))} 2^{1-k} = \frac{n^2}{(k-1)2^k}(1+o(1))$."

p. 50, exercise 8: in (c) change "$\frac{n^2+O(n)}{2^{k+1}} < 1$" to "$\frac{n^2}{(k-1)2^k}(1+o(1)) < 1$"

p. 50, exercise 8: in (d) change "$n = 2^{k/2}$" to "$n = \sqrt{k-1} \cdot 2^{\frac{k-1}{2}}$"

p. 52, line -3: change "$w(5;2)$" to "$w(5,5;2)$"

CHAPTER 3

p. 61, line -8: change the first occurrence of "t" to "i"

p. 69, Table 3.2: replace the functions in the first column by the functions $x-1, x-2, x-3, x-4$ and $x-5$ (in that order)

p. 71, line 2: change "$x_0 = 0$ and $x_1 = 1$" to "$x_0 = n+1$ and $x_1 = n$"

p. 71, line 4: change "min" to "max" and change "$y - x_i$" to "$x_i - y$"

p. 71, line 5: change "$x_1, x_2, \ldots, x_k$" to "$x_k, x_{k-1}, \ldots, x_1$"

p. 71, line 6: change "and that $x_k \leq n$" to "that is contained in $[1,n]$" (Note: the proof of Theorem 3.21 given in the text is for *ascending* waves.)

p. 84, line 8: change "$(1-a)d$" to "$(1-a)+d$"

p. 88, line 10: change "$AP \cup P_{k-2}$" to "$R(AP \cup P_{k-2}, k)$"

CHAPTER 4

p. 105, line -12: change "k" to "$n+1$"

p. 109, line 9: change "j" to "ℓ"

p. 111, Theorem 4.9: change "$m\mathbb{Z}^+$" to "$A_{m\mathbb{Z}^+}$"

p. 111, Corollary 4.10: change "$\mathbb{Z}^+ - F$" to "$A_{\mathbb{Z}^+-F}$"

CHAPTER 5

p. 136, line 5: change "k" to "3"

p. 137, line 9: change "w" to "w'"

p. 138, proof: there are 3 occurrences of w that should be w'

p. 151, line 3: change "$\leq$" to "$<$"

CHAPTER 7

p. 192, line -7 and -9: delete "$= S_1$" and "$= S_2$"

CHAPTER 8

p. 210, line -9: change "the same color" to "of different colors"

CHAPTER 9

p. 242, line 5: insert "$1 + i_1a + j_1b = 1 + i_2a + j_2b$" after "such that"

p. 251, line 17: change "$\{0, 1, \ldots, r\}$" to "$\{1, 2, \ldots, r\}$"

p. 252, line 6: change "There" to "For every r-coloring of $\mathbb{Z}^+$, there"

CHAPTER 10

p. 261, last line: change "if $t \in T$, then $2t \notin T$" to "$t \in T$ does not imply that $2t \in T$"

p. 262, line 2: insert "obtained only" after "sum"

p. 263, lines 3, 5, and 8: change "$n(k; r)$" to "$kn(k; r)$"

p. 263, line 8: change "$\left[\frac{m}{2}, m\right]$" to "$\left(\frac{m}{2}, m\right]$,"

p. 263, lines 17 and 19: change "$(a + d) + D$" to "$(a + d) + \sum_{r\in R\subseteq[1,k]} x_r$"

p.268, Lemma 10.20: before "then" insert "and if either $D + E = \{d + e : d \in D, e \in E\} \subseteq D$ or $D + E \subseteq E$,"

p. 270, replace the first paragraph of the proof of Theorem 10.24 with the following:

We first show that $doa(D) \geq 3$. Assume, for a contradiction, that $\gamma : \mathbb{Z}^+ \to \{0, 1, 2\}$ is a 3-coloring without arbitrarily long monochromatic D-diffsequences. Let $s_1 < s_2 < \cdots < s_m$ be a monochromatic D-diffsequence of maximal length. We may assume this diffsequence has color 2. Then $S = \{s_m + j : j \text{ odd}\}$ is void of color 2. Let $S = \bigcup_{i\geq 0} S_i$, where $S_i = \{s_m + 2i(m + 1) + j : j \in \{1, 3, 5, \ldots, 2m + 1\}\}$ are sets of $m + 1$ elements. Define, for $i \geq 0$, $T_i = \{s_m + 2i(m + 1) + j : j \in \{2, 4, 6, \ldots, 2m + 2\}\}$, which are also sets of $m + 1$ elements. Each T_i must contain an element of either color 0 or color 1, for otherwise T_i would be an $(m + 1)$-term D-diffsequence of color 2, contradicting the choice of m. Furthermore, each S_i must contain elements of both color 0 and color 1 (since it is void of color 2), for otherwise S_i would be an $(m + 1)$-term monochromatic D-diffsequence. Since some color, say color 0, must occur an infinite number of times in the T_i's, there exist $x_{i_1} \in T_{i_1}, x_{i_2} \in T_{i_2}, x_{i_3} \in T_{i_3}, \ldots$, where $i_{j+1} > i_j + 1$, all of color 0. For each $j \geq 1$, let $y_{i_j} \in S_{i_j+1}$ be of color 0. Then $x_{i_1}, y_{i_1}, x_{i_2}, y_{i_2}, \ldots$ is an infinitely long D-diffsequence of color 0, contradicting the existence of m.

p. 275, line 15: insert "long" after "arbitrarily"

p. 285, line 7: change "$2(m - 1)p - 1$" to "$(2m - 1)p - 1$"

p. 288, exercise 9: change each occurrence of "F_0" to "F_1" and each occurrence of "F_1" to "F_2"

p. 288, exercise 10: change "S_3" to "V_3" and "S_4" to "V_4"

p. 290, exercises 10 and 11: change "f" to "Δ" in each problem