Algorithms and Combinatorics 5

Jaroslav Nešetřil Vojtěch Rödl (Eds.)

Mathematics of Ramsey Theory

With 15 Figures

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona

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Mathematics Subject Classification (1980): 00-02, 00 A 10, 03 F 30, 05 A 99, 05 B 25, 05 C 15, 05 C 55, 05 C 65, 05 C 80, 11 B 13, 11 B 25, 28 D 99, 34 C 35, 51 E 10, 51 E 15, 51 M 99, 54 H 20, 68 R 10

ISBN 3-540-18191-1 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-18191-1 Springer-Verlag New York Berlin Heidelberg

Library of Congress Cataloging-in-Publication Data
Mathematics of Ramsey theory / J. Nešetřil, V. Rödl (eds.) p. cm. - (Algorithms and combinatorics; 5)
Includes bibliographical references (p.) and indexes.
ISBN 3-540-18191-1 (Berlin). - ISBN 0-387-18191-1 (New York)
1. Ramsey theory. I. Nešetřil, J. (Jaroslav) II. Rödl, V. (Vojtěch), 1949-. III. Series.
QA166.M37 1990 511'.5-dc20 90-10186 CIP

Printed in the United States of America

Printing and binding: Edwards Brothers Inc., Ann Arbor, Michigan
2141/3140-543210 - Printed on acid-free paper

Preface

During the last Prague Symposium on Graph Theory several people suggested collecting papers which would exhibit diverse techniques of contemporary Ramsey Theory.

The present volume is an outgrowth of this idea. Contemporary research related to Ramsey Theory spans many and diverse areas of mathematics and it has been our intention to demonstrate it. We decided not merely to collect papers but also to edit the volume as a whole. In several instances we asked for specific contributions.

Admittedly this was a bit ambitious project and as a result it took us several years to complete it. But perhaps the time was worth it: we are pleased that we have among the contributors many leading mathematicians.

We thank all the authors for the excellent job they have done.

We thank Jan Kratochvíl and Yvonne Holá (Charles University, Prague) for technical assistance in editing this volume. The book has been TEX-processed in the Institut für Diskrete Mathematik (Bonn) and we thank Mrs. H. Higgins and B. Schaschek.

Bonn, March 13, 1990

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Introduction
Ramsey Theory Old and New

Jaroslav Nešetřil and Vojtěch Rödl

The purpose of this introduction is to outline scope and intentions of this volume. We also state some classical results. This historical perspective will be of some use to a (non-specialist) reader.
Ramsey theory stems from the following result:

Ramsey Theorem (infinite case). *For every finite partition of the set of all p-subsets of an infinite set one of the classes of the partition contains all p-subsets of an infinite set.*

This fundamental result changed and in a sense created combinatorial word as we know it today. Historically, it was also one of the first combinatorial results which attracted attention of mathematicians in general.

Ramsey theorem is a structural generalization of anglo-saxon pidgeonhole – and continental Dirichlet's "Schubfach" – principle (for $p = 1$) and it is important that it admits a finite version:

Ramsey Theorem (finite case). *For every choice of positive integers p, k, n there exists an integer N with the following property:*
For every set X of size at least N and for every partition $A_1 \cup \ldots \cup A_k$ of the set $\binom{X}{p}$ of all p-subsets of X there exists a homogeneous subset Y of X of size at least n. Here homogeneous means that $\binom{Y}{p}$ is a subset of one of the classes of the partition.

Investigations of the finite version are predominant in the history of this subject. This is perhaps due to the many combinatorial applications of Ramsey theorem which started with an independent discovery of finite Ramsey theorem by Erdös and Szekerés. They arrived to it in the following geometrical context:

Erdős-Szekerés Theorem. *For every n there exists N with the following property: Let X be a set of N points of Euclidean plane containing no 3 colinear points. Then X contains n independent points (i.e. n-points which form vertices of a convex n-gon).*

These early results motivated future research. These motivations are persistent until now.

In the more than fifty years since the publication of F. P. Ramsey (1930) and P. Erdös, R. Rado (1952) the whole subject formed one of the most developed combinatorial theories which transcendents by far the original motivation. It is one of the goals of this volume to include modern aspects and more recent approaches to this subject. The monograph by R. Graham, B. Rothschild and J. Spencer gives an introduction to this subject. Nevertheless we feel that recent development merits a volume which would complement their excellent book. Admittedly this was a bit ambitious project and as a result of it it took several years to complete a representative list of contributions.

Let us list several particular aspects of Ramsey theory (moreless in a chronological order) most of which are relevant to articles in this collection.

1. Ramsey Numbers

One of the oldest areas of the Ramseyan research is the study of Ramsey numbers $r(p,k,n)$: $r(p,k,n)$ is the smallest value N for which finite Ramsey theorem is valid.

It is known that there are positive constants c_p and c'_p such that

$$c'_p n^2 \leq log^{(p-1)}\ r(p,2,n) \leq 2^{c_p n}$$

(here $log^{(p-1)}$ denotes $(p-1)$-times iterated logarithm). The lower bound was proved in (Erdös, Rado 1952) and the upper bound (Erdös, Hajnal, Rado 1965). The famous exponential lower bound $r(2,2,n) > 2^{\frac{n}{2}}$ for Ramsey numbers was established first by Erdös in 1947 by a probabilistic method which itself developed into one of the most active areas of combinatorics. These results stimulated extensive research. It is difficult to evaluate Ramsey numbers both exactly (just a handful of non-trivial cases is known) and asymptotically. Recently there has been progress in asymptotical estimates of Ramsey numbers. See the paper by J. Spencer in this volume and, for a more extensive survey, article (Graham, Rödl 1987) which covers great part of the research related to Ramsey numbers. As the original setting presents somehow untractable problems Ramsey numbers were generalized to other homogeneous graphs then complete ones. In the last 15 years there has been a lot of activity in this area and several deep results were obtained.

We are happy to include a paper in this area namely the paper by J. Beck. The paper by S. Burr in this volume investigates the generalized Ramsey numbers from the non-traditional computational point of view.

2. Transfinite Ramsey Theory

Ramsey theorem may be generalized to sets of arbitrary size. This project was developed by Erdös, Rado and Hajnal to the *partition calculus* of cardinal numbers. They also introduced a concise notation of the (relatively complex) statement of Ramsey type theorem. This so called *arrow notation* has many variants. In its most standard form the Infinite Ramsey theorem reads

$$\omega \to (\omega)^p_k$$

for every choice of positive integer p and k. Using this one can formulate fundamental result of Erdös and Rado as follows:

Theorem (Erdös, Rado 1952). *For every finite p and infinite cardinals α, β there exists a cardinal γ such that $\gamma \to (\alpha)^p_\beta$.*
Particularly, $(2^\alpha)^+ \to (\alpha^+)^2_\alpha$.

A monograph devoted so this subject is the book by P. Erdös, A. Hajnal, A. Maté and R. Rado (1984). One of the main uses of transfinite Ramsey theory is set theory, topology and more recently functional analysis.

While most of the contributions to this volume are finitistic the paper by W. Weiss on Ramsey topological spaces is related mostly to this area.

Another related contribution included in this volume is a paper by R. Rado which gives a new proof of well known Erdös-Rado canonization lemma.

3. Chromatic Number

It is well known that one may express a Ramsey type result as a statement about chromatic numbers of special classes of hypergraphs.

Explicitly, given a set X let $\binom{X}{p}$ be the set of all p-element subsets of X. Denote by X^n_p the set system (V, E) where $V = \binom{X}{p}$ and $E = \{\binom{Y}{p} \; ; \; Y \in \binom{X}{n}\}$.

Then $|X| \to (n)^p_k$ if and only if the chromatic number $\chi(X^n_p)$ exceeds k. Every Ramsey type statement may be reformulated in this way. Much of the research on chromatic numbers was motivated by questions of Ramsey type. The paper by P. Erdös in this volume written in the classical Erdösian style suggests many selected problems which indicate how lively is the subject today.

4. Classical Theorems

As expected the Ramsey theorem fails to be chronologically the first statement of Ramsey type. One may speculate that several classical theorems fall within general framework outlined by Mirsky and Burkill philosophical lines (Burkill, Mirski 1973) "every system of a certain class possesses a large subsystem with a higher degree of organization than the original system". One may note that Bolzano-Weierstrass theorem and many compactness type results may be interpreted in this context.

However it is customary to cite the following two theorems as the first examples:

Hilbert Theorem (Hilbert 1892). *For every positive integers k, n there exists N with the following property: For every partition of the power set $P(X)$ into k classes of a set of size $\geq N$ there exist distinct subsets $A_1, \ldots, A_n$ of X which have pairwise identical intersections (i.e. $A_i \cap A_j = A_{i'} \cap A_{j'}$) such that all $2^n - 1$ non-empty unions belong to the same class of the partition.*

Schur Theorem (Schur 1916). *For every positive integers k, n there exists N with the following property:*
For every partition of $\{1, 2, \ldots, N\}$ into k classes one of the classes contains two numbers together with their sum.

There are much more general results known today and also easy proofs of these results are available. The area is still active and there exists relationship to other branches of mathematics. An example of such research motivated by Pisier problem from harmonic analysis is included in this volume (contribution by P. Erdős and the editors).

5. Other Classical Theorems

Apart from the Ramsey theorem itself no other result in this area is as popular as Van der Waerden theorem.

Van der Waerden Theorem *(Van der Waerden 1927) For every k, n there exists N with the following property: For every partition $\{1, 2, \ldots, N\}$ into k classes one of the classes contains an arithmetical progression of length n. This deep result found a proper combinatorial setting by means of Hales-Jewett theorem:*

Hales-Jewett Theorem (Hales, Jewett 1963). *For every k, n there exists N with the following property:*
For every partition of $\{1, \ldots, n\}^N$ into k classes one of the classes contains a combinatorial line.
Here we think of $\{1, \ldots, n\}^N$ as of a cube and a combinatorial line determined by $\omega \subseteq \{1, \ldots, N\}$ and $(x_1^0, \ldots, x_N^0)$ is a set of the form

$$\begin{aligned}\{(x_1, \ldots, x_N);\ & x_i = x_i^0 \ for\ i \notin \omega\} \\ & x_i = x_j \ for\ i, j \in \omega.\end{aligned}$$

Hales-Jewett theorem is presently one of the most useful techniques in Ramsey theory.
We are happy to include in this volume a very nice paper by H. J. Prömel and B. Voigt which surveys the fascinating development related to Hales-Jewett

theorem. The paper is called Parameter sets according to an approach due to Graham and Rothschild (Graham, Rothschild 1971). We are happy to include in the volume a paper by A.Nilli (N. Alon) which presents a particularly concise version of the recent striking new proof of Van der Waerden's theorem due to S. Shelah.

6. Structural Generalizations

Positive examples of Ramsey type statements which were found in the sixties and early seventies encouraged attempts to generalize and to abstract properties of classes of structures with Ramsey properties (see Graham, Rothschild 1971, Graham, Leeb, Rothschild 1972, Deuber 1973, Leeb 1973, Nešetřil, Rödl 1975 for early examples). The whole research attained the level of generality which is in this volume mirrored by papers by Prömel and Voigt and by the editors. Perhaps the main feature of this development was the sharp definition of the whole subject which e.g. led to the project of investigating Ramsey structures and to the following key concepts: parameter sets, Ramsey class, induced and restricted theorems, Euclidean Ramsey theorems, Ramsey property, ordering property, selective and canonical theorems. Several old problems were solved such as Ramsey property of finite vector spaces (conjecture of Rota; Graham, Leeb, Rotschild 1972) and Ramsey property of partition regular sets (conjecture of Rado, Deuber 1973). When dealing with complex statements of Ramsey type it seems that it was useful to apply some kind of a general formalism; (such as category theory) see the paper by Prömel and Voigt and by editors in this volume. Euclidean Ramsey theorey is surveyed in this volume by a paper of R. Graham.

Perhaps the most exciting line of research in Ramsey theorey is related to the validity of the density version of Van der Waerden theorem. The related conjecture proved to be one of the main motivations of the subject:
If A is a set of positive integers with positive upper density, that is, satisfying

$$lim\ sup_n \frac{A \cap \{1, \ldots, n\}}{n} > 0$$

then A contains arbitrary long arithmetical progressions. This conjecture was settled affirmatively in 1974 by E. Szemerédi in perhaps the most difficult paper in the subject. Recently the Erdös-Turán problem and its higher dimensional analogue have been solved by quite different techniques from ergodic theory and topological dynamics by Fürstenberg (1981) and (1981). These techniques are in this volume illustrated by paper by Fürstenberg and Katznelson. Note that these techniques yield results which are (at least presently) not obtainable by combinatorial methods.

7. Infinite Ramsey Theorem

It is well known that one cannot partition infinite subsets of an infinite set in order to obtain an infinite homogeneous set. (Explicitely, $k \not\rightarrow (\omega)_2^\omega$ for every k.) However axiom of choice is needed for this fact and this led R. Solovay to reverse the question : He asked whether one can find an infinite homogeneous set $\binom{\omega}{\omega}$ if only "nice partitions" are considered. This is indeed the case as shown by Nash-Williams, Galvin-Prikry and others. This development culminated in the proof of infinite dual Ramsey theorem which generalizes many results of this type.

The paper by Carlson and Simpson (and partially by Prömel and Voigt) is a survey of this development.

These theorems are related to the theory of Well-Quasi Ordered Sets (WQO). Ramsey theory and theory of WQO share many similarities and the paper by Kříž and Thomas is a recent attempt to study countable ordinal types related to both Ramsey and WQO theory.

8. Unprovability Results

(When available) countable Ramsey type statement generalizes corresponding finitistic statement. This is easy to see by a compactness type argument (applying e.g. the reformulation via the chromatic number). Recently, it has been shown that infinite Ramsey theorem is strictly more powerfull as it may be used (again by compactness) to prove the following:

Paris-Harrington Theorem *(Paris,Harrington 1977). For every k, p, n there exists N with the following property:*
For every partition $\binom{\{1,\dots,N\}}{p}$ into k classes one of the classes contains $\binom{Y}{p}$ where Y is a relatively large set with at least n vertices.

Here Y is *relatively large* if $|Y| \geq minY$.
Paris and Harrington proved that this result fails to be true within theory of finite sets thus giving perhaps the first mathematically interesting example for the Gödel's incompleteness theorem. This result gained an instant popularity as it nicely fits to our common sense. Since then the progress has been quick and several other examples has been found and the paper by Paris included in this volume surveys these developments. Another feature of these indecidability results is that it again relates Ramsey theory and theory of Well Quasi Orderings. These two subjects have many common aspects see e.g. (Nešetřil 1984, Leeb 1973, Nešetřil, Thomas 1987) and the paper by Kříž and Thomas in this volume.

9. Non-Standard Applications

Recently the whole field of Ramsey theory expanded in various directions. Several very deep and far reaching generalizations of the original theorems were found. The interested reader may consult survey articles by Graham and Rödl (1987), Nešetřil and Rödl (1979), Graham (1983). He may also compare them to get a clearer picture of the development of the subject. There are traditional areas of Ramsey applications: these include geometry, number theory, set theory. There are more recent applications some of them quite unexpected: functional analysis, theórey of ultrafilters (Baumgartner, Taylor 1978), mathematical logic (Abramson, Harrington 1978), and especially ergodic theory (Fürstenberg 1981, Fürstenberg 1981). Another of these new trends are the applications of Ramsey type results in theoretical computer science.

One may ask how one can use a typically "non constructive" theorem in the computer science context.

The answer is given e.g. by articles by Yao (1981), Alon, Maass (1986), Moran, Snir, Manber (1985), Vilfan (1976), Pudlák in this volume; see also survey article Nešetřil (1984).

These applications use both lower and upper bounds for Ramsey functions. Using complex examples of graphs (which are obtained by lower bound in corresponding Ramsey-type statement) one may obtain lower bounds for various measures of complexity; see the paper by Pudlák in this volume. The upper bounds, which essentially mean that we apply the existence of Ramsey numbers only, yield a canonical form of various procedures when applied to a very large object (such as sorting a large set). This was in fact the original motivation of F. Ramsey for his discovery. It is surprising how persistent is this motivation.

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Part I

Classics

Problems and Results on Graphs and Hypergraphs: Similarities and Differences

Paul Erdös

Many papers and also the excellent book of Bollobás, recently appeared on extremal problems on graphs. Two survey papers of Simonovits are in the press and Brown, Simonovits and I have several papers, some appeared, some in the press and some in preparation on this subject.

Much fewer papers have appeared on extremal problems on hypergraphs, not because there are no interesting and challenging problems but because none of us could make any significant progress.

In this survey, I will hardly give any new results but will try to emphasize the new difficulties which arise in the study of hypergraphs. In the first chapter, I discuss classical extremal problems of Turán type. In the second chapter, I deal with density problems (generalizations and extensions of the theorem of Stone, Simonovits and myself). In the third chapter, I deal with Ramsey's theorem, here an excellent book of Graham, Rothschild and Spencer recently appeared and also a shorter book of Graham.

In the fourth chapter, I discuss Ramsey–Turán type problems and results. This subject was initiated a few years ago by V.T. Sós and myself and in the last chapter I discuss problems dealing with chromatic numbers of graphs and hypergraphs.

Not to make the paper too long, I do not discuss generalized Ramsey problems. Burr has two excellent and comprehensive papers on this subject and I also omit digraphs and multigraphs. These are discussed in our paper with Brown and Simonovits. I apologize in advance if I omitted to mention a person or result which should have been mentioned. Limitations of time and memory are, I hope, adequate excuses. (I had to finish the paper in a hurry to meet a deadline.)

Many of the references which I do not give here can be found in B. Bollobás', Extremal graph theory, Academic Press, 1978.

1. Extremal Problems of Turán Type

Let $G^{(r)}(h,\ell)$ be an r-uniform hypergraph of h vertices and ℓ edges (hyperedges i.e., r–tuples). $F_n(G^{(r)}(h,\ell))$ is the smallest integer for which every $G^{(r)}(n; F_n(G^{(r)}(h,\ell)))$ contains $G^{(r)}(h,\ell)$ as a subgraph. Turán in his classical paper determined $F(K^{(2)}(m))$ fo every $m \geq 3$ ($K^{(r)}(m)$ denotes the complete r graph of m vertices) and raised the famous problems on $F_n(K^{(r)}(m))$, $r > 2, m > r$. None of these values are known. It is easy to see that the value of the limit

$$(1) \qquad \lim_{n\to\infty} F_n(K^{(r)}(m))\binom{n}{r}^{-1} = c(m,r)$$

exists but its values are not known for any $r > 2$, $m > r$. Turán conjectured

$$(2) \qquad c(4,3) = \frac{5}{9}, \quad c(5,3) = \frac{3}{4}$$

and in general, he made plausible conjectures for the value of $c(m,r)$.

In memory of Turán I offered (and offer) \$ 1000 for settling these problems.

As far as I know, the only extremal problem on hypergraphs which was completely solved is due to Katona: Let $r = 3$, $|S| = 3n$. Let there be given a system $A_1, \ldots, A_m$ of triples in S. Assume that no A_i contains the symmetric difference of two $A's$. Is it then true that $\max m = n^3$? Bollobás proved Katona's conjecture. Frankl and Füredi strengthened Bollobás's result for $n > 1000$, by showing that if $m > n^3$ then there are three triples isomorphic to $\{\{1,2,3\},\{1,2,4\},\{3,4,5\}\}$. The analogous problems for $r > 3$ are open. In view of the theorem of Stone–Simonovits and myself it follows that if $G^{(2)}(k,\ell)$ is t–chromatic then

$$(3) \qquad F_n(G^{(2)}(k,\ell)) = (1 + o(1))(1 - \frac{1}{t-1})\frac{n^2}{2}.$$

Thus $F_n(G^{(2)}(k,\ell))$ is asymptotically determined for $t > 2$. For $T = 2$ (3) only gives $F_n(G^{(2)}(k,\ell)) = o(n^2)$, and indeed most of the asymptotic problems are still open for bipartite graphs. Here I only state a few conjectures of Simonovits and myself. Let $1 < \alpha < 2$ be any rational number. Then there always is a bipartite graph $G^{(2)} = G$ for which

$$(4) \qquad \lim_{n\to\infty} F_n(G)/n^{\alpha} = c_G, \quad 0 < c_G < \infty$$

and conversely, the exponent $\alpha = \alpha(G)$ must always be rational. We consider this conjecture an interesting and challenging problem and I offer a reward of \$ 250 for settling this problem. We have no guess about the possible values of c_α — no doubt c_α must be algebraic.

Let $G_1^{(2)}, \ldots$ be a finite or infinite sequence of graphs, $F_n((G_1, \ldots)$ be the smallest integer for which every $G(n, F_n(G_1, \ldots))$ contains at least one of the

$G's$ as a subgraph. As long as the number of $G's$ is finite, we suspect that (4) remains true with a rational α. In fact perhaps

$$F_n(G_1,\ldots,G_k) = \min_{1\le i\le k} F_n(G_i). \tag{5}$$

Though a forthcoming paper of Faudree and Simonovits throws considerable doubt on (5).

Simonovits just observed that (4) completely fails if the number of $G's$ is infinite. To see this let $n_1 < n_2 < \ldots$ tend to infinity sufficiently fast. The graphs $G_1,\ldots$ are all the graphs of n_i $(i = 1,2,\ldots)$ vertices and cn_i edges. Clearly, $F_{n_i}(G_1,\ldots) \le cn_i$. On the other hand, it easily follows from our work with A. Rényi (on the evolution of random graphs) that if $c > c_0(\epsilon)$ and the n_i tend to infinity sufficiently fast then

$$F_{n_{i+1}-1}(G_1,\ldots) > n_{i+1}^{2-\epsilon}.$$

Perhaps (4) can be "saved" if we insist that for every n there is only one (or a bounded number) of forbidden graphs of n vertices.

For hypergraphs nothing like (4) holds. Let, in fact, $G_1^{(3)}$ consist of two triangles having an edge in common and $G_2^{(3)}$ has three triangles (x_1,x_2,x_3), (x_3,x_4,x_5), (x_2,x_4,x_6). Ruzsa and Szemerédi proved that for every $\epsilon > 0$

$$n^{2-\epsilon} < F_n(G_1^{(3)}, G_2^{(3)}) = o(n^2). \tag{6}$$

It seems certain that one can disprove (4) for $r > 2$ for a single hypergraph $G^{(r)}$, though as far as I know this has not yet been done.

This has been recently proved by Füredi.

In fact, Ruzsa and Szemerédi settled a conjecture of Brown, V.T. Sós and myself, according to

$$F_n(G^{(3)}(6,3)) = o(n^2)$$

The more general conjecture

$$F_n(G^{(3)}(k,k-3)) = o(n^2)$$

is still open for $k > 3$.

To end this chapter, I state a few of our more striking conjectures. Is it true that if $G^{(2)}$ is such that it has no subgraph of minimum valency > 2 then

$$F_n(G^{(2)}) < cn^{3/2}? \tag{7}$$

and perhaps if $G^{(2)}$ has a subgraph each vertex of which has valency > 2 then $F_n(G^{(2)})/n^{3/2} \to \infty$ in fact perhaps $F_n(G^{(2)}) > n^{3/2+\epsilon}$. Is it true that

$$c_2 n^{2-1/r} < F_n(K(r,r)) < c_1 n^{2-1/r}? \tag{8}$$

The upper bound is a well known result of Kövári, V.T. Sós and P. Turán. The lower bound is known only for $r = 2$ and $r = 3$.

Is it true that ($C^{(3)}$ is the graph of the three dimensional cube) $F_n(C^{(3)}) > cn^{8/5}$? And in fact is it true that for some $0 < c < \infty$

$$F_n(C^{(3)}) = (c + o(1))n^{8/5}?$$

$F_n(C^{(3)}) \leq cn^{8/5}$ is a well known result of Simonovits and myself.

I proved (1965) that for every r and t there is an $\epsilon = \epsilon(r,t)$ for which

$$F_n(K^{(r)}(t,\ldots,t)) < n^{r-\epsilon(r,t)} \tag{9}$$

(8) is a generalization of (7). The exact value of the exponent is not known for $r \geq 3$.

Finally, I proved the following extension of (3) for $r > 2$. Denote by $K_\ell^{(r)}(t)$ the r–graph whose vertices are $x_i^{(j)}$, $1 \leq i \leq t$, $1 \leq j \leq \ell$ and whose $t^r\binom{\ell}{r}$ edges are $\{x_{i_1}^{(j_1)}, \ldots, x_{i_r}^{(j_r)}\}$, $1 \leq j_1 < \ldots < j_r \leq \ell$, $1 \leq i_1 \leq t, \ldots,$ $1 \leq i_r \leq t$. Then for every r, ℓ and t

$$\lim_{n\to\infty} F_n(K_\ell^{(r)}(t))\binom{n}{r}^{-1} = c(r,\ell) \tag{10}$$

Just as in (3), the value of the limit does not depend on t.

2. Density Problems

Let $n_1 < n_2 < \ldots$ be a sequence of integers and $G^{(r)}(n_i)$ a sequence of r–uniform hypergraphs. We say that the edge density of this sequence is α, if α is the largest real number for which there is a sequence $m_i \to \infty$, $m_i \leq n_i$ so that for infinitely many indices i, $G^{(r)}(n_i)$ has a subgraph $G^{(r)}(m_i; (\alpha + o(1))\binom{m_i}{r})$ i.e., it has a subgraph of m_i vertices and $(\alpha + o(1))\binom{m_i}{r}$ edges. If there is no danger of misunderstanding, one can speak of the density α if α is the largest real number for which $G^{(r)}(n)$ has a large subgraph $G^{(r)}(m)$ with $(\alpha + o(1))\binom{m}{r}$ edges.

(3) can be stated in this language in the following elegant form: The only possible values of the density of $G^{(2)}(n)$ are 1 or $(1 - \frac{1}{r})$, $r = 1, 2, \ldots$.

For $r \geq 3$ very much less is known. (9) can be restated as follows: No r–graph can have a density α for $0 < \alpha < \frac{r!}{r^r}$, i.e., if the density is positive, then it is at least $\frac{r!}{r^r}$. The weakest form of my "jumping constant conjecture" states: There is an absolute constant c_r such that if $G^{(r)}(n)$ has more than $(1+\epsilon)n^r/r^r$ edges, then its density is $\geq \frac{n!}{r^r} + c_r$. In other words, the density $G^{(r)}$ can never be α where $\frac{n!}{r^r} < \alpha < \frac{n!}{r^r} + c_r$. This modest looking conjecture seems to present great difficulties and it is not even known for $r = 3$. I offer \$ 500 for a proof or disproof. More generally I conjectured that the density can take only denumerably many possible values and if these values are ordered by size then they form a well ordered set. This was one of my favorite conjectures and I offered \$ 1000 for a proof or disproof. P. Frankl and V. Rödl (1984) disproved this conjecture and showed that for $n \geq 3$ the set of these values is not well–ordered. For more general conjectures for multigraphs and digraphs, see our papers with Brown and Simonovits and the paper of Brown and Simonovits.

Several related questions can be asked. Suppose that our $G^{(r)}(n,\ell)$ are not considered to be subgraphs of $K^{(r)}(n)$ but of some suitable subgraph of it. For example, if $G^{(r)}(n,\ell)$ is a subgraph of $K^{(r)}([\frac{n}{r}])$ (the complete r–partite r–graph where each class has $\frac{n}{r}$ vertices), then it easily follows from (9) that the density is 0 or 1. If the graphs are subgraphs of $K_\ell^{(r)}(n)$ then the density can take on only finitely many different values — we leave the details to the reader. On the other hand, let us try to consider subgraphs of $G^{(3)}(3n, n\binom{2n}{2})$ defined as follows: The vertices are $x_1,\dots,x_n; y_1,\dots,y_{2n}$ and the $n\binom{2n}{2}$ triples are $\{x_i, y_j, y_k\}$. Now it is easy to see that every subgraph of positive density contains a subgraph of density 1/2, i.e., it contains a complete tripartite 3–graph $K^{(3)}(t,t,t)$, and it is not hard to see that the only possible densities are in fact $1-\frac{1}{k}$, $k = 1,2,\dots$. For $r = 3$ and even more for $r > 3$, many similar questions can be asked. I leave the formulation to the reader. In our triple papers with Brown and Simonovits, we restrict ourselves to $r = 2$ but as stated before we worked with digraphs and multigraphs. If we restrict ourselves to $r = 2$ and to ordinary graphs, it seems harder to find non–trivial questions. F. Chung and W. Trotter considered the following problem: Let the vertices of $G(n,t)$ be the integers $1,2,\dots,n$, and join two vertices i and j if $|i-j| \leq t$. Here t is large (and fixed) and $n \to \infty$. Their problem is to determine the smallest c such that every subgraph of cnt edges of $G(n,t)$ contains a triangle. V.T. Sós suggested that if $(c+\epsilon)nt$ edges are given then this subgraph perhaps contains a large complete tripartite subgraph. This would be an Erdös–Stone–Simonovits type result. Chung and Trotter proved good inequalities for c but its exact value has not yet been determined as far as I know the question of V.T. Sós has not yet been seriously attacked. Clearly many related questions can be asked here.

It might be worthwhile to investigate from this point of view the subgraphs of the n–dimensional cube $C^{(n)}$. I hope to return to this question at another occasion.

Bollobás, Simonovits, Szemerédi and I proved (among others) the following theorem: $C^{(k)}(t)$ is a cycle of length k where each vertex is multiplied by t, i.e., the vertices are $x_i^{(j)}$, $1 \leq i \leq t$; $1 \leq j \leq k$. Two $x_i^{(j)}$'s are joined if and only if their upper indices differ by one. We proved that to every $c > 0$, there is an $\ell(c)$ such that if $G(n)$ contains no $C^{(2r+1)}(t)$ for $1 \leq r \leq \ell(c)$ then for $n > n_0(c,t)$ $G(n)$ can be made bipartite by the omission of at most cn^2 edges. We conjectured that $\ell(c) = o(c^{-1/2})$ which if true is best possible. We further conjectured: for every $c > 0$ there exists an $\ell = \ell(c)$ so that if $G(n)$ cannot be made 3–chromatic by the omission of cn^2 edges $G(n)$ contains a four–chromatic subgraph $H(r)$ of $r \leq \ell$ vertices. Unfortunately, this attractive conjecture and its obvious generalizations to higher chromatic numbers are still open. This conjecture was very recently proved by Duke and Rödl.

Probably these results can (and should) be generalized to hypergraphs. The first step is to prove or perhaps disprove the following conjecture: For every $c > 0$ there exists an $\ell = \ell(c)$ such that if $G^{(3)}(n)$ cannot be made tripartite (i.e., the vertices are divided into three disjoint classes A, B, C, so that all the triples meet A, B, and C) by omitting cn^3 edges (i.e., triples) then there is a non-tripartite $G^{(3)}(r)$ $r \leq \ell$ so that our $G^{(3)}(n)$ contains a $G^{(3),(t)}(r)$ for $n > n_0(\ell, t, c)$. $G^{(3),(t)}(r)$ is of course a $G^{(3)}(r)$ each vertex of which is multiplied by t. Clearly (unless all our conjectures are wrong here), many interesting and perhaps deep results can be found here.

Finally, I state the following theorem, which is an extension of my theorem with Stone, for hypergraphs: Every $G^{(r)}(n; (c(m,r) + \epsilon)\binom{n}{r})$ contains a $K_m^{(r)}(t)$, i.e., a set of tm vertices $x_i^{(j)}$, $1 \leq j \leq m$; $1 \leq i \leq t$ and $\binom{m}{r}t^r$ edges $\{x_{i_1}^{(j_1)}, \ldots, x_{i_r}^{(j_r)}\}$, $1 \leq j_1 < \ldots < j_r \leq m$; $1 \leq i_1 \leq t, \ldots, 1 \leq i_r \leq t$. Probably many extensions of this result can be proved but as far as I know this has not yet been done.

For ordinary graphs, Bollobás, Simonovits and I published several sharpenings of my theorem with A. Stone. The strongest result of this type is at present a theorem of Chvátal and Szemerédi: Every $G^{(2)}(\frac{n^2}{2}(1 - \frac{1}{d}) + cn^2)$ contains a complete $(d+1)$–partite graph with t vertices in each part where $t > \frac{\log n}{500 \log(1/c)}$. Bollobás and I showed that $\frac{1}{500}$ cannot be replaced by 5.

3. Ramsey's Theorem

In this chapter we discuss Ramsey type problems. The well known arrow symbol of Rado

$$n \to (G_1^{(r)}, \ldots, G_\ell^{(r)})_\ell^r \tag{11}$$

means that if we split the r–tuples of an n–element set F into ℓ classes, then for some i, $1 \leq i \leq \ell$ the r–tuples of the i–th class contain $G_i^{(r)}$ as a subgraph. As stated in the introduction, we will not study (11) in its most general form. Most of the time we restrict ourselves to the cases when $G_i^{(r)}$ is a complete graph $K_i^{(r)}$. Denote the smallest integer n for which (11) holds for $G_i^{(r)} = K_{k_i}^{(r)}$ $1 \leq i \leq \ell$ by $F_r(k_1, \ldots, k_\ell)$. We know of course most if $r = \ell = 2$. It is well known that

$$c_1 k 2^{k/2} < F_2(k,k) < \binom{2k-2}{k-1}. \tag{12}$$

$F_2(k,k) < \binom{2k}{k}/k^\alpha$ has recently been proved by Thomasson.

I offer \$ 100 for a proof that $\lim_{k\to\infty} F_2(k,k)^{1/k}$ exists and \$ 250 for its value. This value if it exists is between $2^{1/2}$ and 4 and any improvement of these bounds would be of great interest and will receive an "appropriate" financial reward. ("Appropriate" I am afraid is not the right word, I do not have enough money to give a really appropriate award.) Further it is well known that

$$(13) \qquad c_1k^2/(\log k)^2 < F_2(3,k) < \frac{c_2k^2}{\log k}$$

The upper bound in (13) is due to Graver and Yackel with the extra $\log\log k$ in the denominator. Ajtai, Komlós and Szemerédi got rid of the factor $\log\log k$ by a new method which was a great breakthrough and was already used successfully in many other problems. The lower bound in (12) and both bounds in (13) use the so–called probability method.

Just a few words about the results of Ajtai, Komlós and Szemerédi. They first of all proved the following theorem: If $G(n)$ is a graph of n vertices and kn edges which has no triangle then it contains at least $\frac{cn\log k}{k}$ independent vertices (cn/k is easy and their essential gain is the extra factor $\log k$). Using this they obtained the improved upper bound in (13) and many other interesting results. Komlós, Pintz and Szemerédi applied the same method to a problem of Heilbronn.

It would be very desirable to get an asymptotic formula for $F(3,k)$ (an exact formula might "not exist" in the same sense as there is no exact (and useful) formula for the n–th prime). Also

$$(14) \qquad F_2(4,k) > c_1k^3(\log k)^{-c_2}$$

should be proved. I offer for both of these problems \$ 250. The current best result $F_2(4,k) > ck^{5/2}$ is due to Joel Spencer.

For $r > 2$, the situation is much less satisfactory. Hajnal, Rado and I proved

$$(15) \qquad 2^{c_1k^2} < F_3(k,k) < 2^{2^{c_2k}}$$

and

$$(16) \qquad \exp_{r-2} c''k < F_r(k,k) < \exp_{r-1} c'k$$

where $\exp_\ell k$ denotes the ℓ–fold iterated exponential. We are sure that the estimation on the right side is the correct one. Hajnal in fact proved

$$(17) \qquad \exp_{r-1} c_1k < F_r(k,k,k,k) < \exp_{r-1} c_2k.$$

On the other hand, J. Beck's surprising results on Ramsey games give the following surprising and beautiful result: Let $|S| = n$ and two players alternatively choose r–tuples from S. If the first player wants to ensure the existence of a large subset X of S so that all the r–tuples of X are chosen by him, the second player cannot prevent him from doing so for $|X| < c(\log n)^{1/(r-1)}$ (and the first player also cannot prevent the second player from getting all the r–tuples of a set of essentially the same size). It is not clear if the exponent $1/(r-1)$ is best possible.

In view of the not quite satisfactory situation of our knowledge of the growth of $F_r(k,k)$ Hajnal and I in a very little known paper started the following investigations: For given n, k, u, v, and r the relation

$$n \to \left(k, \begin{bmatrix} u \\ v \end{bmatrix}\right)^r \tag{18}$$

denotes the truth of the following statement: Split the r–tuples of a set of n elements into two classes I and II. Then either there are k elements all whose r–tuples are in class I or there is a set of u elements which contains at least v elements of class II. We studied (18) in our triple paper with Rado if n is an infinite cardinal, but it seemed to us that interesting and deep questions can be asked in the finite case too. Denote by $h_r(n,u,v)$ the largest value of k for which (18) holds. $h_r(n,u,\binom{u}{r})$ is of course our old Ramsey function. It is the largest integer k for which $F_r(k,u) \le n$.

Our main conjecture is that as v increases from 1 to $\binom{u}{r}$, $h_r(n,u,v)$ grows first like a power of n, then at a well–defined value $L_1^{(r)}(u)$ of v grows like a power of $\log n$, i.e., $h_r(n,u,L_1^{(r)}(u)-1) > n^{c_1}$ but $h_r(n,u,L_1^{(r)}(u)) < (\log n)^{c_2}$. Then as v increases further $h_r(n,u,L_2^{(r)}(u))$ suddenly increases only like a power of $\log\log n$, and finally there is an $L_{r-2}^{(r)}(u)$ for which $h_r(n,u,L_{r-2}^{(r)}(u))$ grows like a power of $\log_{r-2} n$ ($\log_t n$ denotes the t times iterated logarithm).

For $r=2$ no great mysteries remain; only the exact value of the exponent of n of $h_2(n,u,v)$ are in doubt. For $r \ge 3$ (and especially for $r \ge 4$), the situation is much less satisfactory. We proved that for every $r \ge 3$

$$c_r' \log n/\log\log n < h_r(n,r+1,3) < c_r \log n \tag{19}$$

We have a good guess about the value of $L_1^{(3)}(u)$ and more generally about $L_1^{(r)}(u)$, we know nothing about $L_2^{(r)}(u)$ $(r>3)$ and cannot even prove its existence. Put $g_1^{(3)}(u) = x+y+z+xyz$ where $x+y+z=u$ and x,y,z are as nearly equal as possible. We proved

$$h_3(n,u,g_1^{(3)}(u)) > n^{\epsilon_u} \tag{20}$$

and conjecture

$$h_3(n,u,g_1^{(3)}(u)+1) < c\log n \tag{21}$$

where perhaps in (21) $\log n$ has to be replaced by $(\log n)^c$. In any case (20) and the conjecture (21) is equivalent to $L_1^{(3)}(u) = g_1^{(3)}(u)+1$.

Let us color the edges of the complete graph whose vertices are the integers $1,2,\ldots,u$ by three colors I, II, III. We wish to maximize the number of triangles (a,b,c), $a<b<c$ where the edge (a,b) has color I, the edge (b,c) has color II and the edge (a,c) has color III. Denote by $F_1^{(3)}(u)$ the value of this

maximum. It is easy to see that $F_1^{(3)}(u) \geq g_1^{(3)}(u)$ and we, in fact, conjecture

$$F_1^{(3)}(u) = g_1^{(3)}(u). \tag{22}$$

Unfortunately, we have no real evidence for our conjecture (22), except that it is easy to verify it for small values of u. The importance of our conjecture lies in the fact that we proved

$$L_1^{(3)}(u) = F_1^{(3)}(u) + 1, \tag{23}$$

thus if (22) holds then $L_1^{(3)}(u) = g_1^{(3)}(u) + 1$ is proved.

In view of the fact that our paper with Hajnal was completely forgotten (it was almost forgotten by the authors too) we state two more problems from this paper. Color the edges of the complete graph $K^{(2)}(u)$ whose vertices are the integers $\leq u$ by two colors so that the number of triangles (a, b, c), $a < b < c$ (a, b) and (b, c) are colored I and (a, c) is colored II is maximum. Perhaps this maximum is $F_1^{(3)}(u)$ (trivially it is $\geq F_1^{(1)}(u)$).

An older problem of V.T. Sós and myself states: color the edges of $K(n)$ by three colors so that the number of triangles all whose edges get a different color is maximal. Denote this maximum by $F_3(n)$. $F_3(1) = F_3(2) = 0$, $F_3(3) = 1$, $F_3(4) = 4$. We conjectured that

$$\begin{aligned} F_3(n) = &F_3(u_1) + F_3(u_2) + F_3(u_3) + F_3(u_4) \\ &+ u_1u_2u_3 + u_1u_2u_4 + u_1u_3u_4 + u_2u_3u_4 \end{aligned} \tag{24}$$

where $u_1 + u_2 + u_3 + u_4 = n$ and the u's are as nearly equal as possible. We made no progress with this problem. Clearly many generalizations are possible. Rosenfeld and I posed the following related question: For which r is it possible to color the r–tuples of a set S, $|S| = 2r$ by $r+1$ colors so that the r–tuples of each $(r+1)$–tuple of S have all different colors? It is easy to see that this is no longer possible for $|S| > 2r$. As far as I know it is not known if this is possible for any $r > 2$. For $r = 2$ the answer is positive, for $r = 3$ negative.

Let us now consider the cases $r > 3$. We conjectured

$$L_i^{(r)}(r+1) = i + 2. \tag{25}$$

We can prove (25) only for $i = 1$. We can prove

$$h_r(n, r+1, i+2) > c_{r,i}(\log_i n)^{\epsilon_{r,i}} \tag{26}$$

but we do not have a good upper bound of $h_r(n, r+1, i+2)$.

Now we investigate $L_1^{(r)}(k)$. Define $g_1^{(r)}(k)$ as follows: $g_1^{(r)}(k) = 0$ for $k < r$, $g_1^{(r)}(r) = 1$. Assume that $g_1^{(r)}(t)$ has already been defined for all $t < k$. Put

$$g_1^{(r)}(k) = \sum_{i=1}^{r} g_1^{(r)}(u_i) + \prod_{i=1}^{r} u_i$$

where $\sum_{i=1}^{k} u_i = k$ and the u's as nearly equal as possible. We proved

$$h_r(n,k,g_1^{(r)}(k)) > n^{\epsilon_{k,r}}. \tag{27}$$

The proof of (27) is similar to that of (20). We conjectured: $L_1^{(r)}(k) = g_1^{(r)}(k)+1$. In other words, we conjecture

$$h_r(n,k,g_1^{(r)}(k)+1) < c_1(\log n)^{c_2} \tag{28}$$

Finally we conjectured that for every $\epsilon > 0$, there is a $k_0 = k_0(\epsilon)$ such that for every $k > k_0$

$$n \not\to ((\log n)^{\epsilon}, k)^3.$$

At the end of the paper, we say: If we live we hope to investigate these questions, but hope the others will do it before us. Of all these hopes only the first (least important) was fulfilled — we live. I offer \$ 500 for a proof or disproof of these conjectures.

In another somewhat later paper (which also was forgotten and ignored by everybody), I investigate related but nevertheless significantly different problems. Denote by $F_k^{(r)}(n,\alpha)$ the smallest integer for which it is possible to split the r–tuples of a set $|S| = n$ into k classes so that for every $S_1 \subset S$, $|S_1| \geq F_k^{(r)}(n,\alpha)$ every class contains more than $\alpha\binom{|S_1|}{r}$ r–tuples of S_1. The probability method easily gives that for every $0 \leq \alpha \leq \frac{1}{k}$

$$c'_k(\alpha)\log n < F_k^{(2)}(n,\alpha) < c''_k(\alpha)\log n. \tag{29}$$

$c'_k(\alpha) \to \infty$ as $\alpha \to 1/k$. Thus again no great mysteries remain for $r = 2$ though it would be nice to sharpen (29) and prove that

$$F_k^{(2)}(n,\alpha) = (c_k + o(1))\log n \tag{30}$$

The case $r > 3$ is much more interesting and mysterious. For simplicity, let us restrict ourselves to the case $k = 2$.

It is well–known that for α sufficiently close to $1/2$

$$c'(\alpha)(\log n)^{1/(r-1)} < F_2^{(r)}(n,\alpha) < c''(\alpha)(\log n)^{1/(r-1)} \tag{31}$$

The upper bound is a result of Spencer and myself and the lower bound follows easily from an old result of mine. For $\alpha = 0$, we get the "old" Ramsey–function. Our conjecture with Rado and Hajnal (see Erdös, Hajnal, Rado 1965 and Erdös, Hajnal 1972) implies

$$c_2 \log_{r-1} n < F_2^{(r)}(n,0) < c_1 \log_{r-1}(n) \tag{32}$$

Now by (31) and (32) it follows (we assume our conjecture to be true) that as α increases from 0 to 1/2, $F_2^{(r)}(n,\alpha)$ increases from $\log_{r-1} n$ to $(\log n)^{(1/(r-1)}$. Does the change occur continuously or are there jumps? Is there only one jump? I do not know and feel that this question also deserves more careful investigation that it has received so far. If I can hazard a guess completely unsopported by evidence, I am affraid that the jump occurs all in one step at 0. It would be much more interesting if my conjecture would be wrong and perhaps there is some hope for this for $r > 3$. I know nothing and offer \$ 500 to anybody who can clear up this mystery.

Finally, I want to state one more old result and an old problem of mine: Let $|S| = n$ and divide the triples of S into two classes. Then there are always sets A, B, $(|A| = |B| = c(\log n)^{1/2})$ such that all the triples (x, y, z), $x \in A$, $y \in A$, $z \in B$ are in the same class. First of all observe that this is a genuine Ramsey type result, if we only assume that one of the classes has $(1 + o(1))\binom{n}{3}$ triples, this clearly does not imply that this class contains such a system. It seems to me to be an important and difficult question to decide if this theorem can be strengthened to imply that all the triples (x, y, z) which meet both sets A and B all belong to the same class. At present, I cannot decide this question.

Now let $r > 3$. Split the r–tuples of a set $|S| = n$ into two classes. Our proof gives that there always are $r-1$ sets $A_1, \ldots, A_{r-1}$, $|A_i| = c(\log n)^{1/(r-1)}$ so that all the r–tuples $\{x_1, \ldots, x_r\}$, $x_i \in A_i$, $1 \le i \le r-2$, $x_{r-1} \in A_{r-1}$, $x_r \in A_{r-1}$ are in the same class. $c(\log n)^{1/(r-1)}$ is best possible apart from the value of c.

Several generalizations and extensions seem possible, but I had no success so far.

The difference between graphs and hypergraphs is nicely illustrated by the following recently published result of Hajnal and myself. The probability method easily gives that one can color the edges of a $K(r)$, $r = 2^{kf(k)^{-1}}$, $f(k)$ tends to infinity as slowly as we please by two colors so that every $K(k)$ (complete subgraph of (k)) vertices contains $(1/2 + \sigma(1))\binom{k}{2}$ edges of both colors. In other words uniformity of the edge coloring can persist until nearly the Ramsey bound. We proved that if we color the triples of a $K^3(2^{k^2})$ by two colors there always is a set of size k on which the coloring is not uniform i.e. one of the colors gets more than $(\frac{1}{2} + \epsilon)$ of the triples, we believe that the Ramsey bound nevertheless is at $2^{2^{ck}}$! Many problems remain but we have to refer to our paper.

4. Ramsey-Turán Type Problems

Now I turn to Ramsey–Turán type questions. These investigations were started by V.T. Sós and myself. The classical theorem of Turán determines the largest graph on n vertices which contains no $K^{(2)}(r)$. Observe that in this graph, there is a very large independent set. Our problem was: What happens if we restrict the size of the largest independent set? Does this decrease the number of edges of the Turán graph? Here I do not want to state and discuss our most general problem but just want to illustrate clearly the diference between $r = 2$ and $r > 2$. Denote by $F_n(G^{(r)}, \ell)$ the largest integer for which there is an r–uniform hypergraph on n vertices and $F_n(G^{(r)}, \ell)$ edges which does not contain $G^{(r)}$ as a subgraph and the largest independent set of which has size $\leq \ell$. $F_n(G^{(r)})$ is the largest integer for which there is a hypergraph on n vertices and $F_n(G^{(r)})$ edges which does not contain $G^{(r)}$ as a subgraph. The determination of $F_n(G^{(r)})$ is of course the classical problem of Turán. Clearly (by Ramsey's theorem) ℓ cannot be too small (for otherwise no such graphs exist) and usually we just assume $\ell = o(n)$ and we investigate $F_n(G^{(r)}, o(n))$. In our first paper, we proved that for odd t

$$F_n(K^{(2)}(t), o(n)) = (1 + o(1))\frac{n^2}{4}\frac{t-3}{t-1} \tag{33}$$

By the way, for $K^{(2)}(3)$, we trivially have $F_n(K^2(3), \ell) \leq \frac{n\ell}{2}$. For even t the problem is much harder. Bollobás, Szemerédi and I proved

$$F_n(K^{(2)}(4), o(n)) = (1 + o(1))\frac{n^2}{8}.$$

and in paper Erdös et al. (1983) Hajnal, V.T. Sós, Szemerédi and I proved that for even t

$$F_n(K^{(2)}(t), o(n)) = (1 + o(1))\binom{n}{2}\frac{3t-10}{2(3t-4)}. \tag{34}$$

Turán's theorem asserts $F_n(K^{(2)}(t)) = (1 + o(1))\binom{n}{2}\frac{t-2}{2(t-1)}$.

Thus, (33) and (34) show that the condition $\ell = o(n)$ significantly changes the constant in Turán's theorem. We did not investigate the case when $G^{(2)}(t)$ is bipartite. Perhaps here

$$F_n(G^{(2)}(t)) = (1 + o(1))F_n(G^{(2)}(t), o(n))$$

always holds. In fact perhaps $F_n(G^{(2)}(t)) = F_n(G^{(2)}(t), o(n))$.

The contrast in (34) with hypergraphs is sharp and striking. V.T. Sós and I prove first of all that for every $t > r \geq 3$

$$F_n(K^{(r)}(t), o(n)) = (1 + o(1))F_n(K^{(r)}(t)) = (1 + o(1))c_{r,t}\binom{n}{r} \tag{35}$$

In other words, the condition $\ell = o(n)$ has a very much smaller effect for hypergraphs. More generally, we prove the following theorem: Let $G^{(r)}$ be an r-graph which is not r-partite, i.e., we cannot divide the vertex set S into r disjoint sets S_i, $1 \leq i \leq r$ so that every edge of $G^{(r)}$ intersects every S_i $1 \leq i \leq r$ in (exactly) one vertex. This implies $F_n(G^{(r)}) = (c + o(1))\binom{n}{r}$. Assume further that for every edge h_1 of $G^{(r)}$ there is another edge h_2 for which $(h_1 \cap h_2) \geq 2$. Then

$$F_n(G^{(r)}, o(n)) = F_n(G^{(r)})(1 + o(1)) = (1 + o(1))c\binom{n}{r}.$$

In other words, the condition $\ell = o(n)$ has very little effect in this case. On the other hand, we prove that if $G^{(r)}$ is such that its vertex set can be decomposed into r disjoint sets $S_1, \ldots, S_r$ so that the edges of $G^{(r)}$ can be decomposed into two classes E_1 and E_2 so that every edge of E_1 meets all the S_i (in exactly one point of course). The edges of E_2 are all contained in S_1 and can be written in a sequence $h_1, h_2, \ldots, h_s$ so that for every k

$$\left| h_k \cap \bigcup_{1 \le i < k} h_i \right| \le 1.$$

Then

$$F_n(G^{(r)}, o(n)) = o(n^r).$$

We do not at present know if these theorems are best possible and in particular we do not know if there is any $G^{(r)}$ for which

$$F_n(G^{(r)}, o(n)) = (c_1 + o(1))\binom{n}{r}, \; F_n(G^{(r)}) = (c_2 + o(1))\binom{n}{r} \text{ and } 0 < c_1 < c_2$$

This is one of the outstanding problems of this subject. Recently, V. Rödl and P. Frankl found an example of such r–graphs. Another difference between $r = 2$ and $r > 2$ is as follows: It is easy to see that if t and $\eta > 0$ are fixed then there exists $\epsilon > 0$ such that every G with n vertices for which every set of m vertices $m > \epsilon n$ spans a subgraph having more than $\eta\binom{m}{2}$ edges, must contain a $K(t)$, i.e., there is no Ramsey–Turán phenomena if we insist that every large spanned subgraph should contain many edges. On the other hand we easily show that for every $\eta > 0$ there is an $\epsilon > 0$ and a $G^{(3)}(n) = T_n^{(3)}$ for which every induced subgraph of more than ηn vertices contains more than $\epsilon\binom{\eta n}{3}$ edges and which contains no $K^{(3)}(4)$ and contains not even a $G^{(3)}(4;3)$. This $T_n^{(3)}$ can be defined as follows: The vertices are the integers $1 \le t \le n$, the edge (x, y, z) is in $T_n^{(3)}$ if the first digits written in ternary system where x, y and z differ are $0, 1$, and 2 respectively. It is easy to see that the edge density of every induced subgraph of $m > \eta n$ vertices of our $T_n^{(3)}$ is positive but not uniformly positive, i.e., as $\eta \to 0$, the number of edges of a vertex set of ηn vertices can be less than $\epsilon\binom{\eta n}{3}$ if $\eta = \eta(\epsilon)$ is small enough.

Is it true that every such graph $G^{(3)}(n, cn^3)$ of positive (but not necessarily uniformly positive) edge density contains every fixed subgraph of our $T_n^{(3)}$?

There is a further problem here which seemed interesting to us. Let us assume that the edge density of our $G(n)$ is uniformly positive. In other words for every $m > \eta n$ every induced subgraph of m vertices contains more than $c\binom{m}{3}$ edges. Is it then true that if $n > n_0(c, t, \eta)$ then our $G^{(3)}(n)$ contains a $K^3(t)$? We cannot prove this even for $t = 4$ and what is more we do not know if our $G^{(3)}(n)$ must contain $G^{(3)}(4, 3)$. Recently this question has been disproved by A. Hajnal, V.T. Sós, M. Simonovits and myself and using a different example by V. Rödl. One can construct a triple system of n points so that if $t_n / \log n \to \infty$ then every set of t_n vertices spans $(1/2 + o(1))\binom{t_n}{3}$ triples but the system does not contain 4 points with all its triples, perhaps $1/2$ cannot be replaced by $(1/2+\epsilon)$. Also there is an example where every set of t_n elements satisfying $t_n/\log n \to \infty$ has $(\frac{1}{4}+o(1))\binom{t_n}{3}$ triples but there is no set of 4 vertices which contains 3 triples. $1/4$ can perhaps not be replaced by $1/4+\epsilon$.

We couldn't even prove that a graph $G^{(3)}(n)$ with uniformly positive edge density must necessarily contain a $G^{(3)}(7,11)$ of vertices $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ and edges (x_1,x_2,x_3), (x_4,x_5,x_6) and the 9 triples (x_7,x_i,x_j) $1 \le i \le 3$, $4 \le j \le 6$, i.e., must it contain some *fixed* $G^{(3)}$ which is a complete tripartite graph with some additional edges in two vertex sets S_1 and S_2. The positive answer to a general form of this problem was given in Frankl, Rödl (to appear).

Finally to end this long chapter, I state some of our further unsolved problems (for more details, see our papers referenced at the end of this chapter).

Is it true that

$$F_n(K(2,2,2), o(n)) = o(n^2)?$$

Is it true that if $n > n_0(r,c)$ then every $G(n, cn^2)$ for which the largest independent set is $o(n)$ either contains a $K(4)$ or a $K(r,r,r)$?

These two problems seemed fundamental to us. V. Rödl disproved recently the last conjecture and constructed a graph with n vertices, $\frac{n^2}{8}(1+o(1))$ edges, not containing both $K(3,3,3)$ and $K(4)$ and having the largest independent set of size $o(n)$.

One final problem: In view of $F_n(K^{(3)}(4), o(n)) = (c_{3,4}+o(1))\binom{n}{3}$ we tried with V.T. Sós to determine or estimate the largest $h(n)$ for which

$$F_n(K^{(3)}(4), h(n)) = o\left(\binom{n}{3}\right).$$

By Ramsey's theorem such an $h(n)$ exists and we could not get a very good estimation for $h(n)$ but hope to return to this problem.

5. Chromatic Numbers

In this final chapter, I discuss miscellaneous problems. It is well known and easy to see that every graph $G^{(2)}$ of chromatic number k has at least $\binom{k}{2}$ edges, equality only for $K^{(2)}(k)$.

On the other hand for $r > 2$, the situation seems to change completely. It was observed long ago by Hajnal and myself that for $r = 3$, the smallest $G^{(r)}$ of chromatic number 3 is given by the 7 lines (edges) of the Fano plane, whereas $K^{(3)}(5)$ has 10 edges. Perhaps for large chromatic numbers, this difference again disappears and for every r there is a $k_0(r)$ so that for every $k > k_0(r)$ every $G^{(r)}$ of chromatic number k has at least $(k-1)r + \binom{1}{r}$ edges with equality only for $K^{(r)}((k-1)r+1)$. If true the proof will perhaps not be difficult. As far as I know this problem has never been investigated carefully. Very recently the problem has been solved negatively by N. Alon.

Denote by $m_3(r)$ the smallest integer for which there is 3–chromatic $G^{(r)}$ of $m_3(r)$ edges. We have

$$c_1 r^{1/3} 2^r < m_3(r) < c_2 r^2 2^r. \tag{36}$$

The lower bound in (36) is due to Beck and the upper bound to me. An asymptotic formula for $m_3(r)$ seems to be beyond reach at present.

G. Dirac called a k–chromatic graph critical if the omission of any edge decreases its chromatic number. Let $G^{(2)}(n,e)$ be k–chromatic and critical, what is the smallest possible value of $e = e(n,k)$? Perhaps this question will have a simple answer. Thomassen and Dirac conjectured $e(n,4) = \frac{5n}{3} + o(1)$.

Hajnal and I proved that every $G^{(2)}$ of chromatic number $\geq \aleph_1$ contains all finite bipartite graphs as subgraphs (in fact, we show that our $G^{(2)}$ must contain a $K(n,\aleph_1)$ for every integer n. On the other hand, if $G^{(2)}$ is an arbitrary graph which is not bipartite, then for every infinite cardinal number m there is an m chromatic graph of power m which does not contain our $G^{(2)}$ as a subgraph. The situation for $r \geq 3$ is very much more complicated. In a long, exhaustive and carefully written difficult paper, Galvin, Hajnal, and I tried to investigate the finite subgraphs of uncountable hypergraphs. The situation is much more complicated than for $r = 2$. We do not know which finite graphs have the property that every $G^{(3)}$ of chromatic number $\aleph_1$ must contain $H^{(3)}$. We have many special results of this type but no general theorem. In an earlier paper of Hajnal, Rothschild, and myself proved that every graph $G^{(3)}$ of power and chromatic number $\aleph_1$ must contain two edges e_1 and e_2, satisfying $|e_1 \cap e_2| = 2$, but there are graphs of power $> \aleph_1$ which have chromatic number $\aleph_1$ any two edges of which have at most one element in common. This phenomenon of course cannot occur for $r = 2$. To finish this paper, I just state two more unsolved problems from the Galvin–Hajnal–Erdös paper: Let $H_1^{(3)}$ and $H_2^{(3)}$ be two finite graphs. Assume that there exists $G_1^{(3)}$ and $G_2^{(3)}$ of chromatic number $\aleph_1$ not containing $H_1^{(3)}$ and $H_2^{(3)}$ respectively. Is it then true that there is $G^{(3)}$ of chromatic number $\aleph_1$ not containing both $H_1^{(3)}$ and $H_2^{(3)}$?

Is there a finite $H^{(3)}$ so that every G of power $\aleph_2$ and chromatic number $\aleph_1$ must contain $H^{(3)}$ but this is not true for graphs of chromatic number $\aleph_1$ and power $> \aleph_2$? Probably such an $H^{(3)}$ does not exist. Many more deep and interesting problems can be found in our paper which I feel has been unduly neglected.

Final Remarks: I just learned that Hajnal and Komjáth proved that every $G^{(2)}$ of chromatic number $\aleph_1$ contains a half–graph and an extra vertex which is joined to every vertex of infinite valence of the half–graph. A half–graph is defined as follows: The vertices are: $x_1, \ldots, ; y_1, y_2, \ldots$ and x_i is joined to y_j for every $j > i$. On the other hand, assuming $2^{\aleph_0} = \aleph_1$ they constructed a graph of chromatic number $\aleph_1$ which does not contain a half–graph and two extra vertices which are joined to every vertex of infinte valence. This certainly is an astonishingly accurate result.

During our meeting in Prague at a party at the Nešetřils, Mihók and I formulated the following question: Can one characterize the sequences $n_1 < n_2 < \dots$ so that for every k there should exist a graph G of chromatic number $\geq k$ which contains no circuit C_{n_i}, $i = 1, 2, \dots$. If the sequence $n_1 < \dots$ increases sufficiently fast such graphs clearly exist. In particular let $n_i = 2^{i+1}$. We do not know if for sufficiently large k there is a graph of chromatic number k without circuits of size C_{n_i}.

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Note on Canonical Partitions[1]

Richard Rado

To B. L. van der Waerden on his 80. birthday

1. For every set X and every cardinal number r we put

$$[X]^r = \{P \subseteq X : |P| = r\}.$$

Let $A = \{0,1,2,\ldots\}$ and $r \in \{1,2,\ldots\}$. A partition, or coloring, of $[A]^r$ is a function $f : [A]^r \to F$, where F is a set. Let $L \subseteq \{0,1,\ldots,r-1\}$. The partition f is called L–canonical on B if $B \subseteq A$ and, for

$$\{x_0, x_1, \ldots, x_{r-1}\}, \{y_0, \ldots, y_{r-1}\} <\subseteq B,$$

we have $f\{x_0, \ldots, x_{r-1}\} = f\{y_0, \ldots, y_{r-1}\}$ if and only if $x_\lambda = y_\lambda$ for $\lambda \in L$. In Erdős, Rado (1950) the following result was proved:

Theorem 1. *Given any partition $f : [A]^r \to F$, there is an infinite set $B \subseteq A$ and a set $L \subseteq \{0, \ldots, r-1\}$ such that f is L–canonical on B.*

The object of this note is (i) to give a new proof of Theorem 1 which is in some ways simpler than the proof in Erdős, Rado (1950) (ii) to discuss connections between canonicity and some other properties of a partition. If $X \in [A]^r$ we sometimes write

$$X = \{X^0, X^1, \ldots, X^{r-1}\}_<,$$

and similarly for letters other than X.

2. We begin by showing that for every L there exists a L–canonical partition of $[A]^r$.

Theorem 2. *Given any set $L \subseteq \{0, \ldots, r-1\}$, there exists a L–canonical partition of $[A]^r$.*

Proof. We define f by putting, for every $P \in [A]^r$,

$$fP = \{Q \in [A]^r : Q^\lambda = P^\lambda \text{ for } \lambda \in L\}.$$

1 Originally printed in: Bull. London Math. Soc. 18 (1986), 123–126. Reprinted by courtesy of the author and of the London Math. Society. Without a contribution by R. Rado this volume would be incomplete. During the final stage of preparation of this book we learned that R. Rado died. Without him the whole Ramsey theory seems to be incomplete.

We now show that f is L canonical. We shall apply the definition of f repeatedly without referring to this fact.

(i) Let $fP = fQ$. Then $Q \in fQ = fP$; $Q^\lambda = P^\lambda$ for $\lambda \in L$.
(ii) Let $P^\lambda = Q^\lambda$ for $\lambda \in L$, for some $P, Q \in [A]^r$. Consider any set $R \in fP$.

We have, for $\lambda \in L$,

$$R^\lambda = P^\lambda = Q^\lambda; \; R \in fQ.$$

Since R is arbitrary, we have $fP \subseteq fQ$. By symmetry, $fQ \subseteq fP$, and Theorem 2 follows.

3. *Proof of Theorem 1.* For $\{x_0, \ldots, x_{2r-1}\}_< \subset A$ put

$$\begin{aligned} g\{x_0, \ldots, x_{2r-1}\} &= \{(\alpha_0, \ldots, \alpha_{2r-1}) : \alpha_0 < \ldots < \alpha_{r-1} < 2r; \\ &= \alpha_r < \ldots < \alpha_{2r-1} < 2r; \\ f\{x_{\alpha_0}, \ldots, x_{\alpha_{r-1}}\} &= f\{x_{\alpha_r}, \ldots, x_{\alpha_{2r-1}}\}. \end{aligned}$$

The range of the function g is finite. Hence, by Ramsey's theorem (Ramsey 1930), there is an infinite set $B' \subseteq A$ such that g is constant on $[B']^{2r}$. Let $B' = \{b_0, b_1, b_2, \ldots\}_<$ and $B = \{b_0, b_2, b_4, \ldots\}$. Let L be the set of all numbers $p_0 < r$ such that, whenever

$$\{y_0, \ldots, y_{r-1}\}_<, \; \{y'_0, \ldots, y'_{r-1}\}_< \subset B',$$

$y_p = y'_p$ for $p \neq p_0$ and $y_{p_0} \neq y'_{p_0}$, then $f\{y_0, \ldots, y_{r-1}\} \neq f\{y'_0, \ldots, y'_{r-1}\}$. To complete the proof of Theorem 1 we show that f is L–canonical on B.

(a) Let $\{y_0, \ldots, y_{r-1}\}_<$, $\{y'_0, \ldots, y'_{r-1}\}_< \subset B$ and

$$(1) \qquad y_\lambda = y'_\lambda \;\text{ for }\; \lambda \in L.$$

We have to show that $f\{y_0, \ldots, y_{r-1}\} = f\{y'_0, \ldots, y'_{r-1}\}$. To this end we define an operator T thus: Let

$$\{y_0, \ldots, y_{r-1}\}_<, \{y'_0, \ldots, y'_{r-1}\}_< \subset B.$$

If $y_p = y'_p$ for $p < r$ then put

$$T(\{y_0, \ldots, y_{r-1}\}, \{y'_0, \ldots, y'_{r-1}\}) = (\{y_0, \ldots, y_{r-1}\}, \{y'_0, \ldots, y'_{r-1}\}).$$

Now let $y_p \neq y'_p$ for at least one p. Let $p_0 = min\ \{p : y_p \neq y'_p\}$. Then, by our assumption (1), $p_0 \notin L$. Put $z_p = y_p$ and $z'_p = y'_p$ for $p \neq p_0$, and $z_{p_0} = z'_{p_0} = min\ \{y_{p_0}, y'_{p_0}\}$. It follows from $p_0 \notin L$ and the definition of L that

$$\begin{aligned} f\{z_0, \ldots, z_{r-1}\} &= f\{y_0, \ldots, y_{r-1}\}, \\ f\{z'_0, \ldots, z'_{r-1}\} &= f\{y'_0, \ldots, y'_{r-1}\}. \end{aligned}$$

We put

$$T(\{y_0,\ldots,y_{r-1}\},\{y'_0,\ldots,y'_{r-1}\}=(\{z_0,\ldots,z_{r-1}\},\{z'_0,\ldots,z'_{r-1}\}).$$

We iterate T r times and obtain

$$T^r(\{y_0,\ldots,y_{r-1}\},\{y'_0,\ldots,y'_{r-1}\})=(\{w_0,\ldots,w_{r-1}\}_<,\{w_0,\ldots,w_{r-1}\}).$$

Then $f\{y_0,\ldots,y_{r-1}\}=f\{w_0,\ldots,w_{r-1}\}=f\{y'_0,\ldots,y'_{r-1}\}$, as required.

(b) Let $\{x_0,\ldots,x_{r-1}\}_<,\{x'_0,\ldots,x'_{r-1}\}_< \subset B$; $p_0\in L$;

$$(2) \qquad x_{p_0}<x'_{p_0}.$$

To complete the proof of Theorem 1, we now proceed to deduce that $f\{x_0,\ldots,x_{r-1}\}\neq f\{x'_0,\ldots,x'_{r-1}\}$. Let us assume that $f\{x_0,\ldots,x_{r-1}\}=f\{x'_0,\ldots,x'_{r-1}\}$. We have to deduce a contradiction.

For $P,P',Q,Q'\in[B]^r$ let $(P,P')\equiv(Q,Q')$ mean that there is an order preserving bijection $\phi:P\cup P'\to Q\cup Q'$ such that $\phi\ P=Q$ and $\phi P'=Q'$.

Lemma. *Let $P,P',Q,Q'\in[B]^r; fP=fP'; (P,P')\equiv(Q,Q')$. Then $fQ=fQ'$.*

Proof of the Lemma. There is a set $E\in[B]^{2r-|P\cup P'|}$ such that $x<y$ whenever $x\in P\cup P'\cup Q\cup Q'$ and $y\in E$. Then

$$P\cup P'\cup E,\ Q\cup Q'\cup E\in[B]^{2r}$$

and hence $g(P\cup P'\cup E)=g(Q\cup Q'\cup E)$. It follows from the definition of g that $fQ=fQ'$, and the Lemma is proved.

To continue the proof of Theorem 1 put, for $t\in\{1,2,3,\ldots\}$,

$$B(t)=\{b_0,b_t,b_{2t},\ldots\}.$$

Let $r<s\in\{2,3,\ldots\}$. There are sets $X_0,x_1\in[B(r^s)]^r$ such that

$$(X_0,X_1)\equiv(\{x_0,\ldots,x_{r-1}\},\{x'_0,\ldots,x'_{r-1}\}).$$

Then there is a set $X_2\in[B(r^{s-1})]^r$ such that $(X_0,X_1)\equiv(X_1,X_2)$. There is a set $X_3\in[B(r^{s-2})]^r$ such that $(X_1,X_2)\equiv(X_2,X_3)$, and so on until there is a set $X_s\in[B(r)]^r$ such that $(X_{s-2},X_{s-1})\equiv(X_{s-1},X_s)$. We have $X_\sigma\{X^0_\sigma,\ldots,X^{r-1}_\sigma\}_<$ for $\sigma\le s$. Then, by (2) and the definition of $\equiv$, we have

$$X_0^{p_0}<X_1^{p_0}<\ldots<X_s^{p_0}.$$

In view of $s>r$ there is σ_0 with $1\le\sigma_o\le s$ such that

$$(3) \qquad X^{p_0}_{\sigma_0}\neq X^p_0 \quad \text{for} \quad p<r.$$

There is a number π such that $X^{p_0}_{\sigma_0}=b_{2\pi}$. Put $Z_{\sigma_0}=\{Z^0_{\sigma_0},\ldots,Z^{r-1}_{\sigma_0}\}_<$, where $Z^p_{\sigma_0}=X^p_{\sigma_0}$ for $p\neq p_0$ and $Z^{p_0}_{\sigma_0}=b_{2\pi+1}$. Since $p_0\in L$ we have $fX_{\sigma_0}\neq fZ_{\sigma_0}$.

On the other hand, we have, by choice of σ_0 and the definition of Z_{σ_0}, that

$$(X_0, X_1) \equiv (X_{\sigma_0 - 1}, X_{\sigma_0}); \ (X_0, X_{\sigma_0}) \equiv (X_0, Z_{\sigma_0}).$$

We have $fX_0 = fX_1 = \ldots = fX_{\sigma_0}$. Hence, by (3) and the Lemma, $fX_{\sigma_0} = fX_0 = fZ_{\sigma_0}$, which yields the required contradiction. This proves Theorem 1.

4. We now consider connections between canonicity and some other properties of partitions. Let A and B denote infinite subsets of $\{0,1,\ldots\}$. Consider a partition $f : [A \cup B]^r \to F$. We require some definitions.

(f, A) is called invariant if, whenever $P, Q, P', Q' \in [A]^r$ and $(P; Q) \equiv (P', Q')$, then $fP = fQ$ if and only if $fP' = fQ'$.

(f, A) is called isomorphic to (f, B) [in symbols $(f, A) \cong (f, B)$] if, whenever $P, Q \in [A]^r$ and $\phi : A \to B$ is an order preserving bijection, then $fP = fQ$ if and only if $f\phi P = f\phi Q$.

(f, A) is called stationary if, whenever $B \subseteq A$ then $(f, B) \cong (f, A)$.

Theorem 3. *The following three conditions are equivalent:*
(i) (f, A) *is invariant,*
(ii) (f, A) *is stationary,*
(iii) (f, A) *is L–canonical for some L.*

Proof of (i) ⇒ (ii). (f, A) is invariant. Let $B \subseteq A$. There is an order preserving bijection $\phi : A \to B$. Let $P, Q \in [A]^r$. Then $(P, Q) \equiv (\phi\ P, \phi\ Q)$. By invariance we have $fP = fQ$ if and only if $f\phi P = f\phi Q$, and (ii) holds.

Proof of (ii) ⇒ (iii). (f, A) is stationary. By Theorem 1 there is an infinite set $B \subseteq A$ such that (f, B) is L–canonical for some L. Then, by (ii), $(f, B) \cong (f, A)$ which implies that (f, A) is L–canonical, and (iii) holds.

Proof of (iii) ⇒ (i). (f, A) is L–canonical for some L. Let $P, Q, P', Q' \in [A]^r$ and $(P, Q) \equiv (P', Q')$. Then we have

$$(fP = fQ) \Leftrightarrow (P^\lambda = Q^\lambda \text{ for } \lambda \in L) \Leftrightarrow (P'^\lambda = Q'^\lambda for \lambda \in L) \Leftrightarrow (fP' = fQ'),$$

and (i) holds. This proves Theorem 3.

The author would like to thank the referee.

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Part II

Numbers

On Size Ramsey Number of Paths, Trees and Circuits. II

József Beck

1. Introduction

In this paper we shall demonstrate that random graphs satisfy some interesting Ramsey type properties. This paper deals with finite, simple and undirected graphs only. If G and H are graphs, write $G \longrightarrow H$ to mean that if the edges of G are coloured by two colours, then G contains a monochromatic copy of H.

Erdős, Faudree, Rousseau and Schelp (1978) were the first to consider the unconvenient question of how few *edges* G can have, given that $G \longrightarrow H$. Following them, by the *size Ramsey number* $\hat{r}(H)$ we mean the least integer $\hat{r}$ such that there exists a graph G with $\hat{r}$ edges for which $G \longrightarrow H$, i.e., $\hat{r}(H) = \min |G| : G \longrightarrow H$ (here, as usual, $|\cdot|$ means the cardinality).

We mention some known results concerning size Ramsey number. Clearly $\hat{r}(K_{1,n}) = 2n - 1$ where $K_{1,n}$ denotes the star of n edges. Moreover, for every sufficiently large value of n,

$$\hat{r}(P_n) < 900n \tag{1}$$

where P_n denotes the path of length n (see Beck 1983, actually it was proved that the "greater colour" contains a copy of P_n). It was also shown there that there exists a "universal" graph $G = G(n, D)$ with less than $D.n.(\log n)^{12}$ edges, such that colouring the edges of G by two colours in any fashion, one of the colours contains all trees with $\leq n$ edges and maximal degree $\leq D$ (note that here n is sufficiently large and the upper bound cannot be replaced by $D(n - D)/4$).

As a corollary of it we get that for any tree T_n of n edges,

$$\hat{r}(T_n) < D \cdot n \cdot (\log n)^{12} \tag{2}$$

where D denotes the maximal degree of $T_n (n > n_0)$. Recently we realized that a slight modification of the proof of (2) gives an asymptotically good estimation for the size Ramsey number of any individual tree.

We need some notations. Let T be an arbitrary tree. Since T is bipartite, there is a unique bipartition (A_1, A_2) of the vertex-set $V(T)$ of T such that for any edge $\{u,v\} \in T$ either $u \in A_1, v \in A_2$ or $u \in A_2, v \in A_1$.

Let $D_1 = \max_{v \in A_1} d_T(v)$ and $D_2 = \max_{v \in A_2} d_T(v)$ where $d_T(v)$ denotes the degree of the vertex v in T. The key quantity is as follows

$$\Delta(T) = |A_1| \cdot D_1 + |A_2| \cdot D_2$$

(we recall that $|A|$ denotes the number of elements of the set A). The main result of this paper is

Theorem 1. *For any tree T_n of n edges*

$$\frac{\Delta(T_n)}{4} < \hat{r}(T_n) < C_0 \cdot \Delta(T_n) \cdot (\log n)^{12}$$

where C_0 is a universal constant.

Note that our proof will be nonconstructive, we shall use random bipartite graphs. Unfortunately, we *cannot* prove here *density* theorem (density theorem means that the "greater colour" contains T_n). From Theorem 1 immediately follows that for a large class of trees T the size Ramsey number $\hat{r}(T)$ is much less than the trivial upper bound $\binom{r(T)}{2}$, where $r(T)$ denotes the traditional (vertex) Ramsey number of T. We state the following

Conjecture. *There is an absolute constant c_1 such that $\hat{r}(T) < c_1 \cdot \Delta(T)$ holds true for all trees T.*

We have some results on the size of Ramsey number of any *induced* tree as well (the problem is due to P. Erdös, communication by letter).
We write

$$G \xrightarrow{ind} H$$

if any two-colouring of the edges of G yields a monochromatic induced copy of H. Furthermore, let

$$\hat{r}(ind\, H) = \min |G| : G \xrightarrow{ind} H.$$

We cannot determine the correct order of magnitude of $\hat{r}(ind\, T_n)$, but we have some results

Theorem 2. *There is a graph $G = G(n)$ with less than $n^3 \cdot (\log n)^4$ edges such that $G \xrightarrow{ind} T_n$ for every tree T_n of n edges ($n > n_0$).*

Actually, we shall prove that the "greater colour" contains an induced copy of any tree T_n. Here the upper bound cannot be essentially less than n^2. Indeed, it follows easily combining the trivial inequality

$$\hat{r}(ind\, H) \geq \hat{r}(H)$$

and the fact that for some tree T_n^* the size Ramsey number $\hat{r}(T_n^*)$ is constant times n^2. We mention that in contrast to stars the size Ramsey number of the path P_n cannot be "as small as possible", that is, greater than 2n-1.

Theorem 3.

$$lim\ inf_{n\to\infty}\frac{\hat{r}(P_n)}{n}\geq\frac{9}{4}$$

Note that the problem of estimating the size Ramsey number of more complex graphs seems to be very hard. As an example of unsolved questions we mention the following

Problem. *Let $G_{n,D}$ be a graph of n edges and maximal degree D. Decide whether $\hat{r}(G_{n,D}) < c_2(D)\cdot n$ where the constant $c_2(D)$ depends only on D.*

We remark that recently Chvátal, Rödl, Szemerédi and Trotter succeeded in proving the analogous linear upper bound for Ramsey number.

Finally, we mention that each of Theorem 1–3 can be generalized for more than two colours without any difficulty. We leave the details to the reader.

2. Proof of Theorem 1 – Part One

We start with the lower bound. Let there be given a tree T_n with bipartition (A_1, A_2). We may assume that $|A_1|D_1 \geq |A_2|\cdot D_2$. Suppose $G \longrightarrow T_n$, and let $V = V(G)$ denote the set of vertices of G. Let

$$V_1 = \{v\in V : d_G(v)\geq D_1\} \text{ and } V_2 = V\backslash V_1 = \{v\in V : d_G(v) < D_1\}$$

Now we define a two-colouring of the edges of G as follows: Let $e = \{u, v\}\in G$ be *red* if either $u\in V_1, v\in V_2$ or $u\in V_2, v\in V_1$; otherwise let e be coloured by *blue*. We distinguish two cases.

Case 1. $D_1 > D_2$

If G contains a blue copy of T_n, then clearly all vertices of this monochromatic copy belong to V_1, and so

$$|G|\geq(n+1)\cdot D_1/2\geq\Delta(T_n)/2.$$

If G contains a red copy of T_n, then exactly $|A_1|$ vertices of this monochromatic copy belong to V_1, and so

$$|G|\geq|A_1|\cdot D_1/2\geq\Delta(T_n)/4.$$

Case 2. $D_1\leq D_2$

Then G contains a blue copy of T_n such that all vertices of this copy belong to V_1. Therefore,

$$|G|\geq(n+1)\cdot D_1/2\geq|A_1|\cdot D_1/2\geq\Delta(T_n)/4,$$

which completes the proof of the lower bound. □

The proof of the upper bound is much harder. Let T_n be a tree of n edges with bipartition $V(T_n) = A_1 \cup A_2$, and let

$$D_i = \max_{v \in A_i} d_{T_n}(v), \quad i = 1, 2.$$

Without loss of generality we may assume

$$|A_1| \cdot D_1 \geq |A_2| \cdot D_2. \tag{3}$$

Let

$$n_1 = |A_1| \text{ and } n_2 = D_1 \cdot |A_1| / D_2. \tag{4}$$

We may further assume that $\min_{i=1,2} D_i \geq 2$, since in the opposite case T_n is a star and we know the exact value of $\hat{r}(T_n)$. We shall use the following notation: For any graph F and subsets X, Y of the vertex-set $V(F)$ of F, let

$$F[X, Y] = \{\{u, v\} \in F : u \in X \text{ and } v \in Y\}.$$

The proof of the upper bound is based on the following rather difficult and technical lemma.

Main Lemma. *Let F be a bipartite graph with bipartition $V(F) = V_1 \cup V_2$ such that*

$$|V_i| \leq n_i \cdot (\log n)^5 \text{ and } \max_{v \in V_i} d_F(v) \leq D_i \cdot (\log n)^8, \quad i = 1, 2. \tag{5}$$

Let

$$\overline{d}_i(F) = \frac{1}{|V_i|} \sum_{v \in V_i} d_F(v), \ i = 1, 2.$$

Suppose F satisfies the following three properties:

(α_1) *For any two sets $X \subset V_i, X^* \subset V_{3-i}$ $(i = 1, 2)$ with $1 \leq |X| \leq n_{3-i} \cdot (\log n)^{5/2}/D_i$, $|X^*| = D_i \cdot (\log n)^{9/4} \cdot |X|$, or with $n_{3-i} \cdot (logn)^{5/2}/D_i < |X| \leq n_{3-i} \cdot (\log n)^{5/2}, |X^*| = n_{3-i}(\log n)^{19/4}$, we have $|F[X, X^*]| < \overline{d}_i(F) \cdot |X|/4$.*

(α_2) *For any two sets $Y_1 \subset V_1, Y_2 \subset V_2, |Y_i| \geq n_i \cdot (\log n)^{5/2}$ $(i = 1, 2)$ the induced bipartite subgraph $\hat{F} = F[Y_1, Y_2]$ has the property that for any two sets $Z \subset Y_i, Z^* \subset Y_{3-i}$ $(i = 1, 2)$ with $1 \leq |Z| \leq 2n_{3-i}/D_i, |Z^*| = D_i \cdot (\log n)^{9/4} \cdot |Z|$, or with $2n_{3-i}/D_i < |Z| \leq n_{3-i}, |Z^*| = n_{3-i} \cdot (\log n)^{9/4}$, we have $|F[Z, Z^*]| < \overline{d}_i(\hat{F}) \cdot |Z|/4$ where $\overline{d}_i(\hat{F}) = \frac{1}{|Y_i|} \sum_{v \in Y_i} d_{\hat{F}}(v)$.*

(α_3) *For any two disjoint set $U_i \subset V_i, |U_i| \geq n_i \cdot (\log n)^{1/5}$ $(i = 1, 2)$ there exists an edge of F going from U_1 to U_2.*

Moreover, assume that n is sufficiently large. Then two-colouring the edges of F in any fashion one of the colours contains a copy of T_n, i.e., $F \longrightarrow T_n$.

Proof of the Main Lemma: Let $F = F_1 \cup F_2$ be an arbitrary two-colouring of the edges of F. Suppose $|F_1| \geq |F|/2$. We need a simple lemma. For notational convenience, write $G(S) = G[S, V(G)]$.

Lemma 2.1. *Let H be a bipartite graph with bipartition $V(H) = W_1 \cup W_2$, and let*

$$\overline{d}_i(H) = \frac{1}{|W_i|} \sum_{v \in W_i} d_H(v), \ i = 1,2.$$

Then there exists an induced subgraph G of H such that for each vertex-set $S \subset W_i$ $(i = 1,2), |G(S)| \geq \overline{d}_i(H) \cdot |S|/2$.

We postpone the proof of Lemma 2.1 later. Applying Lemma 2.1 with $H = F_1$, we get the existence of an induced subgraph G of F_1 such that for each $S \subset W_i$, where (W_1, W_2) denotes the bipartition of $V(G), |G(S)| \geq \overline{d}_i(F_1) \cdot |S|/2 \geq \overline{d}_i(F) \cdot |S|/4$. For any $S \subset V(G)$, write

$$\Gamma_G(S) = \{v \in V(G) : \text{ there exists } u \in S \text{ such that } \{u,v\} \in G\}.$$

We conclude that

(6) if $S \subset W_i$, $1 \leq |S| \leq n_{3-i} \cdot (\log n)^{5/2}/D_i$ then $|\Gamma_G(S)| > D_i \cdot (\log n)^{9/4} \cdot |S|$, and if $S \subset W_i, n_{3-i} \cdot (\log n)^{5/2}/D_i < |S| \leq n_{3-i}(\log n)^{5/2}$ then $|\Gamma_G(S)| > n_{3-i} \cdot (\log n)^{19/4}$.

Indeed, setting $X = S, X^* = \Gamma_G(S)$ we know

$$|G(S)| = |G[S_1, \Gamma_G(S)]| \geq \overline{d}_i(F) \cdot |S|/4.$$

Comparing it to (α_1) we get (6). Next we need

Lemma 2.2. *Let G be a bipartite graph with bipartition $V(G) = W_1 \cup W_2$. Suppose $|W_i| \leq k_i \cdot t^3$, $\max_{v \in W_i} d_G(v) \leq D_i \cdot t^4$ and for each $S \subset W_i$,*

if $|S| \leq 2k_{3-i}/D_i$ then $|\Gamma_G(S)| \geq (c_3 \cdot t \cdot \log t) \cdot D_i \cdot |S|$,
if $2k_{3-i}/D_i < |S| \leq k_{3-i}$ then $|\Gamma_G(S)| \geq (c_3 \cdot t \cdot \log t) \cdot 2k_{3-i}$

where c_3 is a sufficiently large absolute constant and $t, k_i, D_i \geq 2$ are unspecified parameters $(i = 1,2)$. Assume further that k_i is greater than a threshold depending only on c_3. Then either G contains any tree T with bipartition $V(T) = A_1 \cup A_2$ such that

$$|A_i| \leq \min\{k_i, 2^{\sqrt{(t-1)/8}}\}$$

and

$$\max_{v \in A_i} d_T(v) \leq D_i \ \ (i = 1,2),$$

or there are two sets $Y_1 \subset W_1, Y_2 \subset W_2, |Y_i| \geq k_i$ $(i = 1,2)$ such that no edge of G goes from Y_1 to Y_2.

We postpone the proof of Lemma 2.2 later.

By (5) and (6) we see that the hypotheses of Lemma 2.2 certainly hold for G with $k_i = n_i \cdot (\log n)^{5/2}$ and $t = 8 \cdot (\log_2 n)^2 + 1$ where $\log_2 n$ denotes the binary logarithm of n. Thus Lemma 2.2 gives that either G contains a copy of T_n (see (4)), or there are two disjoint sets $Y_1 \subset W_1, Y_2 \subset W_2, |Y_i| \geq$

$n_i(\log n)^{5/2}(i = 1, 2)$ such that no edge of G goes from Y_1 to Y_2, or equivalently, the induced bipartite graph $\hat{F} = F[Y_1, Y_2]$ is entirely contained in the "smaller colour" F_2. In the first case we are done. In the second case we conclude, similarly as above, by property (α_2) that Lemma 2.2 can be applied for $\hat{F}$ with $k_i = n_i \cdot (\log n)^{1/5}$ and $t = 8 \cdot (\log_2 n)^2 + 1$. Lemma 2.2 gives that either $\hat{F}$ contains a copy of T_n, and again we are done, or there are two sets $U_1 \subset Y_1, U_2 \subset Y_2, |U_i| \geq n_i \cdot (\log n)^{1/5}(i = 1, 2)$ such that *no* edge goes from U_1 to U_2. Comparing the last case with (α_3) we get a contradiction. This proves the Main Lemma assuming the validity of Lemma 2.1 and Lemma 2.2. □

Proof of Lemma 2.1: Let us consider the following "truncating" operation. Let $H' \subset H$ be a subgraph. Assume that one can find a non-empty subset $S \subset W_i \cap V(H')(i = 1, 2)$ such that $|H'(S)| < \bar{d}_i(H) \cdot |S| /2$. Then let

$$\beta(H') = H'[V(H') \backslash S]$$

(note that the operation $\beta(\cdot)$ is not uniquely determined). Otherwise, let $\beta(H') = H'$. Any sequence $H, \beta(H), \beta^2(H) = \beta(\beta(H)), \ldots, \beta^{N+1}(H) = \beta(\beta^N(H)), \ldots$ will certainly be stabilized within a finite number of steps, say $\beta^M(H) = \beta^{M+1}(H) = \beta^{M+2}(H) = \ldots$. If $\beta^M(H)$ is non-empty, then we are done. But in the opposite case we have

$$|H| = \sum_{i=0}^{M}(|\beta^i(H)| - |\beta^{i+1}(H)|) < \bar{d}_1(H) \cdot |W_1| /2 + \bar{d}_2(H) \cdot |W_2| /2 = |H|$$

a contradiction, which completes the proof of Lemma 2.1. □

The proof of Lemma 2.2 proceeds exactly along the same lines as that of Lemma 3.5 in Beck (1983). We need the following lemmas which are the "asymmetric bipartite" versions of Lemmas 3.1, 3.2 and 3.3 of that paper in this order. Their proofs are just the same as those of the analogous Lemmas 3.1, 3.2 and 3.3 in there, and so we omit them.

Lemma 2.3. *Given any tree T_n of n edges we can "build it up" within $2 \cdot (\log_2 n)^2 + 1$ steps as follows:*

$$T_n = G_0 \otimes G_1 \otimes G_2 \otimes \ldots \otimes G_q, \quad q \leq 2(\log_2 n)^2 + 1$$

where G_0 is one vertex, $G_i(i \leq q)$ is a system of vertex-disjoint paths having equal length, and the operation $\otimes$ means that each path in G_i is "glued" by one of its endpoints to not necessarily distinct vertices of the same class of the bipartition of $G_0 \otimes G_1 \otimes \ldots \otimes G_{i-1}$. □

We recall the notation: if $X \subset V(G)$ then

$$\Gamma_G(X) = \{v \in V(G) : \text{ there exists } u \in X \text{ such that } \{u, v\} \in G\}.$$

Lemma 2.4. *Let $D_1 \geq 2$ and $D_2 \geq 2$ be natural numbers. Let G be a bipartite graph with bipartition $V(G) = W_1 \cup W_2$, and let there be given a partition*

$V(G) = S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4$ *such that each* $X \leq V(G)$, $i = 1,2$ *and* $j = 1,2,3,4$ *have the following "expandity" property: There are integers* k_1, k_2 *such that*

if $X \subset W_i$ *and* $|X| \leq 2k_{3-i}/D_i$ *then* $|\Gamma_G(X) \cap S_j| \geq D_i$, *and*
if $X \subset W_i$ *and* $2k_{3-i}/D_i < |X| \leq k_{3-i}$ *then* $|\Gamma_G(X) \cap S_j| \geq 2 \cdot k_{3-i}$.

Let $i \in \{1,2\}$. *Let* $v_1, v_2, \ldots, v_k$ *be arbitrary vertices in* $S_0 \cap W_i$ *and let there be given arbitrary integers* $D^{(1)}, D^{(2)}, \ldots, D^{(h)}$ *and* l *such that* $D^{(j)} \leq D_i (j = 1, \ldots, h)$ *and*

$$(l-1) \cdot (\sum_{j=1}^{h} D^{(j)}) \leq k_i.$$

Then either one can find paths $P^{j,\nu}, 1 \leq j \leq h, 1 \leq \nu \leq D^{(j)}$ *in* G *such that* $P^{j,\nu}$ *starts from the vertex* v_j, $P^{j,\nu}$ *and* $P^{j,\nu_0} (\nu \neq \nu_0)$ *are vertex-disjoint except* v_j, $P^{j,\nu}$ *and* $P^{j_0,\nu_0} (j \neq j_0)$ *are vertex-disjoint, and all the paths* $P^{j,\nu}$ *have equal length* l; *or there exist two sets* $Y_1 \subset W_1, Y_2 \subset W_2, |Y_i| \geq k_i$ *such that no edge of* G *goes from* Y_1 *to* Y_2. □

Lemma 2.5. *Under the hypotheses of Lemma 2 the vertex-set* $V(G)$ *of* G *can be partitioned into* t *parts* $(V^{(1)}, V^{(2)}, \ldots, V^{(t)})$ *in such a way that for each* $i = 1,2$ *and* $j = 1, 2, \ldots, t$:

$\left|\Gamma_G(X) \cap V^{(j)}\right| \geq D_i \cdot |X|$ *whenever* $X \subset W_i$ *and* $|X| \leq 2k_{3-i}/D_i$, *and*
$\left|\Gamma_G(X) \cap V^{(j)}\right| \geq 2k_{3-i}$ *whenever* $X \subset W_i$ *and* $2k_{3-i}/D_i < |X| \leq k_{3-i}$. □

Repeating now the proof of Lemma 3.5 in Beck (1983) one can easily prove Lemma 2.2, but instead of Lemma 3.1–2–3 of that paper one has to use the analogous Lemma 2.3–4–5 in this order. We leave the details to the reader. Thus the proof of the Main Lemma is complete. □

3. Proof of Theorem 1 – Part Two

We recall Lemma 2.1 from Beck (1983).

Lemma 3.1. *If* $R_{m,p}$ *is a random variable with binomial distribution* $B(m,p)$ *then*

$$Prob(R_{m,p} \geq k) \leq (m \cdot p \cdot e/k)^k \tag{7}$$

and

$$Prob(R_{m,p} \leq m \cdot p/2) \leq (2/e)^{m \cdot p/2}. \tag{8}$$ □

Let us consider the complete $n_1 \cdot (\log n)^5 \times n_2 \cdot (\log n)^5$ bipartite graph and choose its edges at random, independently of each other, with common probability

$$p = \min\{1, \frac{D_1}{n_2} \cdot (\log n)^2\} = \min\{1, \frac{D_2}{n_1} \cdot (\log n)^2\}.$$

Let $RG(n_1, n_2, p)$ denote this random bipartite graph. Let (V_1, V_2) denote the bipartition of the vertex-set of $RG(n_1, n_2, p)$. By the Main Lemma it suffices to verify that $F = RG(n_1, n_2, p)$ satisfies properties (5), (α_1),(α_2),(α_3) and that the number of edges is less than $c_0 \cdot \Delta(T_n) \cdot (\log n)^{12}$ with probability approaching one as n tends to infinity.

First we state that for any two sets $Y_1 \subset V_1, Y_2 \subset V_2$ with $|Y_i| \geq n_i \cdot (\log n)^{5/2} (i = 1, 2)$, the induced subgraph $\hat{F} = F[Y_1, Y_2]$ (we recall that $F = RG(n_1, n_2, p)$) have average degrees

$$\bar{d}_i(\hat{F}) = \frac{1}{|Y_i|} \sum_{v \in Y_i} d_{\hat{F}}(v) > |Y_{3-i}| \cdot p/2 \quad (i = 1, 2) \tag{9}$$

with probability $1 - \epsilon_n$ (here $\epsilon_n \to 0$ as $n \to \infty$).

Indeed, the probability of the complementary event can be estimated from above by

$$\sum_{l_1 = n_1 \cdot (\log\ n)^{5/2}}^{N_1} \; \sum_{l_2 = n_2 (\log\ n)^{5/2}}^{N_2} \binom{N_1}{l_1} \cdot \binom{N_2}{l_2} \cdot Prob(R_{l_1 \cdot l_2} \leq l_1 \cdot l_2 \cdot p/2) \tag{10}$$

where $N_i = |V_i| = n_i \cdot (\log n)^5$, $l_i = |Y_i|$ and R_m denotes the random variable with binomial distribution $B(m, p)$(throughout this section p is fixed and so we omit the lower-index p from $R_{m,p}$). Applying estimate (8) and the elementary fact $\binom{N}{l} < (N \cdot e/l)^l$ it is easy to see that the expression in (10) tends to zero as $n \to \infty$.

Under the assumption that (9) is true, the probability of the event that F is false to satisfy property (α_2) can be estimated from above by

(11)

$$\sum_{i=1}^{2} \left\{ \sum_{k=1}^{2n_{3-i}/D_i} \binom{N_i}{k} \cdot \binom{N_{3-i}}{K^*} \sum_{l_1 = n_1 (\log\ n)^{5/2}}^{N_1} \; \sum_{l_2 = n_2 (\log\ n)^{5/2}}^{N_2} \right.$$

$$Prob(R_{k \cdot K^*} > l_{3-i} \cdot p \cdot k/8) + \sum_{n_{3-i} \geq k > 2n_{3-i}/D_i} \binom{N_i}{k} \cdot \binom{N_{3-i}}{K^{**}} \sum_{l_1 = n_1 (\log\ n)^{5/2}}^{N_1}$$

$$\left. \sum_{l_2 = n_2 (\log\ n)^{5/2}}^{N_2} Prob(R_{k \cdot K^{**}} > l_{3-i} \cdot p \cdot k/8) \right\}$$

where $N_i = |V_i| = n_i \cdot (\log n)^5$, $k = |Z|, K^* = D_i \cdot (\log n)^{9/4} \cdot k, K^{**} = n_{3-i} \cdot (\log n)^{9/4}$ and $l_i = |Y_i|$.

Applying estimate (7) and the fact $\binom{N}{k} < (N \cdot e/k)^k$ one can easily see by some elementary calculation that (11) tends to zero as $n \longrightarrow \infty$.

Summarising, we obtain that $F = RG(n_1, n_2, p)$ satisfies property (α_2) with probability $1 - \epsilon_n$. A similar (but simpler) calculation shows that F satisfies property (α_1) as well with probability $1 - \epsilon_n$. Next we estimate the event that F is false to satisfy property (α_3) from above:

$$(12) \qquad \binom{N_1}{n_1 \cdot (\log n)^{1/5}} \cdot \binom{N_2}{n_2 \cdot (\log n)^{1/5}} \cdot (1-p)^{n_1 \cdot n_2 \cdot (\log n)^{2/5}}$$

Since

$$(1-p)^{n_i} \leq n^{-D_{3-i} \cdot \log n}$$

it follows easily that (12) tends to zero as $n \longrightarrow \infty$. That is, property (α_3) is also settled.

The probability of the event that $F = RG(n_1, n_2, p)$ is false to satisfy the degree condition in (5) is clearly less than

$$(13) \qquad \sum_{i=2}^{2} \cdot Prob(R_{N_{3-i}} > D_i \cdot (\log n)^8) \text{ where } N_i = |V_i|\,.$$

Here we can apply (7), since

$$N_{3-i} \cdot p = n_{3-i} \cdot (\log n)^5 \cdot \frac{D_i}{n_{3-i}} (\log n)^2 = D_i (\log n)^7 .$$

By (7) it follows that the expression in (13) tends to zero as $n \longrightarrow \infty$. This settles the degree condition. Finally, the expected value of the number of edges in our random graph equals $N_1 \cdot N_2 \cdot p$. Clearly

$$N_1 \cdot N_2 \cdot p \leq n_1 \cdot D_1 \cdot (\log n)^{12} \leq \Delta(T_n) \cdot (\log n)^{12} .$$

Thus, by Chebyshev's inequality the probability of the event

$$\{|RG(n_1, n_2, p)| < 2 \cdot \Delta(T_n) \cdot (\log n)^{12}\}$$

tends to 1 as $n \to \infty$. Consequently, for every sufficiently large value of n, there exists a deterministic graph F_0 with $|F_0| < 2 \cdot \Delta(T_n) \cdot (\log n)^{12}$ such that $F_0 \to T_n$. This completes the proof of the upper bound, and so Theorem 1 follows. □

4. Proof of Theorem 2

We say that a graph G has property (β) if for every vertex $v \in V(G)$ and for every less than n element subset $S \subset V(G) \backslash \{v\}$, $|\Gamma_G(v) \cap \Gamma_G(S)| \leq \bar{d}(G)/4$ where $\bar{d}(G)$ denotes the average degree of G (we recall that for $S \subset V(G), \Gamma_G(S) = \{u \in V(G) :$ there exists $v \in S$ such that $\{u, v\} \in G\}$).

Lemma 4.1. *Let G be a graph of property (β). Then every subgraph $H \subset G$ with $|H| \geq |G|/2$ contains all trees T_n of n edges.*

Proof. Let H be an arbitrary subgraph of G containing at least the half of the edges of G. Let $X \subset V(H)$ be the smallest vertex-subset of H such that the induced subgraph $F = H[X]$ has at least $\bar{d}(H) \cdot |X|/2$ edges. We claim that every vertex of F has degree greater than $\bar{d}(H)/2$. Assume, in contrary, that there exists a vertex $u \in X$ such that $d_F(u) \leq \bar{d}(H)/2$. Then the induced subgraph $H[X \setminus \{u\}]$ contains $\geq \bar{d}(H) \cdot (|X| - 1)/2$ edges, which contradicts the minimality of X. Therefore, for every vertex $v \in V(F), d_F(v) > \bar{d}(H)/2 \geq \bar{d}(G)/4$. Combining this with property (β) we see that given any vertex $v \in V(F)$, the endpoint-set $\Gamma_F(v)$ *cannot* be covered by the neighbourhood $\Gamma_F(S)$ of any less than n element subset $S \subset V(F) \setminus \{v\}$. From this property of F it follows that one can successively embed every tree T_n of n edges to F. Since $F \subset H$, the lemma follows. □

Let us consider a complete graph of $N = n^2.(\log n)^2$ vertices and choose its edges at random, independently of each other, with common probability $p = 1/(18n)$. Let $RG(N,p)$ denote this random graph. We shall show that with probability tending to one, the random graph $RG(N,p)$ satisfies property (β). First we claim that the probability of the event

$$\{\text{the maximal degree of } RG(N,p) < 3 \cdot N \cdot p\} \tag{14}$$

tends to zero as $n \to \infty$. Indeed, the probability of the complementary event can be estimated from above by

$$N \cdot Prob(R_{N-1} \geq 3 \cdot N \cdot p)$$

where R_{N-1} is a random variable with binomial distribution $B(N-1,p)$ (p is fixed throughout this section). From (7) it follows by some elementary calculation that this upper bound tends to zero as $n \to \infty$.
Moreover, the expected value of the number of edges in $RG(N,p)$ equals

$$\binom{N}{2} \cdot p \approx N^2 \cdot p/2.$$

Thus by Chebyshev's inequality the probability of the event

$$\{N^2 \cdot p/4 < |RG(N,p)| < N^2 \cdot p\} \tag{15}$$

tends to zero as $n \to \infty$.

Therefore, we may assume that both events (14) and (15) hold true. Under these assumptions, the probability of the event that our random graph is false to satisfy property (β) is clearly less than

$$\binom{N}{n-1} \cdot (N-n) \cdot Prob(R_{D \cdot (n-1)} > N \cdot p/2) \text{ where } D = 3 \cdot N \cdot p. \tag{16}$$

Since

$$D \cdot (n-1) \cdot p < \frac{1}{3} \cdot (N \cdot p/3),$$

we can here apply (7) and by some elementary calculations we get that expression (16) tends to zero as $n \to \infty$.

Summarising, we obtain that for every sufficiently large value of n, there exists a deterministic graph G with less than

$$N^2 \cdot p = n^4(\log n)^4 \cdot 1/(18n) < n^3(\log n)^4$$

edges such that G satisfies property (β). This completes the proof of Theorem 2. □

5. Proof of Theorem 3

We need a simple lemma.

Lemma 5.1. *Given any graph H and any natural number $t < N = |V(H)|$, there exists a t-element subset $S \subset V(H)$ such that the induced subgraph $H[S]$ contains at most $\frac{t(t-1)}{N(N-1)} \cdot |H|$ edges.*

Proof. We use the standard probabilistic method. Let $RS \subset V(H)$ be a "random t-element subset" of $V(H)$, i.e., for any $S_0 \subset V(H)$ with $|S_0| = t$,

$$Prob(R\,S = S_0) = \binom{N}{t}^{-1}.$$

The expected value of the number of edges of the random induced subgraph $H[RS]$ equals

$$\binom{N}{t}^{-1} \sum_{\substack{S \subset V(H): \\ |S|=t}} \sum_{\{u,v\} \in H[S]} 1 = \sum_{\{u,v\} \in H} \sum_{\substack{S \subset V(H): \\ |S|=t \text{ and } \{u,v\} \subset S}} \binom{N}{t}^{-1}$$
$$= |H| \cdot \binom{N-2}{t-2} \cdot \binom{N}{t}^{-1} = |H| \frac{t(t-1)}{N(N-1)}.$$

Thus there must exist a t-element subset $S^* \subset V(H)$ such that $|H[S^*]| \leq |H| \cdot \frac{t(t-1)}{N(N-1)}$. □

Now let G be a graph such that $G \to P_n$ and $|G|$ is minimal. Let

$$V_1 = \{v \in V(G) : d_G(v) = 1\}, V_2 = \{v \in V(G) : d_G(v) = 2\}$$

and

$$V_3 = \{v \in V(G) : d_G(v) \geq 3\}.$$

Since each vertex in V_3 has degree ≥ 3, we have

$$N = |V_3| \leq \frac{2}{3} |G|. \tag{17}$$

Let $t = N - [n/2] + 2$ and apply Lemma 8 to $H = G[V_3]$. We obtain the existence of a t-element subset $S^* \subset V_3$ such that

$$(18) \qquad |G[S^*]| \leq |G[V_3]| \cdot \frac{t(t-1)}{N(N-1)} \leq |G| \cdot \frac{t(t-1)}{N(N-1)}.$$

Let us now consider the following two-colouring of the edges of G. If $e = \{u,v\} \in G[V_3]$ and $\{u,v\} \subset S^*$ or $\{u,v\} \subset S^{**} = V_3 \backslash S^*$, then let the edge e be red. If $e = \{u,v\} \in G[V_3]$ and $u \in S^*, v \in S^{**}$ or $u \in S^{**}, v \in S^*$, then let the edge e be blue. Moreover, the edges of the subgraph $\hat{G} = G[V_1 \cup V_2, V_1 \cup V_2 \cup V_3]$ can be two-coloured so that for any $v \in V_2$ the exactly two edges containing v have different colours. Indeed, it easily follows from the fact that $\hat{G}$ is the union of vertex-disjoint paths starting from V_3 and terminating at $V_3 \cup V_1$, or circuits containing at least one point of V_3 (note that by the minimality of G, V_2 cannot contain circuits).

Since $G \to P_n$, there must exist a monochromatic copy of P_n. From the colouring strategy immediately follows $G[V_3] \to P_{n-2}$. This monochromatic copy of P_{n-2} in V_3 cannot be blue, since the bipartite graph $G[S^*, S^{**}]$ cannot contain a path of length $> 2\,|S^{**}|+1 = 2[n/2]-3$. Consequently, $G[S^*] \supset P_{n-2}$, and so $|G[S^*]| \geq n-2$. Comparing it with (18) we get the inequality

$$(19) \qquad n-2 \leq |G| \frac{t(t-1)}{N(N-1)} \text{ where } t = N - [n/2] + 2.$$

Let $N = c \cdot (n-2)\, (c \geq 1)$, then (19) gives $|G| \geq ((1 - \frac{1}{2c})^{-2} - \epsilon)(n-2)$ where $\epsilon \to 0$ as $n \to \infty$. On the other hand, by (17)

$$|G| \geq \frac{3}{2}N \geq \frac{3}{2} \cdot c(n-2).$$

Combining the last two inequalities we obtain

$$|G| \geq \frac{1}{2}\{(1 - \frac{1}{2c})^{-2} - \epsilon + \frac{3}{2}c\} \cdot (n-2).$$

Since $(1 - \frac{1}{2c})^{-2} + \frac{3}{2}c \geq 9/2$ for every real $c \geq 1$, we get

$$|G| > (\frac{9}{4} - \epsilon)n.$$

This completes the proof of Theorem 3. □

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On the Computational Complexity of Ramsey–Type Problems

Stefan A. Burr

Abstract

If F, G and H are graphs, write $F \rightarrow (G, H)$ to mean that if the edges of F are colored red and blue, either a red G or a blue H must occur. It is shown that, if G and H are fixed 3–connected graphs (or triangles), then deciding whether $F \not\rightarrow (G, H)$ is an NP–complete problem. On the other hand, if G and H are arbitrary stars, or if G is fixed matching and H is any fixed graph, the complexity of the problem is polynomial bounded.

1. Introduction

If F, G, and H are (simple) graphs, write $F \rightarrow (G, H)$ to mean that if the edges of F are colored red and blue, either the red subgraph contains a copy of G or the blue subgraph contains a copy of H. (Of course, we write $F \not\rightarrow (G, H)$ if the above does not hold.) This relation has been much studied lately; see Burr (1979) for a survey. We will consider here the computational complexity of this relation. To do this we first define our basic problem, in the style of Garey, Johnson (1979). (For the terminology of computational complexity, see that book.) For technical reasons, we must state then the problem in the negative.

NON–ARROWING
Instance: (Finite) graphs F, G, and H.
Question: Is it true that $F \not\rightarrow (G, H)$?

We will prove two theorems about the problem. First we need a definition: Let Γ_3 denote the class of 3–connected graphs, together with the triangle K_3.

Theorem 1. *NON–ARROWING is NP–complete for any fixed graphs G and H that belong to Γ_3.*

In particular, NON–ARROWING is NP–complete when $G = H = K_3$; this case is mentioned in Garey, Johnson (1979). Note that it was important that NON–ARROWING be stated in the negative; "ARROWING" would be co–NP–complete. For Theorems 2 and 3 we would not have needed to state the problem in the negative.

Theorem 2. *If G and H are restricted to be stars, then NON–ARROWING is polynomial–bounded.*

Note that in Theorem 2, G and H need not be fixed. On the other hand, they must be fixed in the next theorem.

Theorem 3. *If G is a fixed matching nK_2, and H is any fixed graph, then NON–ARROWING is polynomial–bounded.*

Sections 2 and 3 will be devoted to proofs of these theorems, while Section 4 will discuss some related questions.

2. NP–Complete Ramsey Problems

Define a (G, H)–*good* coloring of a graph F to be a 2–coloring of the edges of F in such a way that no red G nor blue H occurs. Thus, $F \to (G, H)$ means that F has no (G, H)–good coloring. A (G, H)–*determiner* with *determined edge e* is an F such that $F \not\to (G, H)$, but in any (G, H)–good coloring , e is red. Note that if F_1 is an (H, G)–determiner, then $F_1 \not\to (G, H)$, but in any (G, H)–good coloring its determined edge is blue. Also observe that a (G, H)–determiner cannot exist when $G = H$. A *positive* (G, H)–*sender* with *signal edges e* and f is an F such that $F \to (G, H)$, but in any (G, H)–good coloring, e and f have the same color, and moreover, F is not a determiner for the edges e and f. A *negative* (G, H)–*sender* is the same, but with "same color" replaced by "opposite colors". We call these graphs senders because they, in effect, send a signal between e and f. We will usually drop the (G, H) from the above terms when the meaning is clear. The following lemma, taken from Burr, Nešetřil, Rödl (1985), shows that the above definitions are meaningful.

Lemma 2.1 (Burr, Nešetřil, Rödl). *If $G, H \in \Gamma_3$, then both positive and negative (G, H)–senders exist with the property that the signal edges are farther apart than the order of either G or H. In addition, if $G \neq H$, then (G, H)–determiners exist.*

As an example, the following is a positive sender for $G = H = K_3$. Take two copies of K_5 and identify a K_3 of each. The four points not participating in this identification spann the two signal edges. Determiners and especially senders are very powerful tools in the study of arrow relations. For examples of their use, see Burr, Nešetřil, Rödl (1985); Burr, Erdős, Lovász (1976); Burr, Faudree, Schelp (1977). We will now define, and prove the existence of, a more elaborate type of graph in a similar vein. The existence of this type of graph, called an evaluator, will then lead very directly to a proof of Theorem 1. Basically, an evaluator evaluates a boolean function, with the two colors acting as truth values. A negative sender in effect implements the function "not". For the general case, a rather technical definition is necessary.

Let G and H be graphs, and let Φ be a boolean function of k variables. Then a *(G, H, Φ)–evaluator* is a graph F with a *reference edge f*, *input edges $a_1, \ldots, a_k$*, and an *output edge b*, which has the following properties:

(i) If $G \neq H$, then f is always red in any (G, H)–good coloring. (Whether $G = H$ or not, we regard the color of f as denoting falsehood.)
(ii) For each of the 2^k possible colorings of the input edges, there is a (G, H)–good coloring of F for which the input edges are so colored.
(iii) In any (G, H)–good coloring of F, the color of b is $\Phi(a_1, \ldots, a_k)$, where the color of f represents falsehood and the opposite color represents truth.

We are now ready for a fundamental lemma. Of course, $\wedge$ represents 2–variable logical conjunction.

Lemma 2.2. *If $G, H \in \Gamma_3$, then $(G, H, \wedge)$–evaluators exist.*

Proof. Choose disjoint edges a_1, a_2, b, f. If $G \neq H$, attach a (G, H)–determiner to f, so (i) will be satisfied. Therefore, whether $G = H$ or not, we are assured that no copy of G in the graph to be constructed can be entirely in the color of falsehood in any good (that is, (G, H)–good) coloring. To simplify terminology, call an edge true or false according to its color.

Let G_1, G_2, G_3 be disjoint copies of G. Join one edge of G_1 to a_1 by a positive sender, and all but one of the other edges to f by positive senders. Treat G_2 in the same fashion, but join the first edge to a_2, not a_1. Now join one edge of G_3 to a_1 and another to a_2 by negative senders; again, join all but one of the other edges to f by positive senders. (If $G = K_3$, no edge is joined to f.) Use senders with widely–separated signal edges, so that no copies of G or H occur that are not entirely contained in one sender. Designate the edges of G_1, G_2, G_3 that are not presently joined to senders by e_1, e_2, e_3 respectively.

It is easy to see that in any good coloring of the graph we have built so far e_1 is true if a_1 is false, and is undetermined if a_1 is true. The same holds for e_2 and a_2. Also, in any good coloring e_3 is true if both a_1 and a_2 are true, and underdetermined otherwise (unless $G = K_3$, in which case e_3 is false if both a_1 and a_2 are false).

Now join e_1 and e_2 to b by negative senders, and e_3 to b by a positive sender. It is straightforward that in any good coloring, b is true if and only if a_1 and a_2 are, so (iii) is satisfied. Furthermore, any of the four possible ways of coloring a_1 and a_2 can exist in a good coloring, so (ii) is satisfied. □

The general case is now easy.

Lemma 2.3. *If $G, H \in \Gamma_3$ and Φ is any boolean function, then (G, H, Φ)–evaluators exist. Furthermore, if G and H are fixed, and Φ is expressed in terms of $\sim, \wedge$, and $\vee$ (say), then an evaluator can be constructed in time bounded by a polynomial in the length of the expression for Φ.*

Proof. Lemma 2.2 performs the desired construction for $\Phi = \wedge$. For $\Phi = \sim$, we just use a negative sender (as indicated before), except that, to be consistent with (i) we adjoin a disjoint edge f, and if necessary a determiner on f. (Of course, f serves no real purpose in this case). Once we have $\wedge$ and $\sim$, we can easily construct evaluators for $\vee$ or for any boolean function Φ of any number of variables. All we have to do is the following: Express Φ in terms of $\sim$ and $\wedge$ (which of course can be done), and create an appropriate evaluator for each occurance of $\sim$ or $\wedge$. Connect inputs to outputs in the appropriate manner with positive senders. Take the output edge of the highest–level operation ($a \sim$ or an v) to be b. Finally, connect the edges f with positive senders, designating one such f as the f to be used in the final evaluator. To avoid extraneons occurances of G or H, use senders with widely–separated signal edges.

Moreover, it is obvious that if Φ is expressed in terms of $\sim, \wedge$ and $\vee$, the construction of the evaluator can be performed in polynomial time. □

Proving Theorem 1 now only requires taking account of a few technical details.

Proof of Theorem 1: It is clear that NON–ARROWING $\in NP$, since if a coloring of the edges of a graph F is given, one can obviously check in polynomial time that no red G nor blue H occurs. (Note that, this is true only because G and H are fixed.) To show that NON–ARROWING is NP–complete, we reduce SATISFIABILITY to it. We state SATISFIABILITY as follows; this statement is obviously equivalent to that in Garey, Johnson (1979).

SATISFIABILITY
Instance: A (finite) boolean function Φ, expressed in conjunctive normal form in terms of $\sim, \wedge$, and $\vee$.
Question: Is there any assignment of truth values to the variables for which Φ is true?

Consider any instance of SATISFIABILITY. By Lemma 3, we can construct a (G, H, Φ)–evaluator in polynomial time. Now join its output edge to its reference edge by a negative sender. Clearly, some assignment of variables makes Φ true if and only if the graph we have constructed does not arrow the pair (G, H). Hence, we have reduced SATISFIABILITY to NON–ARROWING in polynomial time; therefore NON–ARROWING is NP–complete. □

3. Polynomial–Bounded Ramsey Problems

We will prove Theorem 2 by reducing NON–ARROWING for stars to the problem of finding a maximum c–matching, a polynomial–bounded problem. We state that problem formally as a decision problem.

C–MATCHING

Instance: A graph $G = (V, E)$ with a capacity $c(v)$ associated with each $v \in V_1$, and an integer K.

Question: Does there exist a set $E' \subset E$ of at least K edges with the property that for each vertex $v \in V$, no more than $c(v)$ edges of E' are incident with v?

This is easily seen to be a polynomial–bounded problem, since it can readily be reduced to the problem of finding a maximum matching in the ordinary sense; see Berge (1973), Chapter 8. (In the ordinary maximum matching problem, $c(v) = 1$ for all $v \in V$.)

Proof of Theorem 2: Let $G = K_{1,k}, H = K_{1,\ell}$; we will show how, for any graph F, to decide in polynomial time whether or not $F \to (G, H)$. If any vertex of F has degree at least $k + \ell - 1$, (which can be checked in polynomial time) it is obvious that $F \to (G, H)$, so we may assume that all vertices have degree no more than $k + \ell - 2$.

Now we construct a new graph F' from F. At each vertex of F which has degree d, attach $k + \ell - 2 - d$ free edges. The new graph F' has vertex set $V \cup V'$, where V is the vertex set of F, and where all vertices of V' have degree 1. Furthermore, all vertices of V have degree $k + \ell - 2$ in F'. It is clear that $F \not\to (G, H)$ if and only if $F' \not\to (G, H)$, since any good coloring of F can be extended to F' by coloring the free edges at each $v \in V$ so that the red and blue degrees become $k - 1$ and $\ell - 1$ respectively. In fact, any good coloring of F' has red degree exactly $k - 1$ at every vertex of V. It is obvious that F' can be constructed in polynomial time.

Consider the following c–matching problem on the graph $F : c(v) = k - 1$ if $v \in V, c(v) = 1$ if $v \in V', K = (k - 1)|V|$. This problem can be solved in polynomial time. Clearly, if the required c–matching exists, it can be taken to be the red subgraph of F', which induces a good coloring of F', and hence of F. On the other hand, if no such c–matching exists, it is clear that F' and F do not have good colorings. □

In the above proof, the transformation from F and F' was suggested by Jack Edmonds (personal communication).

If $G = H = K_{1,k}$, it is even easier to determine if $F \to (G, H)$. By Theorem 9 of Burr, Erdös, Lovász (1976), $F \to (K_{1,k}, K_{1,k})$ if and only if F has a vertex of degree at least $2k - 1$, or if F has a component which is regular of degree $2k - 2$ and has an odd number of points. It is quite possible that the case of stars in general has a similar solution.

Theorem 3 is even easier to prove than Theorem 2. We first need a definition. Say that F is (G, H)*–minimal* if $F \to (G, H)$, but $F' \not\to (G, H)$ for any proper subgraph F' of F.

Proof of Theorem 3: Let G be a matching nK_2, and let H be any graph. In Burr et al (1978) it is proved that only finitely many (G, H)–minimal graphs exist. Clearly, for any F it takes only time polynomial in the size of F to search it for any occurance of one of these (G,H)-minimal graphs. (Of course, it may well take exponential time in the sizes of G and H to find the set of minimal graphs, but since G and H are fixed, this does not matter.) □

4. Discussion

The above results leave open many questions, even aside from the fact that many types of G and H are not considered. For example, what if G and H are members of Γ_3, but are not fixed? By Theorem 1, this problem is NP-hard (again, see Garey, Johnson 1979 for terminology), and indeed it seems likely that this problem is, in some appropriate sense, strictly harder than the problems in NP. The only known methods for testing whether $F \to (G,H)$ when G and H (or even just G) are not fixed involve forming exponentially many colorings of F, and then testing each for a red G or blue H. (This is true even if backtracking is used.) Since testing for a red G, say, is in general an NP-complete problem, this method requires solving exponentially many NP-complete problems. When G and H are not fixed, NON–ARROWING belongs to the class called $\sum_2^p$ in Garey, Johnson (1979). It seems conceivable that NON–ARROWING might be proved $\sum_2^p$-complete. If, as is likely, $NP \neq \sum_2^p$, then such a result would show that NON–ARROWING is, in this sense, strictly harder than the NP-complete problems.

Further evidence for the above idea is that even in the very special case $F = K_n$, NON–ARROWING has been shown to be NP-hard Burr (1984). Indeed, it is shown in Burr (1984) that if H is a path, then the minimum n such that $K_n \to (G,H)$, called the *Ramsey Number*, often depends on the chromatic number of G, and of course, determining the chromatic number of G is NP-hard. It seems likely that in the general case, determining whether $F \to (G,H)$ is far harder than determining the chromatic numbers of the graphs involved.

In Theorem 2, G and H need not be fixed. In Theorem 3, they are fixed, and if this requirement is relaxed it is likely that the difficulty escalates. It is possible, given the present state of the author's knowledge, that the problem becomes $\sum_2^p$-complete, but this seems much less likely than for $G,H \in \Gamma_3$. In fact, if H is fixed, or is an arbitrary matching, this cannot be the case, unless $\sum_2^p = NP$. For in this case, NON–ARROWING is definitely a member of NP, since a proposed good coloring can be checked for forbidden matchings, and for occurrances of a blue H if H is fixed, in polynomial time.

Finally, it is worth mentioning that it is possible to use determiners, senders, and evaluators in a rather straightforward way to show that NON–ARROWING is undecidable for infinite graphs F. For details, see Burr (1984a).

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Constructive Ramsey Bounds and Intersection Theorems for Sets

Peter Frankl

Abstract

A classical result of Erdös says that $R(k,k) > 2^{k/2}$. However, Erdös's proof is probabilistic, and the only known graphs showing that $R(k,k)$ is greater than any polynomial of k were constructed via set-intersections. Here we show, that almost all such constructions yield non-polynomial lower bounds for $R(k,k)$.

1. Introduction

Let us recall that $R(k,k)$ is the minimal integer m, so that if the edges of the complete graph on m vertices are partitioned into 2 classes then one of them contains a complete subgraph on k vertices.

It is a classical result of Erdös (1947) and Erdös-Szekeres (1935) that

$$(1) \qquad 2^{k/2} < R(k,k) \le \binom{2k-2}{k-1} \text{ holds.}$$

Since his proof was probabilistic, Erdös raised the problem (cf. Erdös 1969) of giving explicit constructions showing that $R(k,k)$ grows faster than any polynomial of k. In Frankl (1977) such a construction was given. A similar construction in Frankl, Wilson (1981) provides a lower bound $R(k,k) \ge exp((1+o(1))\log^2 k/4\log\log\ k)$, i.e., rather poor in comparison with (1). Let us describe the construction.

Let X be an n-element set, r a positive integer, $r < n$, and L a subset of $\{0,1,\ldots,r-1\}$. Define a graph with vertex set $V = \binom{X}{r}$ – all r-subsets of X, and $A,B \in V$ forming an edge iff $|A \cap B| \in L$. This graph is denoted by $G(n,r,L)$. E.g. in Frankl, Wilson (1981) $n = p^3$, $r = p^2 - 1$ and $L = \{p-1, 2p-1, \ldots, p^2-p-1\}$ is used.

In what follows we always assume that $n > n_0(r)$. To given r there are 2^r choices for L. When saying almost all choices of L it is understood that r tends to infinity. Similarly, $o(1)$ denotes a quantity tending to zero as $r \to \infty$.

1.1 Theorem. *For almost all choices of L the size of the largest complete or empty subgraph in $G(n,r,L)$ is $n^{o(r)}$.*

2. Families of Sets with Prescribed Intersections

A family $\mathcal{F} \subset \binom{X}{r}$ is called an (n,r,L)-system if for all distinct $F, F' \in \mathcal{F}$ $|F, \cap F'| \in L$ holds. Also $m(n,r,L)$ denotes the maximum size of an (n,r,L)-system. For a survey on (n,r,L)-systems we refer the interested reader to Deza, Frankl (1983). In Frankl (1977) a recurrent bound for $m(n,r,L)$ is obtained. This would be sufficient for the purpose of the present paper. However, it will be simpler for the reader if we use a recent result of Füredi.
Consider families $\mathcal{A} \subset 2^{\{1,2,\ldots,r\}}$ satisfying the following assumptions:

(i) $|A| \in L$ for all $A \in \mathcal{A}$,
(ii) $(A \cap A') \in \mathcal{A}$ for all $A, A' \in \mathcal{A}$, i.e., $\mathcal{A}$ is closed under intersection.

A subset $B \subset \{1,2,\ldots,r\}$ is called free (with respect to $\mathcal{A}$) if $B \not\subset A$ holds for all $A \in \mathcal{A}$. The minimum size of a free subset is denoted by $b(\mathcal{A})$. Note that $0 \le b(\mathcal{A}) \le r$ holds.

2.1 Definition. *Let $a(L)$ denote the maximum of $b(\mathcal{A})$ over all $\mathcal{A} \subset 2^{\{1,\ldots,r\}}$ satisfying (i) and (ii).*

2.2 Theorem. (Füredi (1983)) *There exists a constant $c(r)$ depending only on r so that*

(2) $m(n,r,L) \le c(r)\binom{n}{b(L)}$ holds.

For a family $\mathcal{F}$ and a set S one defines $\mathcal{F}(S) = \{F - S : S \subset F \in \mathcal{F}\}$. Set also $\mathcal{F}_S = \{F \cap S : S \not\subset F \in \mathcal{F}\}$.

In analogy define $L(i) = \{l - i : i \le l \in L\}$. The following assertions are direct consequences of the above definitions.

(3) $b(\mathcal{A}) \le b(\mathcal{A}(S)) + b(\mathcal{A}_S) \le b(\mathcal{A}(S)) + |S|$,
(4) if $\mathcal{A}$ fulfills (i) and (ii), $(\{1,2,\ldots,r\} - S) \in \mathcal{A}$, then for all $B \in \mathcal{A}(S)$ $|B| \in (L \cap L(|S|))$ holds.

Let us state a special case of (3):

(5) $b(\mathcal{A}) \le b(\mathcal{A}_A) + 1$ holds for all maximal (for containment) $A \in \mathcal{A}$.

These propositions will be used to prove the following:

2.3 Theorem. *Suppose $L \subset \{0,\ldots,r-1\}$ and L contains no arithmetic progression of length t. Then $a(L) \le 3r/p$ holds, where p is defined by $r = tp^2 \log\ p$.*

2.4 Corollary. *Suppose L contains no arithmetic progressions of length t, $\frac{r}{t} = p^2 \log\ p$. Then $m(n,r,L) \le c(r)\binom{n}{3r/p}$ holds.*

3. The Proof of Theorem 2.3

Suppose $\mathcal{A} \subset 2^{\{1,\dots,r\}}$, $\mathcal{A}$ satisfies (i) and (ii). We must show that $b(\mathcal{A}) \le 3r/p$ holds. Let us set $r_0 = r$, $\mathcal{A}_0 = \mathcal{A}$, $L_0 = L$. Suppose that we have defined so far r_j, $L_j \subset \{0,\dots,r_j - 1\}$, $\mathcal{A}_j \subset 2^{\{1,\dots,r_j\}}$, $\mathcal{A}_j$ satisfies (i) and (ii) with $L = L_j$. If $b(\mathcal{A}_j) \le r/p$, then we stop. Otherwise let A be a maximal (for containment) member of $\mathcal{A}_j$ and set $S = \{1,\dots,r_j\} - A$.

(a) $|S| > p$, Define $r_{j+1} = |A|$, $L_{j+1} = L_j \cap \{0,\dots,|A|-1\}$, $\mathcal{A}_{j+1} = (\mathcal{A}_j)_A$.
(b) $|S| \le p$. Define $r_{j+1} = |A|$, $L_{j+1} = L_j \cap L_j(|S|), \mathcal{A}_{j+1} = \mathcal{A}_j(S)$.

First note that case a) occurs at most r/p times – in fact r decreases each time by more than p, thus it would become negative otherwise.

Next we claim that b) occurs at most r/p^2 times. To show this we use the easy observation that if a set K contains no arithmetic progression of length t, then for any nonzero integer α one has $|K \cap K(\alpha)| \le (1 - \frac{1}{t})\,|K|$. Therefore after r/p^2 applications of b) the corresponding L_j satisfies

$$|L_j| \le |L|\,(1-\frac{1}{t})^{r/p^2} < r(1-\frac{1}{t})^{t\log\ p} < r/p.$$

Consequently $b(\mathcal{A}_j) < r/p$. Thus we stop. Let $\mathcal{A}_q$ be the family with which we end up and denote by $s(a)$ ($s(b)$) the number of times a) (b)) occurred, respectively. In view of (3) and (5) we have $b(\mathcal{A}) \le b(\mathcal{A}_q) + s(a) + ps(b) \le 3r/p$. □

4. The Proof of Theorem 1.1

First note that the size of the largest complete subgraph in $G(n,r,L)$ is $m(n,r,L)$. Also the size of the largest empty subgraph is $m(n,r,\{0,\dots,r-1\} - L)$.

Recall as well that by a theorem of Erdös and Rado (1952) for almost all choices of L neither L nor its complement contain arithmetic progressions of length at least $3\log\ r$. Set $t = 3\log\ r$ and let K denote L or its complement. A little computation shows that $p \ge \frac{\sqrt{\frac{2}{3}r}}{\log\ r}$ and hence $\frac{3r}{p} \le \sqrt{\frac{27}{2}r\log\ r}$. Thus Corollary 2.4 yields $m(n,r,K) \le c(r)\binom{n}{\sqrt{\frac{27}{2}r\log\ r}} = n^{o(r)}$. □

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Ordinal Types in Ramsey Theory and Well-Partial-Ordering Theory

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There is a gap between the infinite Ramsey's theorem $\omega \to (\omega)^n_k$ and its finite version

$$R(n; \ell_1, \ldots, \ell_k) \to (\ell_1, \ldots, \ell_k)^n_k.$$

The finite Ramsey's theorem is much finer. In this paper we fill in the gap by defining "Ramsey numbers $R(n; \gamma_1, \ldots, \gamma_k)$" for arbitrary ordinals $\gamma_1, \ldots, \gamma_k$; these generalized "Ramsey numbers" are again ordinals, and their estimate is a quantitative strengthening of the infinite Ramsey's theorem. Actually, this is just a special case of our general definition of "Ramsey numbers" which is based on an axiomatic approach. The axioms themselves imply some estimates and other facts. To obtain sharper results, however, we have to consider more concrete situations. Besides the classical one already mentioned we investigate also the Canonical Ramsey Theorem, the Erdös–Szekeres theorem on monotone sequences and the well-partial–ordering (wpo) theory. In the last case, the "Ramsey numbers" generalize the so–caled types of wpo sets, a concept already studied in a great detail.

The "Ramsey numbers" are also closely connected with independence results in finite combinatorics. This fact has been already known for the types of wpo sets. The existence of "$R(n; \omega, \ldots, \omega)$" implies the Paris–Harrington modification of Ramsey's theorem. As one might expect from unprovability of this theorem in PA, it holds

$$\lim_{n \to \infty} R(n; \omega, \ldots, \omega) = \varepsilon_0.$$

Finally, we should remark that our approach is different from what is known as ordinal Ramsey theorems (like e.g. $\omega^2 \to (\omega^2, n)$).

1. Introduction

The well–known Finite Ramsey's Theorem says that, given natural numbers $n, k, \ell_1, \ldots, \ell_k$, there is a number $R(n; \ell_1, \ldots, \ell_k)$, called the *Ramsey number*, with the following property. If r is a coloring of n–element subsets of $\{1, \ldots, N\}$ by k colors (i.e., $r : [\{1, \ldots, N\}]^n \to \{1, \ldots, k\}$) such that every set $E \subseteq \{1, \ldots, N\}$ whose all n–subsets are colored i, has less than ℓ_i elements, then $N < R(n; \ell_1, \ldots, \ell_k)$. The infinite Ramsey's theorem (i.e. $\omega \to (\omega)^2_2$) gives no such number, it simply says that every infinite sequence contains an infinite homogeneous subsequence without indicating how sparse the homogeneous subsequences are. We propose a way to measure this sparsity based on a generalization of the notion of a Ramsey number. We refer to (Graham, Rothschild, Spencer 1980) or (Nešetřil 1987) for an exposition of Ramsey theory.

Our results are motivated by the well–partial–ordering theory, so let us start by recalling its rudiments. Let Q be a partially ordered set. A sequence $q_1, q_2, \ldots$ (finite or not) of elements of Q is called *good* if there are indices i, j such that $i < j$ and $q_i \leq q_j$, and is called *bad* otherwise. The set of all bad sequences of elements of Q is denoted by $\mathrm{Bad}(Q)$. The set Q is called *well–partially–ordered (wpo)* if every infinite sequence of elements of Q is good. Let us remark that this theory is often called the well–quasi–ordering one, because it is usually sufficient to work with quasi–orderings (i.e. reflexive and transitive relations) rather than with partial orderings. But since every quasi–ordering becomes a partial–ordering after identifying all elements x, y with $x \leq y \leq x$ we found it more convenient to work with partial orderings, and hence we call the existing theory the well–partial–ordering theory.

The well–partially-ordered sets have been studied for a while (see e.g. (Higman 1952), (Kruskal 1960), (Nash–Williams 1963) or (Kruskal 1972) for a survey). A recent major breakthrough was done by Robertson and Seymour (Robertson, Seymour) who proved the so–called Wagner's conjecture, an outstanding problem in the area which has been open for many decades.

The usual method in wpo theory is a minimal bad sequence argument, basically due to Nash–Williams. It is an induction–like argument, but it is highly nonconstructive. Trying to find a more constructive proofs for some wpo theorems we rediscovered the theory of types of wpo sets, initiated by de Jongh, Parikh (de Jongh, Parikh 1977) and Schmidt (Schmidt 1978), (Schmidt 1979). The constructive approach is as follows: To find an ordinal γ and a function $f : \mathrm{Bad}(Q) \to \gamma$ such that

$$f(q_1, \ldots, q_{n-1}) > f(q_1, \ldots, q_n)$$

for every $(q_1, \ldots, q_n) \in \mathrm{Bad}(Q)$. The least ordinal for which such a function exists, called the type of Q and denoted by $c(Q)$, turns out to be an interesting invariant which reflects the complexity of the wpo set and also provability and nonprovability of some wpo statements in certain logical systems.

There is another way of expressing $c(Q)$. A partially ordered set $(Q, \leq)$ is wpo if and only if every linear extension of $\leq$ is a well–ordering, in which case it has an ordinal type. It is a nontrivial fact that among all these linear extensions there is a maximal one and its ordinal type is exactly $c(Q)$. This may be viewed as a minimax theorem in wpo theory (see Section 4). De Jongh, Parikh (de Jongh, Parikh 1977) and Schmidt (Schmidt 1979) have computed the types of some wpo sets.

The above facts led us to introduce the type in a more general setting which includes both the Ramsey theory and well–partial–ordering theory. This is a bit more involved. The type of a Ramsey result is not a single number, but an ordinal function of the complexity of the partition. To clarify it let us consider the simplest example. Let $\mathcal{A} \subseteq [\omega]^2$ and assume that each infinite set $X \subseteq \omega$ contains an infinite subset Y such that $[Y]^2 \subseteq \mathcal{A}$. For γ an ordinal and $g : \omega^{<\omega} \to \gamma$ let us call a set $X \subseteq \omega$ $(\mathcal{A}, g)$–bad if for any $x_1 < x_2 < \ldots < x_n \in X$ such that $[\{x_1, \ldots, x_n\}]^2 \subseteq \mathcal{A}$ we have $g(x_1, \ldots, x_{n-1}) > g(x_1, \ldots, x_n)$. Roughly, the ordinal γ and the function g measure the "killing" of homogeneous subsets of X. Note that, by our assumption about $\mathcal{A}$, there is no infinite $(\mathcal{A}, g)$–bad set. The *type* $c_{\mathcal{A}}(\gamma)$ corresponding to $\mathcal{A}$ and γ is defined to be the least ordinal δ such that for every $g : \omega^{<\omega} \to \gamma$ there is $f : \omega^{<\omega} \to \delta$ such that for every $X \subseteq \omega$ the following holds:

if X is $(\mathcal{A}, g)$–bad, then $f(x_1, \ldots, x_{n-1}) > f(x_1, \ldots, x_n)$ for any $x_1 < \ldots < x_n \in X$.

Hence if the killing of homogeneous parts of X is measured by g, then the killing of X itself is measured by f.

Such a formulation is possible not only for the Ramsey's theorem, but also for the Canonical Ramsey Theorem of Erdös and Rado (Erdös, Rado 1950), for the Nash–Williams' Partition Theorem (Nash–Williams 1965) and in general for every Ramsey type theorem which has an infinitary version and where homogeneity can be recognized from finite segments.

In the following section we introduce the exact definitions. The key notion is that of a *sheaf*, which corresponds to a *partition* in Ramsey theory. We consider some basic examples and prove two theorems on abstract sheaves.

As in Ramsey theory we are not interested in a single partition but in a system of partitions of the same kind. In Section 3 we introduce the corresponding concepts of R–property and strong R–property. These definitions enable us to distinguish between "uniform" and "non–uniform" estimates (with respect to systems of partitions). For a broad class of systems (the so–called standard ones) the uniform and non–uniform cases coincide (under some obvious cardinality assumptions).

Section 4 is devoted to the wpo theory considered from our point of view. Some of the theorems presented in this section were known, but the proofs are new and simpler.

In Section 5 we investigate two possible generalizations of the Erdös–Szekeres theorem on monotone sequences. Similarly as in the finite case, the "Ramsey function" reveals as a product of its arguments, suitably defined for ordinals.

In Section 6 we give upper and lower bounds to the "Ramsey function" of the k–system which corresponds to classical Ramsey theory. As in the finite case the bounds are of the form of iterated exponentiation and in fact are obtained by similar methods. The lower bound requires a somewhat tricky modification of the Stepping-Up Lemma from Ramsey theory (see Graham, Rothschild, Spencer 1980).

In Section 7 we give upper bounds for the Canonical Ramsey Theorem of Erdös and Rado (Erdös, Rado 1950).

Let us introduce some terminology. If U is an arbitrary set, then $U^{<\omega}$, U^{ω}, U^n and $[U]^n$ denote the set of nonempty finite sequences of elements of U, the set of infinite sequences of elements of U, the Cartesian product of n copies of U and the set of n–element subsets of U, respectively. If $a \in U^{<\omega}$, then $|a|$ is the length of a. For $a = (a_1, a_2, \ldots)$, $b = (b_1, b_2, \ldots) \in U^{<\omega} \cup U^{\omega}$ we write $a \subseteq b$ if there are $j_1 < j_2 < \ldots$ such that $(a_1, a_2, \ldots) = (b_{j_1}, b_{j_2}, \ldots)$ and $a \ll b$ if $a \neq b$ and there is n such that $a = (a_1, \ldots, a_n) = (b_1, \ldots, b_n)$. In particular $a \subseteq a$, but not $a \ll a$. If $a \subseteq b$ we say that a is a *subsequence* of b and if $a \ll b$ we say that a is a *segment* of b. For $a \in U^{<\omega}$ we put $\downarrow a := \{b \in U^{<\omega} \mid b \subseteq a\}$. If $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_m) \in U^{<\omega}$, then

$$a.b := (a_1, \ldots, a_n, b_1, \ldots, b_m) \in U^{<\omega}.$$

If $f : X \to Y$ is any function and $M \subseteq X$, then $f \restriction M$ denotes the restriction of f to M. If $f : X \to Y$ is any function and $(x_1, \ldots, x_n) \in X$, then the value of f at $(x_1, \ldots, x_n)$ will be denoted by $f(x_1, \ldots, x_n)$, to avoid cumbersome notation like $f((x_1, \ldots, x_n))$.

A *tree* is a couple $(T, \leq)$, where T is a set and $\leq$ is a partial ordering on T such that for every $t \in T$ the set $\{t' \in T \mid t' \leq t\}$ is a finite chain. A *subtree* of T is a subset S of T such that $s_1 \leq s_2 \leq s_3$ and $s_1, s_3 \in S$ imply $s_2 \in S$, together with the restriction of $\leq$ to S. A frequently used tree will be $(U^{<\omega}, \leq)$, where $a \leq b$ iff either $a = b$ or $a \ll b$. We make the convention that subsets of $U^{<\omega}$ will be regarded as trees with this ordering. If $(T, \leq)$ is a tree and t, t' are distinct elements of T, we say that t' is a *successor* of t if $t \leq t'$ and there is no t'', distinct from t and t' such that $t \leq t'' \leq t'$.

Of great interest will be trees without an infinite chain: to such a tree one can find the least ordinal $\gamma_T < |T|^+$ such that there is a function $\psi_T : T \to \gamma_T$ satisfying $\psi_T(t) > \psi_T(t')$ for all $t, t' \in T$ with $t < t'$. The ordinal γ_T is called the *type* of T and the function ψ_T is called a *character* on T. If T is a tree, then T_t denotes the subtree of all $t' \in T$ such that $t \leq t'$. Let S, T be trees. A mapping $f : S \to T$ is called a *tree homomorphism* if it is strictly increasing, that is, if $s < s'$ implies $f(s) < f(s')$ for all $s, s' \in S$. If T contains no infinite chain and there is a tree homomorphism $f : S \to T$, then it can be seen by induction on γ_T that S contains no infinite chain and $\gamma_S \leq \gamma_T$.

The terminology about ordinals is a standard. We identify each ordinal with the set of its predecessors. If $\alpha, \beta \in On$, the class of all ordinals, and

$$\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \quad (\alpha_1 \geq \ldots \geq \alpha_n),$$

$$\beta = \omega^{\beta_1} + \ldots + \omega^{\beta_m} \quad (\beta_1 \geq \cdots \geq \beta_m)$$

are their Cantor's normal forms, then the *natural sum* of α, β is defined by

$$\alpha \oplus \beta := \omega^{\gamma_1} + \ldots + \omega^{\gamma_{n+m}},$$

where $\gamma_1 \geq \ldots \geq \gamma_{n+m}$ is a nonincreasing rearrangement of $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots \beta_m$ An equivalent definition is

$$\alpha \oplus \beta = \sup\{\alpha' \oplus \beta + 1, \alpha \oplus \beta' + 1 \mid \alpha' < \alpha,\ \beta' < \beta\}.$$

The *natural product* is defined by

$$\alpha \otimes \beta := \bigoplus\{\omega^{\alpha_i \oplus \beta_j} \mid i = 1, \ldots, n,\ j = 1, \ldots, m\}.$$

If λ is an ordinal, then a set $M \subseteq \lambda$ is called *cofinal* in λ if for every $\alpha \in \lambda$ there exists $\beta \in M$ such that $\beta \geq \alpha$. The *cofinality* of λ, denoted by $cf(\lambda)$ is the least ordinal α such that there exists a cofinal set $M \subseteq \lambda$ of order type α. If X is a set then by $|X|$ we denote the least ordinal which has the same cardinality as X, and by $|X|^+$ we denote the least ordinal which has cardinality bigger than X.

We list below some properties of $\oplus$ and $\otimes$ which will be used without any further reference.

(i) $\alpha \oplus \beta = \beta \oplus \alpha$, $\alpha \otimes \beta = \beta \otimes \alpha$,
(ii) $\alpha \oplus 1 = \alpha + 1$,
(iii) $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$, $\alpha \otimes (\beta \otimes \gamma) = (\alpha \otimes \beta) \otimes \gamma$,
(iv) if $\alpha \leq \gamma$, $\beta \leq \delta$ and one inequality is strict, then $\alpha \oplus \beta < \gamma \oplus \delta$ and $\alpha \otimes \beta < \gamma \otimes \delta$,
(v) if $\beta < \omega^\alpha$ and $\gamma < \omega^\alpha$, then $\beta \oplus \gamma < \omega^\alpha$,
(vi) $\alpha \otimes (\beta \oplus \gamma) = (\alpha \otimes \beta) \oplus (\alpha \otimes \gamma)$,
(vii) if $\beta < \omega^{\omega^\alpha}$ and $\gamma < \omega^{\omega^\alpha}$ then $\beta \otimes \gamma < \omega^{\omega^\alpha}$.

2. Sheaves

Definition 2.1. Let U be an infinite set. A *sheaf* in U is a set $A \subseteq U^{<\omega}$ such that $a \in A$ and $b \ll a$ implies $b \in A$. A *k-sheaf* in U is a k–tuple of sheaves and it is convenient to identify sheaves and 1–sheaves. A sheaf A is said to have the *R-property* (short for Ramsey property) if every infinite sequence in U contains an infinite subsequence each finite segment of which belongs to A. A k–sheaf $(A_1, \ldots, A_k)$ is said to have the *R-property* if the sheaf $A_1 \cup \ldots \cup A_k$ has the R–property. Equivalently $(A_1, \ldots, A_k)$ has the R–property if and only if for every infinite sequence p in U there are an index i with $1 \leq i \leq k$ and an infinite subsequence of p each finite segment of which belongs to A_i.

Example 2.2. For $r : U^n \to \{1, \dots, k\}$ and $i \in \{1, \dots, k\}$ we define

$$R_i^r := \{a \in U^{<\omega} \mid r(b) = i \text{ for every sequence } b \subseteq a \text{ of length } n\}.$$

Clearly $R^r := (R_1^r, \dots, R_k^r)$ is a k–sheaf, it will be called the *Ramsey k-sheaf* corresponding to the coloring r. By Ramsey's theorem this k–sheaf has the R–property.

Definition 2.3. An ordinal–valued function f defined on a set $M \subseteq U^{<\omega}$ is called a *killing* on M if $a, b \in M$ and $a \ll b$ imply $f(a) > f(b)$.

Let $\gamma_1, \dots, \gamma_k$ be ordinals. A k–tuple of functions $g = (g_1, \dots, g_k)$ is called a $(\gamma_1, \dots, \gamma_k)$*-testing* if $g_i : U^{<\omega} \to \gamma_i$. Let $A = (A_1, \dots, A_k)$ be a k–sheaf, let $\gamma_1, \dots, \gamma_k$ be ordinals and let $g = (g_1, \dots, g_k)$ be a $(\gamma_1, \dots, \gamma_k)$–testing. A sequence $a \in U^{<\omega}$ is called (A, g)*-bad* if each g_i is a killing on $\downarrow a \cap A_i$. The tree of (A, g)-bad sequences will be denoted by $\mathrm{Bad}(A, g)$.

Definition 2.4. Let $A = (A_1, \dots, A_k)$ be a k–sheaf, and let $\gamma_1, \dots, \gamma_k$ be ordinals. The *R–ordinal* $\Phi_A(\gamma_1, \dots, \gamma_k)$ is defined as the minimum ordinal γ such that for each $(\gamma_1, \dots, \gamma_k)$–testing g there exists a function $f : U^{<\omega} \to \gamma$, called the *R-character corresponding to A and g* (or simply corresponding to g if it is clear which k–sheaf is ment), such that one of the following equivalent conditions is satisfied.

(2.4a) If a is (A, g)–bad, then f is a killing on $\downarrow a$ for every $a \in U^{<\omega}$.
(2.4b) If $b \ll a$ and a is (A, g)–bad, then $f(b) > f(a)$.
(2.4c) f is a character on the tree of (A, g)–bad sequences.

If no such ordinal exists then the R–ordinal $\Phi_A(\gamma_1, \dots, \gamma_k)$ is undefined. In other words $\Phi_A(\gamma_1, \dots, \gamma_k)$ is well-defined if and only if for no $(\gamma_1, \dots, \gamma_k)$–testing g the tree $\mathrm{Bad}(A, g)$ contains an infinite chain and equals the least upper bound of the types of $\mathrm{Bad}(A, g)$ taken over all $(\gamma_1, \dots, \gamma_k)$–testings g.

Definition 2.5. The above least upper bound may be attained for some g. Such a g will be called the *universal* $(\gamma_1, \dots, \gamma_k)$*-testing*. If $\gamma_1, \dots, \gamma_k$ are finite then the supremum is always attained, namely for $g = (g_1, \dots, g_k)$ defined by $g_i(a) = (\gamma_i - |a|)^+$. We make the convention that for $\gamma_1, \dots, \gamma_k$ finite we shall understand by a universal $(\gamma_1, \dots, \gamma_k)$–testing the one defined above.

Theorem 2.6. *For a k-sheaf $A = (A_1, \dots, A_k)$, the following conditions are equivalent.*

(i) A has the R-property.
(ii) The R–ordinal $\Phi_A(\gamma_1, \dots, \gamma_k)$ is well-defined and is $\leq |U|^+$ for all ordinals $\gamma_1, \dots, \gamma_k$.
(iii) The R–ordinal $\Phi_A(|U|^+, \dots, |U|^+)$ is well-defined.
(iv) The R–ordinal $\Phi_A(\gamma_1, \dots, \gamma_k)$ is well-defined for all ordinals $\gamma_1, \dots, \gamma_k < |U|^+$.

Proof. (i) $\Rightarrow$ (ii): Let $\gamma_1, \dots, \gamma_k$ be given and let $g = (g_1, \dots, g_k)$ be a $(\gamma_1, \dots, \gamma_k)$–testing. By assumption, $\mathrm{Bad}(A, g)$ contains no infinite chain. Thus $f : U^{<\omega} \to |U|^+$ defined by

$$\begin{aligned} f(a) &= \psi_{\mathrm{Bad}(A,g)}(a) \quad \text{for } a \in \mathrm{Bad}(A,g) \\ &= 0 \qquad\qquad\qquad \text{otherwise} \end{aligned}$$

is the desired R–character corresponding to A and g.
(ii) $\Rightarrow$ (iii): Obvious.
(iii) $\Rightarrow$ (iv): Obvious.
(iv) $\Rightarrow$ (i): Let (iv) hold and suppose for a contradiction that A does not have the R–property. Then there exists an infinite sequence p in U such that for $i = 1, \dots, k$ the subtree S_i of $\mathrm{Bad}(A,g)$ consisting of all $\{a \subseteq p \mid a \in A_i\}$ contains no infinite chain. Put, for $i = 1, \dots, k$,

$$\begin{aligned} g_i(a) &= \psi_{S_i}(a) \quad \text{for } a \in S_i \\ &= 0 \qquad\quad\ \ \text{otherwise.} \end{aligned}$$

Then $g = (g_1, \dots, g_k)$ is a $(\gamma_{S_1}, \dots, \gamma_{S_k})$–testing such that every finite sequence $a \subseteq p$ is (A,g)–bad. By (iv) there exists an R–character f corresponding to A and g. Let $p_1 \ll p_2 \ll \dots \ll p$, then

$$f(p_1) > f(p_2) > \dots$$

is an infinite decreasing sequence of ordinals, a contradiction. □

Definition 2.7. Let $A = (A_1, \dots, A_k)$ be a k–sheaf, let $\gamma_1, \dots, \gamma_k$ be ordinals and let $a \in U^{<\omega}$. An $(A;\gamma_1, \dots, \gamma_k)$*-germ* on a is a k–tuple $g = (g_1, \dots, g_k)$ of functions $g_i : \downarrow a \to \gamma_i$ such that each g_i is a killing on $\downarrow a \cap A_i$. If $g = (g_1, \dots, g_k)$ is a $(\gamma_1, \dots, \gamma_k)$–testing, we define

$$g \restriction\!\downarrow a := (g_1 \restriction\!\downarrow a, \dots, g_k \restriction\!\downarrow a).$$

Thus if a is (A,g)–bad, then $g \restriction\!\downarrow a$ is an $(A;\gamma_1, \dots, \gamma_k)$–germ on a.

Theorem 2.8. *If a k–sheaf A has the R–property, then $\Phi_A(\gamma_1, \dots, \gamma_k) < |U|^+$ for all ordinals $\gamma_1, \dots, \gamma_k < |U|^+$.*

Proof. Let $A = (A_1, \dots, A_n)$. Consider the tree $(S, \leq)$ defined by
$S := \{(a,g) \mid a \in U^{<\omega}$ and g is an $(A;\gamma_1, \dots, \gamma_k)$–germ on $a\}$,
$(a,g) < (b,h)$ if $a \ll b$ and $g = h \restriction\!\downarrow a$.
We claim that S has no infinite chain. Indeed, let $\{(a^i, g^i)\}_{i=1}^{\infty}$ be an infinite chain in S. Let $a \in U^{\omega}$ be such that $a^i \ll a$ for every $i = 1, 2, \dots$. From the fact that A has the R–property it follows that there exist an integer j with $1 \leq j \leq k$ and an infinite subsequence $s \subseteq a$ each finite segment of which lies in A_j. Let $s_1 \ll s_2 \ll \dots \ll s$ and assume that $s_i \subseteq a^{j(i)}$, where $j(1) < j(2) < \dots$. By the definition of $\leq$ we obtain

$$\gamma_j > g_j^{j(1)}(s_1) > g_j^{j(2)}(s_2) > \dots,$$

which is a contradiction showing that S has no infinte chain. Now $\gamma_S < |U|^+$.

Let g be a $(\gamma_1,\ldots,\gamma_k)$–testing, we define $f:\ U^{<\omega}\to\gamma_S$ by

$$\begin{aligned} f(a) &= \psi_S(a, g \restriction\downarrow a) && \text{if } a \text{ is } (A,g)-\text{bad} \\ &= 0 && \text{otherwise.} \end{aligned}$$

Obviously, f is an R–character corresponding to A and g. □

3. Ramsey Systems

Definition 3.1. A *k–system* $\mathcal{M}$ in U is a set of k–sheaves. A k–system $\mathcal{M}$ is said to have the *R–property*, if each $A\in\mathcal{M}$ has the R-property. In that case we define the *R–ordinals*

$$\Phi_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) := \sup\{\Phi_A(\gamma_1,\ldots,\gamma_k) \mid A\in\mathcal{M}\}.$$

Definition 3.2. Let $r: U^n\to\{1,\ldots,k\}$. The Ramsey k–sheaf $R^r=(R^r_1,\ldots,R^r_k)$ was defined in 2.2. We put

$$\mathcal{R}^n_k := \{R^r \mid r: U^n\to\{1,\ldots,k\}\}.$$

Clearly, $\mathcal{R}^n_k$ has the R–property. It will be called the *Ramsey k-system.* We shall write $\Phi_n(\gamma_1,\ldots,\gamma_k)$ instead of $\Phi_{\mathcal{R}^n_k}(\gamma_1,\ldots,\gamma_k)$.

Proposition 3.3. *If $\gamma_1,\ldots,\gamma_k$ are finite, then*

$$\Phi_n(\gamma_1,\ldots,\gamma_k)+1 = R(n;\gamma_1+1,\ldots,\gamma_k+1),$$

the Ramsey number.

Proof. Let us consider the universal $(\gamma_1,\ldots,\gamma_k)$–testing g. Let $m=R(n;\gamma_1+1,\ldots,\gamma_k+1)-1$, let $a_1,\ldots,a_m$ be distinct elements of U and let

$$r' : [\{a_1,\ldots,a_m\}]^n\to\{1,\ldots,k\}$$

be such that there is no $E\subseteq\{a_1,\ldots,a_m\}$ with $|E|>\gamma_i$ and $r'(X)=i$ for every $X\in[E]^n$. Let $r: U^n\to\{1,\ldots,k\}$ be such that if $(a_{i_1},\ldots,a_{i_n})\subseteq(a_1,\ldots,a_m)$, then $r(a_{i_1},\ldots,a_{i_n})=r'(\{a_{i_1},\ldots,a_{i_n}\})$. Then $a=(a_1,\ldots,a_m)$ is (R^r,g)–bad; hence there is a killing $f:\ U^{<\omega}\to\Phi_{R^r}(\gamma_1,\ldots,\gamma_k)$ on $\downarrow a$, and thus

$$R(n;\gamma_1+1,\ldots,\gamma_k+1)-1=m\le\Phi_{R^r}(\gamma_1,\ldots,\gamma_k)\le\Phi_n(\gamma_1,\ldots,\gamma_k).$$

To show the converse inequality let a k–sheaf $R^r\in\mathcal{R}^n_k$ be given. We define $f:\ U^{<\omega}\to R(n;\gamma_1+1,\ldots,\gamma_k+1)$ by

$$f(a) := (R(n;\gamma_1+1,\ldots,\gamma_k+1)-|a|-1)^+,$$

and for $a=(a_1,\ldots,a_m)\in U^{<\omega}$ we define $\bar r := [\{1,\ldots,m\}]^n\to\{1,\ldots,k\}$ by $\bar r(\{i_1,\ldots,i_n\})=r(a_{i_1},\ldots,a_{i_n})$. If a is (R^r,g)–bad, then there is no $E\subseteq\{1,\ldots,m\}$ such that $|E|>\gamma_i$ and $\bar r(X)=i$ for every $X\in[E]^n$. Hence $m<R(n;\gamma_1+1,\ldots,\gamma_k+1)$ and thus f is a killing on $\downarrow a$. □

Remark 3.4. The R–ordinal $\Phi_n(\omega,\dots,\omega)$ corresponds to a statement whose finite miniaturization is the Paris–Harrington principle (Paris, Harrington 1977), i.e., the statement $\forall n\ \forall k\ \forall n_1,\dots,n_k\ \exists N$ such that for every k–coloring of $[\{1,\dots,N\}]^n$ there exists $A \subseteq \{1,\dots,N\}$ and $i \in \{1,\dots,k\}$ such that $|A| \geq n_i$ and $[A]^n$ is colored i, and, moreover, A is relatively large, i.e., $|A| > \min A$. Indeed, letting $U = \omega$ and $g = (g_1,\dots,g_k)$, where

$$g_i(a_1,\dots,a_m) = \max(\gamma_i - m, \min\{a_1,\dots,a_m\} - m, 0)$$

we see that the corresponding Ramsey character is a killing on sets of sequences without "monochromatic relatively large" subsequences.

One might expect that because of unprovability of the Paris–Harrington principe from PA it would hold

$$\sup\{\Phi_n(\underbrace{\omega,\dots,\omega}_{k\ \text{times}}) \mid n,k \in \omega\} = \varepsilon_0.$$

This is in fact true, we prove it in Section 6.

In this section we stay on a rather abstract level. Of course, we have an analogy of 2.7 for Ramsey systems, but there is no general analogy of 2.8. We search for a restricted class of Ramsey systems for which such a statement would hold.

Definition 3.5. A k–system $\mathcal{M}$ is said to have the *strong R-property* if for every infinite sequence

$$A^1 = (A^1_1,\dots,A^1_k), A^2 = (A^2_1,\dots,A^2_k),\dots$$

of elemets of $\mathcal{M}$ and for each infinite sequence $p \in U^\omega$ there exists an infinite subsequence $s \subseteq p$ such that for each finite segment $a \ll s$ there exists an $i \geq |a|$ with $a \in A^i_1 \cup \dots \cup A^i_k$. Note that the condition actually implies a to be in A^i_j for $j \in \{1,\dots,k\}$ fixed and infinitely many i.

Definition 3.6. Let $\mathcal{M}$ be a k–system and let $\gamma_1,\dots,\gamma_k$ be ordinals. We define the *germ tree* $T(\mathcal{M};\gamma_1,\dots,\gamma_k) = (T,\leq)$ by

$T := \{(a,g) \mid a \in U^{<\omega}$ and there is $A \in \mathcal{M}$ such that g is an $(A;\gamma_1,\dots,\gamma_k)$–germ on $a\}$,

$(a,g) < (b,h)$ if $a \ll b$ and $g = h \restriction\downarrow a$.

If the germ tree has no infinite chain we define the *strong R-ordinal* by

$$\overline{\Phi}_{\mathcal{M}}(\gamma_1,\dots,\gamma_k) := \gamma_{T(\mathcal{M};\gamma_1,\dots,\gamma_k)}$$

and we write $\overline{\Phi}_n(\gamma_1,\dots,\gamma_k)$ instead of $\overline{\Phi}_{\mathcal{R}^n_k}(\gamma_1,..,\gamma_k)$. In general the existence of $\overline{\Phi}_{\mathcal{M}}$ implies the existence of $\Phi_{\mathcal{M}}$, $\overline{\Phi}_{\mathcal{M}} \geq \Phi_{\mathcal{M}}$, and nothing more holds.

Theorem 3.7. *A k-system $\mathcal{M}$ has the strong R–property if and only if the germ tree $T(\mathcal{M};\gamma_1,\dots,\gamma_k)$ has no infinite chain for all $\gamma_1,\dots,\gamma_k < |U|^+$. In that case we have*

$$\Phi_{\mathcal{M}}(\gamma_1,\dots,\gamma_k) \leq \overline{\Phi}_{\mathcal{M}}(\gamma_1,\dots,\gamma_k)$$

for all ordinals $\gamma_1,\dots,\gamma_k$. *Moreover* $\overline{\Phi}_{\mathcal{M}}(\gamma_1,\dots,\gamma_k) < |U|^+$ *for all ordinals* $\gamma_1,\dots,\gamma_k < |U|^+$.

Proof. The last statement follows by a cardinality argument. To see $\Phi_{\mathcal{M}}(\gamma_1,\dots,\gamma_k) \le \overline{\Phi}_{\mathcal{M}}(\gamma_1,\dots,\gamma_k)$ let a $(\gamma_1,\dots,\gamma_k)$–testing g be given, and let $A \in \mathcal{M}$. We define $f:\ U^{<\omega} \to \overline{\Phi}_{\mathcal{M}}(\gamma_1,\dots,\gamma_k)$ by

$$\begin{aligned} f(a) &= \psi_{T(\mathcal{M};\gamma_1,\dots,\gamma_k)}(a, g \restriction\downarrow a) && \text{if } a \text{ is } (A,g)\text{–bad} \\ &= 0 && \text{otherwise.} \end{aligned}$$

It is easily seen that f is an R–character corresponding to g.

Let us pass to the proof of the equivalence.

$\Rightarrow$: We must prove that $T(\mathcal{M};\gamma_1,\dots,\gamma_k)$ has no infinite chain. Suppose that $(a^1,g^1),(a^2,g^2),\dots$ is an infinite chain in $T(\mathcal{M};\gamma_1,\dots,\gamma_k)$, let $g^i=(g^i_1,\dots,g^i_k)$ be an $(A^i,\gamma_1,\dots,\gamma_k)$–germ on a^i, and let $A^i=(A^i_1,\dots,A^i_k)$. Let $a\in U^\omega$ be such that $a^i \ll a$ for all $i=1,2,\dots$.By the strong R–property there exists an infinite sequence $s\subseteq a$, a sequence $s_1\ll s_2\ll\dots$ of segments of s, an integer $j\in\{1,...,k\}$ and an increasing sequence $n(1)<n(2)<\dots$ of integers such that $s_i\in A^{n(i)}_j$ for $i=1,2,\dots$. By the definition of $T(\mathcal{M};\gamma_1,\dots,\gamma_k)$ we have

$$\gamma_j > g^{n(1)}_j(s_1) > g^{n(2)}_j(s_2) > \cdots ,$$

which is a contradiction.

$\Leftarrow$: Let $A^n=(A^n_1,\dots,A^n_k)$ be a sequence in $\mathcal{M}$ and let $p\in U^\omega$ be an infinite sequence such that each infinite subsequence s of p has a finite segment $a\ll s$ such that $a\in A^n_1\cup\ldots\cup A^n_k$ implies $n<|a|$. For $j=1,\dots,k$ let S_j be the subtree of $U^{<\omega}$ consisting of all $a\subseteq p$ such that $a\in A^n_j$ for some $\gamma_1,\dots,n\ge|a|$. It follows from the assumption that S_j contains no infinite chain. Now for $a\ll p$ let $g^a_j:\downarrow a\to\gamma_{S_j}$ be defined by

$$\begin{aligned} g^a_j &= \psi_{S_j}(b) && \text{if } b\in S_j \\ &= 0 && \text{otherwise.} \end{aligned}$$

We see easily that $g^a=(g^a_1,\dots,g^a_k)$ is an $(A^{|a|};\gamma_{S_1},\dots,\gamma_{S_k})$–germ on a. Thus, $\{(a,g^a)\}_{a\ll p}$ is an infinite chain in $T(\mathcal{M};\gamma_1,\dots,\gamma_k)$, showing that $\overline{\Phi}_{\mathcal{M}}(\gamma_{S_1},\dots,\gamma_{S_k})$ does not exist. □

Proposition 3.8. *The Ramsey* k*–system* $\mathcal{R}^n_k$ *has the strong* R*–property.*

Proof. This follows either from 3.17 or from the estimates of $\overline{\Phi}_n$ given below, but we give a direct proof. Let $(A^{r_j})_{j\in\omega}$ be a sequence of elements of $\mathcal{R}^n_k$. Let us choose a non–trivial ultrafilter $\mathcal{U}$ on ω and define $r: U^n\to\{1,\dots,k\}$ by

$$r(x_1,\dots,x_n)=i \ \text{ iff } \{j \mid r_j(x_1,\dots,x_n)=i\}\in\mathcal{U}.$$

By Ramsey's theorem there is for each infinite sequence p in U an infinite subsequence $s\subseteq p$ such that every finite segment a of s belongs to $A^r_1\cup\ldots\cup A^r_k$.

By the definition of r there is $j \geq |a|$ such that $r_j(x_1,\ldots,x_n) = r(x_1,\ldots,x_n)$ for all subsequences $(x_1,\ldots,x_n) \subseteq a$, and hence a belongs to $A_1^{r_j} \cup \ldots \cup A_k^{r_j}$. $\square$

Theorem 3.9. *If $\mathcal{M}$ is finite and $\gamma_1,\ldots,\gamma_k$ are also finite, then $\Phi_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k)$ exists if and only if $\overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k)$ exists and $\Phi_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) = \overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k)$.*

Proof. Let $\mathcal{M} = \{A_1,\ldots,A_m\}$, let $T = T(\mathcal{M};\gamma_1,\ldots,\gamma_k)$ be the germ tree and for $i = 1,\ldots,m$ let T_i be the subtree of T consisting of all $(a,g) \in T$ such that g is an $(A_i;\gamma_1,\ldots,\gamma_k)$–germ on a. Then each T_i is downwards–closed in T (i.e., $(a,g) \leq (b,h)$ and $(b,h) \in T_i$ imply $(a,g) \in T_i$), and hence

$$\begin{aligned}\overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) = \gamma_T &= \max\{\gamma_{T_i} \mid 1 \leq i \leq m\}\\ &= \max\{\overline{\Phi}_{A_i}(\gamma_1,\ldots,\gamma_k) \mid 1 \leq i \leq j\}.\end{aligned}$$

Thus by Theorem 3.7 it is sufficient to prove that $\Phi_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) \geq \overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k)$ for $|\mathcal{M}| = 1$. We proceed by transfinite induction with the following induction hypothesis.

(H_α) *For any natural number k, any k–sheaf A and any finite numbers $\gamma_1,\ldots,\gamma_k$ such that $\Phi_A(\gamma_1,\ldots,\gamma_k) \leq \alpha$ the inequality $\Phi_A(\gamma_1,\ldots,\gamma_k) \geq \overline{\Phi}_A(\gamma_1,\ldots,\gamma_k)$ holds.*

Assume that (H_β) is true for any $\beta < \alpha$ and let $k, A, \gamma_1,\ldots,\gamma_k$ such that $\Phi_A(\gamma_1,\ldots,\gamma_k) = \alpha$ be given. We denote by T the germ tree $T(\{A\};\gamma_1,\ldots,\gamma_k)$. Let $a \in U^{<\omega}$ be a one–element sequence and let $g = (g_1,\ldots,g_k)$ be an $(A;\gamma_1,\ldots,\gamma_k)$–germ on a. We define a $2k$–sheaf $A^a = (A_1^a,\ldots A_{2k}^a)$ by $A_i^a = A_i$, $A_{k+i}^a = \{b \in U^{<\omega} \mid a.b \in A_i\}$ $(i = 1,\ldots,k)$ and a tree homomorphism

$$H : T_{(a,g)} \to T(\{A^a\};\gamma_1,\ldots,\gamma_k;g_1(a),\ldots,g_k(a))$$

by $H(a.b,h) = (b,\overline{h})$, where $\overline{h}_i(x) = h_i(x)$ and $\overline{h}_{k+i}(x) = h_i(a.x)$ for $x \subseteq b$. Thus

$$\overline{\Phi}_{\{A\}}(\gamma_1,\ldots,\gamma_k) \leq \sup\{\overline{\Phi}_{\{A^a\}}(\gamma_1,\ldots,\gamma_k,g_1(a),\ldots,g_k(a)) + 1 \mid (a,g) \in T\}.$$

Now we are going to estimate the corresponding R–ordinals. Given finite numbers $\gamma_1,\ldots,\gamma_{2k}$ such that $\gamma_i > \gamma_{i+k}$ $(i = 1,\ldots,k)$, let $\overline{h} = (\overline{h}_1,\ldots,\overline{h}_{2k})$ be the universal $(\gamma_1,\ldots,\gamma_{2k})$–testing. We define a $(\gamma_1,\ldots,\gamma_k)$–testing $h = (h_1,\ldots,h_k)$ by

$$\begin{aligned}h_i(x) &= \overline{h}_i(x) && \text{if neither } a = x \text{ nor } a \ll x\\ &= \gamma_{k+i} && \text{if } x = a\\ &= \overline{h}_{k+i}(b) && \text{if } x = a.b.\end{aligned}$$

It is easily seen that if $b \in U^{<\omega}$ is $(A^a,\overline{h})$–bad, then $a.b$ is (A,h)–bad. If f is an R–character corresponding to A and h, we may define $\overline{f} := U^{<\omega} \to f(a)$ by $\overline{f}(b) = f(a.b)$. Then $\overline{f}$ is clearly an R–character corresponding to A and $\overline{h}$, thus showing $\Phi_{\{A^a\}}(\gamma_1,\ldots,\gamma_{2k}) = \gamma_{\mathrm{Bad}(A^a,\bar{h})} < \gamma_{\mathrm{Bad}(A,h)} \le \Phi_{\{A\}}(\gamma_1,\ldots,\gamma_k)$. (Here the finiteness of $\gamma_1,\ldots,\gamma_k$ is crucial.) Hence we may use the induction hypothesis to conclude that (H_α) is true. □

Theorem 3.10. *If $\Phi_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) < \omega$, then $\overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) = \Phi_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k)$.*

Proof. If $g = (g_1,\ldots,g_k)$ is an $(A;\gamma_1,\ldots,\gamma_k)$–germ on $a = (a_1,\ldots,a_m)$, then extending each g_i by $g_i(b) = 0$ for $b \not\in\downarrow a$ we see that a is (A,g)–bad. Hence $m \le \Phi_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k)$ and consequently $\overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) \le \Phi_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k)$. The converse inequality follows from 3.7. □

Remark 3.11. For no k–system $\mathcal{M}$ one can expect $\Phi_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) = \overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k)$ for *all* ordinals $\gamma_1,\ldots,\gamma_k$, since $\Phi_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) \le |U|^+$ by 2.6(ii), while it is an easy exercise that $\overline{\Phi}_{\mathcal{M}}(\gamma_1,\ldots,\gamma_k) \ge \min\{\gamma_1,\ldots,\gamma_k\}$. On the other hand it is easy to construct k–systems such that $\Phi_{\mathcal{M}}(1) < \overline{\Phi}_{\mathcal{M}}(1)$, or that $\Phi_{\mathcal{M}}(1)$ exists and $\overline{\Phi}_{\mathcal{M}}(1)$ does not. Namely, for $a = (a_1,\ldots,a_m) \in U^{<\omega}$ we define

$$A^a := \{(x_1,\ldots,x_n) \in U^{<\omega} \mid \text{ either } x_1 = \ldots = x_n, \text{ or } x_i \ne a_j \text{ for all } i,j\}.$$

Clearly $\Phi_{A^a}(1) = |a|$. Thus letting $\mathcal{M} = \{A^a \mid a \in U\}$ we have $\Phi_{\mathcal{M}}(1) = \omega$, while the corresponding strong R–ordinal does not exist. If $S \subseteq U^{<\omega}$ is a tree without infinite chains, then for $\mathcal{M}(S) := \{A^a \mid a \in S\}$ we have $\Phi_{\mathcal{M}(S)}(1) \le \omega$, while $\overline{\Phi}_{\mathcal{M}(S)}(1) \ge \gamma_S$.

An example of a k–system consisting only of one k–sheaf for which the R–ordinals and strong R–ordinals differ for small ordinals is given in the next section (see Remark 4.12).

In the rest of this section we prove that for a large class of k–systems the R–ordinals and strong R–ordinals coincide for all ordinals $\gamma_1,\ldots,\gamma_k < |U|^+$.

Definition 3.12. We say that a k–system $\mathcal{M}$ is *movable*, if for any k–sheaf $A = (A_1,\ldots,A_k) \in \mathcal{M}$, any sequence $(a_1,\ldots,a_m) \in U^{<\omega}$ and any injective sequence $(b_1,\ldots,b_m) \in U^{<\omega}$ there exists a k–sheaf $B = (B_1,\ldots,B_k) \in \mathcal{M}$ such that if $(b_{i_1},\ldots,b_{i_p}) \in B_j$ for some $1 \le i_1 < i_2 < \ldots < i_p \le m$ and $1 \le j \le k$, then $(a_{i_1},\ldots,a_{i_p}) \in A_j$.

Let $A = (A_1,\ldots,A_k)$, $B = (B_1,\ldots,B_k)$ be two k–sheaves and let $V \subseteq U^{<\omega}$. We say that $A = B$ on V if for every sequence $a \in V$ and any $j \in \{1,\ldots,k\}$ we have $a \in A_j$ if and only if $a \in B_j$. We say that a k–system $\mathcal{M}$ has the *concatenation property*, if for any family $\{A^\alpha\}_{\alpha\in\Lambda}$ of elements of $\mathcal{M}$, any family of subsets $\{V_\alpha\}_{\alpha\in\Lambda}$ of $U^{<\omega}$ such that each V_α is closed under subsequences and $A^\alpha = A^\beta$ on $V_\alpha \cap V_\beta$ for any $\alpha,\beta \in \Lambda$ there exists a k–sheaf $A \in \mathcal{M}$ such that $A^\alpha = A$ on V_α for every $\alpha \in \Lambda$.

A movable k–system which has the concatenation property will be called *standard*. For example, the Ramsey k–system is standard.

Theorem 3.13. *Let $\mathcal{M}$ be a movable k–system satisfying the strong R–property and let V be an infinite subset of U. For $A = (A_1,\ldots,A_k) \in \mathcal{M}$ we put $A^V = (A_1^V,\ldots,A_k^V)$, where*

$$A_i^V := A_i \cap V^{<\omega}$$

and further

$$\mathcal{M}^V := \{A^V \mid A \in \mathcal{M}\}.$$

Then $\mathcal{M}^V$ satisfies the strong R–property (in V) and

$$\overline{\Phi}_{\mathcal{M}}(\gamma_1, \dots, \gamma_k) = \overline{\Phi}_{\mathcal{M}^V}(\gamma_1, \dots, \gamma_k)$$

for any ordinals $\gamma_1, \dots, \gamma_k$.

Proof. Let $a_1, a_2, \dots$ be distinct elements of V. Let $(b,h) \in T(\mathcal{M}; \gamma_1, \dots, \gamma_k)$, where $b = (b_1, \dots, b_m)$ and $h = (h_1, \dots, h_k)$; we define $H(b,h) = (a,g)$ by $a = (a_1, \dots, a_m)$ and $g = (g_1, \dots, g_k)$, where

$$g_i(a_{i_1}, \dots, a_{i_p}) = h_i(b_{i_1}, \dots, b_{i_p}).$$

Let S be the range of H. By movability, S is a subtree of $T(\mathcal{M}^V; \gamma_1, \dots, \gamma_k)$ and H is a tree homomorphism, showing that

$$\overline{\Phi}_{\mathcal{M}}(\gamma_1, \dots, \gamma_k) = \gamma_{T(\mathcal{M};\gamma_1,\dots,\gamma_k)} \le \gamma_S \le \overline{\Phi}_{\mathcal{M}^V}(\gamma_1, \dots, \gamma_k).$$

Since the converse inequality is obvious, we are done. □

Definition 3.14. Let T be a subtree of the germ–tree $T(\mathcal{M}; \gamma_1, \dots, \gamma_k)$. We say that T is *simple* if whenever $(a,g), (b,h) \in T$ are such that a and b have the same last term, then $(a,g) = (b,h)$. Similarly, $S \subseteq U^{<\omega}$ is called *simple* if whenever $a, b \in S$ have the same last term, then $a = b$.

Lemma 3.15. *If $\mathcal{M}$ is a movable k–system satisfying the strong R–property and $\gamma_1, \dots, \gamma_k < |U|^+$, then there exists a simple subtree S of the germ tree $T(\mathcal{M}; \gamma_1, \dots, \gamma_k)$ such that $\gamma_S = \overline{\Phi}_{\mathcal{M}}(\gamma_1, \dots, \gamma_k)$.*

Proof. Let us denote by T the germ tree $T(\mathcal{M}; \gamma_1, \dots, \gamma_k)$, let $I : T \to U$ be defined by $I((a_1, \dots, a_m), g) = a_m$ and let $J : T \to U$ be a bijection. For a $(\gamma_1, \dots, \gamma_k)$–testing $g = (g_1, \dots, g_k)$ we define a $(\gamma_1, \dots, \gamma_k)$–testing $g' = (g'_1, \dots, g'_k)$ by

$$g'_i(a_1, \dots, a_m) = g(IJ^{-1}(a_1), \dots, IJ^{-1}(a_m))$$

and for $(a,g) = ((a_1, \dots, a_m), g) \in T$ we define

$$\begin{aligned} a' =&(J((a_1), g \restriction\downarrow (a_1)), J((a_1,a_2), g \restriction\downarrow (a_1,a_2)), \dots \\ &\dots, J((a_1, \dots, a_m), g \restriction\downarrow (a_1, \dots, a_m))). \end{aligned}$$

Now if g is an $(A; \gamma_1, \dots, \gamma_k)$–germ on a, there exists by movability a k–sheaf $A' \in \mathcal{M}$ such that g' is an $(A'; \gamma_1, \dots, \gamma_k)$–germ on a'. Hence the mapping $H : T \to T$ defined by $H(a,g) = (a', g')$ is a tree homomorphism. Thus denoting by S the image of T under H we see that

$$\overline{\Phi}_{\mathcal{M}}(\gamma_1, \dots, \gamma_k) \ge \gamma_S \ge \gamma_T = \overline{\Phi}_{\mathcal{M}}(\gamma_1, \dots, \gamma_k),$$

moreover S is clearly simple. □

Theorem 3.16. *A k–system which has the R–property and the concatenation property has the strong R–property.*

Proof. Let $\mathcal{M}$ be as above, let $\gamma_1, \ldots, \gamma_k$ be ordinals and suppose that $\{(a^i, g^i)\}_{i=1}^{\infty}$ is an infinite chain in $T(\mathcal{M}; \gamma_1, \ldots, \gamma_k)$. Let $a^i = (x_1, \ldots, x_i)$, and let $g^i = (g_1^i, \ldots, g_k^i)$ be an $(A^i; \gamma_1, \ldots, \gamma_k)$–germ on a^i, where $A^i \in \mathcal{M}$. By an easy compactness argument we may choose an increasing sequence $i_1, i_2, \ldots$ such that

$$A^{i_n} = A^{i_m} \text{ on } \downarrow a^n \text{ for any } m \geq n.$$

Now we apply the concatenation property to the family of sheaves $\{A^{i_n}\}_n$ and to the family of sets $\{\downarrow a^n\}_n$. We obtain a sheaf $A \in \mathcal{M}$ with

$$A = A^{i_n} \text{ on } \downarrow a^n.$$

For $x \subseteq (x_1, \ldots, x_n)$ we put $g_j(x) = g_j^n(x)$ $(j = 1, \ldots, k)$. Then $g = (g_1, \ldots, g_k)$ is well–defined and the sequence $(x_1, \ldots, x_n, \ldots)$ is (A, g)–bad, contrary to the assumption that $\mathcal{M}$ has the R–property. Thus the germ tree $T(\mathcal{M}; \gamma_1, \ldots, \gamma_k)$ has no infinite chain and so $\mathcal{M}$ has the strong R–property by 3.7. □

Theorem 3.17. *If $\mathcal{M}$ is standard and has the R-property, and $\gamma_1, \ldots, \gamma_k < |U|^+$, then*

$$\Phi_{\mathcal{M}}(\gamma_1, \ldots, \gamma_k) = \overline{\Phi}_{\mathcal{M}}(\gamma_1, \ldots, \gamma_k).$$

Proof. Let T be a simple subtree of $T(\mathcal{M}; \gamma_1, \ldots, \gamma_k)$ of type $\overline{\Phi}_{\mathcal{M}}(\gamma_1, \ldots, \gamma_k)$, which exists by 3.15. We shall find a subtree S of T of the same type, a $(\gamma_1, \ldots, \gamma_k)$–testing h and a k–sheaf $A \in \mathcal{M}$ such that h is an $(A; \gamma_1, \ldots, \gamma_k)$–germ on a for every $(a, g) \in S$.

Let T_Π be the tree obtained from T by formally adding a least element Π. For $z \in T_\Pi$ we define

$$T(z) := \{z' \in T \mid z \leq z' \text{ or } z' \leq z\}$$

(so that $T(\Pi) = T$) and $V(z) := \bigcup\{\downarrow a \mid (a, g) \in T(z)\}$. Let us observe that by simplicity if $z = (a, g) \in T$ and z', z'' are its distinct successors, then $V(z') \cap V(z'') = \downarrow a$. We shall construct for every $z \in T_\Pi$ a k–sheaf A^z, a subtree $S(z)$ of $T(z)$ and a k–tuple $g^z = (g_1^z, \ldots, g_k^z)$ of functions such that

(3.17a) for every $(a, g) \in S(z)$, g^z is an $(A^z; \gamma_k < |U|^+$. In that case we have $\gamma_1, \ldots, \gamma_k)$–germ on a, and

(3.17b) $\gamma_{S(z)} = \gamma_{T(z)}$.

If $z = (a, g) \in T$ is such that $\psi_T(z) = 0$, then let $S(z) = T(z)$, $g^z = g$ and let A^z be such that g is an $(A^z; \gamma_1, \ldots, \gamma_k)$–germ on a. If $\psi_T(z)$ is a successor ordinal, then $\psi_T(z) = \psi_T(z') + 1$ for some $z \leq z'$. We put $S(z) := S(z')$, $A^z := A^{z'}$ and $g^z := g^{z'}$. Finally if $\psi_T(z)$ is a limit ordinal, let $(z_\alpha)_{\alpha \in \Lambda}$ be successors of z in T such that $\sup\{\psi_T(z_\alpha) \mid \alpha \in \Lambda\} = \psi_T(z)$. There is a subset $\Lambda' \subseteq \Lambda$ such that

(3.17c) $\sup\{\psi_T(z_\alpha) \mid \alpha \in \Lambda'\} = \psi_T(z)$

and if $z \neq \Pi$, say $z = (a, g) \in T$, then for every $b \subseteq a$, every $\alpha, \beta \in \Lambda'$ and every $i \in \{1, \ldots, k\}$, $b \in A_i^{z_\alpha}$ iff $b \in A_i^{z_\beta}$. Hence $A^{z_\alpha} = A^{z_\beta}$ on $V(z_\alpha) \cap V(z_\beta)$. By the concatenation property there exists a k–sheaf $A^z \in \mathcal{M}$ such that $A^z = A^{z_\alpha}$ on $V(z_\alpha)$ for every $\alpha \in \Lambda'$. We define $g^z = (g_1^z, \ldots, g_k^z)$ by

$$\begin{aligned} g_i^z(c) &= g_i^{z_\alpha}(c) && \text{if } c \in V(z_\alpha) \text{ for some } \alpha \in \Lambda' \\ &= 0 && \text{if no such } \alpha \text{ exists.} \end{aligned}$$

Then g^z is well–defined and it follows that (3.17a) is satisfied for $S(z) := \bigcup \{S(z_\alpha) \mid \alpha \in \Lambda'\}$; condition (3.17b) follows from (3.17c).

Now put $S := S(\Pi)$, $h := g^\Pi$ and $A := A^\Pi$. Then S, h, A are as claimed. Let $S' = \{\, a \mid (a, g) \in S \text{ for some } g \,\}$. Then $S' \subseteq \mathrm{Bad}(A, h)$, and hence

$$\overline{\Phi}_{\mathcal{M}}(\gamma_1, \ldots, \gamma_k) = \gamma_{S'} \leq \gamma_{\mathrm{Bad}(A,h)} \leq \Phi_{\mathcal{M}}(\gamma_1, \ldots, \gamma_k).$$

The converse inequality follows from 3.7. □

Corollary 3.18 of the proof. *If $\mathcal{M}$ is standard and has the R–property, and if $\gamma_1, \ldots, \gamma_k < |U|^+$, then there exist $A \in \mathcal{M}$, a $(\gamma_1, \ldots, \gamma_k)$–testing g and a simple subtree S of* $\mathrm{Bad}(A, g)$ *such that*

$$\overline{\Phi}_{\mathcal{M}}(\gamma_1, \ldots, \gamma_k) = \Phi_{\mathcal{M}}(\gamma_1, \ldots, \gamma_k) = \Phi_A(\gamma_1, \ldots, \gamma_k) = \gamma_{\mathrm{Bad}(A,g)} = \gamma_S.$$

In particular, there exists a universal $(\gamma_1, \ldots, \gamma_k)$–testing.

4. Well-Partial-Ordering

Definition 4.1. Let $(Q, \leq)$ be a partially ordered set. A sequence $q_1, q_2, \ldots$ (finite or not) of elements of Q is called *good* if there are indices i, j such that $i < j$ and $q_i \leq q_j$ and is called *bad* otherwise. The set Q is called *well-partially-ordered (wpo)* if every bad sequence is finite.

Proposition 4.2. *The following conditions on a partially ordered set $(Q, \leq)$ are equivalent.*

(i) $(Q, \leq)$ *is wpo.*

(ii) *For every infinite sequence $q_1, q_2, \ldots$ of elements of Q there is an increasing sequence $i_1, i_2, \ldots$ of natural numbers such that $q_{i_1} \leq q_{i_1} \leq \ldots$.*

(iii) *There is neither an infinite decreasing sequence in Q nor an infinity of mutually incomparable elements of Q.*

(iv) *Every linear extension of $\leq$ is a well–ordering.*

(v) *Every nonempty subset of Q has at least one but only finitely many minimal elements.*

Proof. Easy consequence of Ramsey's theorem. □

Definition 4.3. Let Q be a partially ordered set. For $q_1, q_2 \in Q$ we write $q_1 < q_2$ if $q_1 \leq q_2$ and $q_1 \neq q_2$, and $q_1 \not\leq q_2$ if not $q_1 \leq q_2$. Let κ be a fixed cardinal.

We put $U := \kappa \times Q$ and we introduce the following sheaves in U :

$$\begin{aligned}
\mathrm{Asc}(U) &:= \{((\alpha_1, q_1), \ldots, (\alpha_m, q_m)) \in U^{<\omega} \mid q_1 \leq q_2 \leq \ldots \leq q_m\},\\
\mathrm{Bad}(U) &:= \{((\alpha_1, q_1), \ldots, (\alpha_m, q_m)) \in U^{<\omega} \mid q_i \not\leq q_j \text{ for } i < j\},\\
\mathrm{Nd}(U) &:= \{((\alpha_1, q_1), \ldots, (\alpha_m, q_m)) \in U^{<\omega} \mid q_i > q_j \text{ for no } i < j\},\\
\mathrm{Dec}(U) &:= \{((\alpha_1, q_1), \ldots, (\alpha_m, q_m)) \in U^{<\omega} \mid q_1 > q_2 < \ldots > q_m\},\\
\mathrm{Inc}(U) &:= \{((\alpha_1, q_1), \ldots, (\alpha_m, q_m)) \in U^{<\omega} \mid q_i \leq q_j \text{ for no } i \neq j\},\\
\mathrm{Comp}(U) &:= \{((\alpha_1, q_1), \ldots, (\alpha_m, q_m)) \in U^{<\omega} \mid \text{ for all } i, j \text{ either } q_i \leq q_j \text{ or } \\
&\qquad q_j \leq q_i\}.
\end{aligned}$$

For $\kappa = 1$ we identify U with Q; thus the sheaves $\mathrm{Asc}(Q), \ldots, \mathrm{Comp}(Q) \subseteq Q^{<\omega}$ are defined. If Q is wpo, then $\mathrm{Bad}(U), \mathrm{Dec}(U)$ and $\mathrm{Inc}(U)$ are trees without infinite chains. Hence they have types, which are denoted by $\gamma_{\mathrm{Bad}(U)}, \gamma_{\mathrm{Dec}(U)}$ and $\gamma_{\mathrm{Inc}(U)}$, respectively. Let g be the universal (1)–testing. Then a sequence $(\alpha_1, q_1), (\alpha_2, q_2), \ldots$ is $(\mathrm{Asc}(U), g)$–bad if and only if the sequence $q_1, q_2, \ldots$ is bad. (This should justify our terminology). Hence the sheaf $\mathrm{Asc}(U)$ has the R–property if and only if Q is wpo. Moreover

$$\overline{\Phi}_{\{\mathrm{Asc}(U)\}}(1) = \Phi_{\mathrm{Asc}(U)}(1) = \overline{\Phi}_{\{\mathrm{Asc}(Q)\}}(1) = \Phi_{\mathrm{Asc}(Q)}(1) = \gamma_{\mathrm{Bad}(U)} = \gamma_{\mathrm{Bad}(Q)}.$$

This ordinal, denoted by $c(Q)$, is called the *type* of the wpo set Q. Similarly, we have

$$\overline{\Phi}_{\{Nd(U)\}}(1) = \Phi_{Nd(U)}(1) = \overline{\Phi}_{\{Nd(Q)\}}(1) = \Phi_{Nd(Q)}(1) = \gamma_{\mathrm{Dec}(U)} = \gamma_{\mathrm{Dec}(Q)}.$$

This ordinal will be called the *height* of Q and will be denoted by $\mathrm{ht}(Q)$. Finally,

$$\begin{aligned}
&\overline{\Phi}_{\{\mathrm{Comp}(U)\}}(1) = \Phi_{\mathrm{Comp}(U)}(1) = \overline{\Phi}_{\{\mathrm{Comp}(Q)\}}(1) = \Phi_{\mathrm{Comp}(Q)}(1) = \\
&= \gamma_{\mathrm{Inc}(U)} = \gamma_{\mathrm{Inc}(Q)}.
\end{aligned}$$

This last ordinal will be called the *width* of Q and will be denoted by $wd(Q)$.

For $q \in Q$ we define

$$\begin{aligned}
\mathrm{ht}(q) &= 0 && \text{if } q \text{ is a minimal element of } Q\\
&= \sup\{\mathrm{ht}(q') + 1 \mid q' < q\} && \text{otherwise.}
\end{aligned}$$

Clearly $\mathrm{ht}(Q) = \sup\{\mathrm{ht}(q) + 1 \mid q \in Q\}$.

Definition 4.4. For $(q_1, \ldots, q_m) \in \mathrm{Bad}(Q)$ we put

$$\begin{aligned}
&Q_{(q_1, \ldots, q_m)} := \{q \in Q \mid (q_1, \ldots, q_m, q) \in \mathrm{Bad}(Q)\},\\
&Q_{q_1} := Q_{(q_1)},\\
&cl(q_1) := \{q \in Q \mid q_1 \leq q\},
\end{aligned}$$

and we denote

$$\begin{aligned}
&\lambda(Q) := \sup\{\alpha \in On \mid \alpha \text{ is the (ordinal) type of a linear extension of } \leq\},\\
&\chi(Q) := \sup\{\alpha \in On \mid \alpha \text{ is the (ordinal) type of a chain in } Q\}.
\end{aligned}$$

Let us remark that $c(Q) = \sup\{c(Q_q) + 1 \mid q \in Q\}$.

If Q_1 and Q_2 are partially ordered sets, then $Q_1 \dot{\cup} Q_2$ denotes the disjoint union of Q_1 and Q_2 whose partial ordering is the disjoint union of the partial orderings on Q_1 and Q_2, and $Q_1 \times Q_2$ denotes the Cartesian product of Q_1 and Q_2 with the partial ordering defined by

$$(q_1, q_2) \leq (q_1', q_2') \text{ iff } q_i \leq q_i' \text{ in } Q_i \ (i = 1, 2).$$

It is easy to see that if Q_1 and Q_2 are wpo, then these constructions define again wpo sets.

Lemma 4.5. *Let Q be wpo and let $(q_\alpha \mid \alpha \in \mathrm{cf}(\lambda))$ be a transfinite sequence of elements of Q. Then there exists an increasing ordinal sequence $(\alpha_\beta \mid \beta \in \mathrm{cf}(\lambda))$ such that $q_{\alpha_\beta} \leq q_{\alpha_{\beta'}}$ for $\beta \leq \beta' \in \mathrm{cf}(\lambda)$.*

Proof. Let $(q_\alpha \mid \alpha \in \mathrm{cf}(\lambda))$ be as above. For a cofinal subset M of $\mathrm{cf}(\lambda)$ we call an ordinal $\alpha \in M$ *terminal* for M if the set

$$\{\beta \in M \mid q_\alpha \not\leq q_\beta\}$$

is cofinal in λ. We claim that there is a cofinal subset of $\mathrm{cf}(\lambda)$ without a terminal element. Suppose not and put $M_0 = \mathrm{cf}(\lambda)$. If $M_0, \ldots, M_n$, $\delta_0, \ldots, \delta_{n-1}$ are defined, we let δ_n be the terminal element for M_n and $M_{n+1} := \{\beta \in M_n \mid q_{\delta_n} \not\leq q_\beta\}$. Then $q_{\delta_1}, q_{\delta_2}, \ldots$ is a bad sequence in Q, a contradiction.

So let $M \subseteq \mathrm{cf}(\lambda)$ be cofinal without a terminal element. We define inductively $\alpha_0 := \min M$ and

$$\alpha_\beta := \sup\{\sup\{\delta \in M \mid q_{\alpha_{\beta'}} \not\leq q_\delta\} + 1 \mid \beta' < \beta\}. \qquad \square$$

Theorem 4.6 (de Jongh, Parikh 1977). *If Q_1 and Q_2 are wpo, then*

$$c(Q_1 \dot{\cup} Q_2) = c(Q_1) \oplus c(Q_2).$$

Proof. By induction on $c(Q_1) \oplus c(Q_2)$

$$\begin{aligned} c(Q_1 \dot{\cup} Q_2) &= \sup\{c((Q_1 \dot{\cup} Q_2)_q) + 1 \mid q \in Q_1 \dot{\cup} Q_2\} = \\ &= \sup\{c((Q_1)_{q_1} \dot{\cup} Q_2) + 1, c(Q_1 \dot{\cup} (Q_2)_{q_2}) + 1 \mid q_1 \in Q_1,\ q_2 \in Q_2\} = \\ &= \sup\{c((Q_1)_{q_1}) \oplus c(Q_2) + 1, c(Q_1) \oplus c((Q_2)_{q_2}) + 1 \mid q_1 \in Q_1,\ q_2 \in Q_2\} = \\ &= c(Q_1) \oplus c(Q_2). \end{aligned} \qquad \square$$

Theorem 4.7. *Let $(Q, \leq)$ be wpo.*

(i) (de Jongh, Parikh 1977) There exists a maximal linear extension of $\leq$. That is, a linear extension of order type $\lambda(Q)$.
(ii) (The First Minimax Theorem) $c(Q) = \lambda(Q)$.

Proof. Clearly $c(Q) \geq c(\lambda(Q)) = \lambda(Q)$. For the converse inequality let $\gamma = c(Q)$ and let $(q_\alpha \mid \alpha \in \mathrm{cf}(\gamma))$ be such that $\sup c(Q_{q_\alpha}) + 1 = \gamma$. By Lemma 4.5 we may assume that $q_\alpha \leq q_\beta$ for $\alpha \leq \beta \in \mathrm{cf}(\gamma)$. We proceed by induction on γ.

Assume first that $\gamma = \xi + \eta$, where $\xi, \eta \neq 0$ and η is a power of ω. Choose α such that $c(Q_{q_\alpha}) \geq \xi$. Since

$$c(Q) = c(Q_{q_\alpha}) \oplus c(\mathrm{cl}(q_\alpha)) \geq c(Q_{q_\alpha}) + c(\mathrm{cl}(q_\alpha)),$$

we have $\gamma > c(\mathrm{cl}(q_\alpha)) \geq \eta$. Hence by the induction hypothesis

$$\lambda(Q) \leq c(Q) = \xi + \eta \leq \lambda(Q_{q_\alpha}) + \lambda(\mathrm{cl}(q_\alpha)) \leq \lambda(Q).$$

Now let γ be a power of ω. Let γ_α converge to γ ($\alpha \in \mathrm{cf}(\gamma)$), and for α with $0 \leq \alpha < \mathrm{cf}(\gamma)$ define inductively

$$\mu(\alpha) := \min\{\beta \in \mathrm{cf}(\gamma) \mid c((\bigcap\{\mathrm{cl}(q_{\mu(\alpha')}) \mid \alpha' < \alpha\})_{q_\beta}) > \gamma_\alpha\}.$$

The function μ is well-defined, since

$$(\bigcap\{\mathrm{cl}(q_{\mu(\alpha')}) \mid \alpha' < \alpha\})_{q_\beta}) \cup \bigcup\{Q_{q_{\mu(\alpha')}} \mid \alpha' < \alpha\} = Q_{q_\beta},$$

and hence

$$c((\bigcap\{\mathrm{cl}(q_{\mu(\alpha')}) \mid \alpha' < \alpha\})_{q_\beta}) \oplus c(\bigcup\{Q_{q_{\mu(\alpha')}} \mid \alpha' < \alpha\}) \geq c(Q_{q_\beta}),$$

which converges to γ. We put

$$Q_\alpha := (\bigcap\{\mathrm{cl}(q_{\mu(\alpha')}) \mid \alpha' < \alpha\})_{q_{\mu(\alpha)}}.$$

Clearly $\gamma > c(Q_\alpha) > \gamma_\alpha$. By the induction hypothesis

$$\lambda(Q) \leq c(Q) \leq \sum_{\alpha \in \mathrm{cf}(\gamma)} c(Q_\alpha) = \sum_{\alpha \in \mathrm{cf}(\gamma)} \lambda(Q_\alpha) \leq \lambda(Q).$$

Thus we have proved (ii) in both cases. A maximal linear extension is, in each case, obtained by concatenating the maximal linear extensions on corresponding subsets of Q. □

Theorem 4.8 (de Jongh, Parikh 1977). *Let Q_1, Q_2 be wpo sets. Then*

$$c(Q_1 \times Q_2) = c(Q_1) \otimes c(Q_2).$$

Proof. Clearly

$$c(Q_1 \times Q_2) = \lambda(Q_1 \times Q_2) \geq \lambda(c(Q_1) \times c(Q_2)) \geq c(Q_1) \otimes c(Q_2).$$

We prove the converse inequality by induction on $c(Q_1) \oplus c(Q_2)$. Let first $c(Q_1) = \alpha_1 + \ldots + \alpha_n$, $c(Q_2) = \beta_1 + \ldots + \beta_m$, $\alpha_1 \geq \ldots \geq \alpha_n$, $\beta_1 \geq \ldots \beta_m$, where each α_i, β_i is a power of ω, and $n > 1$ or $m > 1$. By Theorem 4.7,

$Q_1 = Q_1^1 \cup \ldots \cup Q_1^n$, $Q_2 = Q_2^1 \cup \ldots \cup Q_2^m$, where $c(Q_1^i) = \alpha_i$, $c(Q_2^j) = \beta_j$ ($i = 1, \ldots n$; $j = 1, \ldots, m$). We have

$$\begin{aligned} c(Q_1 \times Q_2) &= c(\bigcup\{Q_1^i \times Q_2^j \mid 1 \le i \le n,\ 1 \le j \le m\}) \le \bigoplus\{c(Q_1^i) \otimes c(Q_2^j) \mid \\ &\quad 1 \le i \le n,\ 1 \le j \le m\} = \\ &= \left(\bigoplus_{1\le i\le n} c(Q_1^i)\right) \otimes \left(\bigoplus_{1 \le j \le m} c(Q_2^j)\right) = c(Q_1) \otimes c(Q_2). \end{aligned}$$

Now let $c(Q_1)$ and $c(Q_2)$ be powers of ω. Then by the induction hypothesis

$$\begin{aligned} & c(Q_1 \times Q_2) = \sup\{c((Q_1 \times Q_2)_{(q_1,q_2)}) + 1 \mid q_1 \in Q_1,\ q_2 \in Q_2\} \le \\ & \le \sup\{c(((Q_1)_{q_1} \times Q_2) \cup (Q_1 \times (Q_2)_{q_2})) + 1 \mid q_1 \in Q_1,\ q_2 \in Q_2\} \le \\ & \le \sup\{c((Q_1)_{q_1} \times Q_2) \oplus c(Q_1 \times (Q_2)_{q_2}) + 1 \mid q_1 \in Q_1,\ q_2 \in Q_2\} = \\ & = \sup\{(c((Q_1)_{q_1}) \otimes c(Q_2)) \oplus (c(Q_1) \otimes c((Q_2)_{q_2})) + 1 \mid q_1 \in Q_1,\ q_2 \in Q_2\} \le \\ & \le c(Q_1) \otimes c(Q_2). \end{aligned}$$ □

Theorem 4.9. *Let Q be wpo.*

(i) (Wolk 1967) There exists a maximal chain in Q. That is, a chain of order type $\chi(Q)$.

(ii) (The Second Minimax Theorem) $\mathrm{ht}(Q) = \chi(Q)$.

Proof. (i) This argument is taken from (Wolk 1967). Let M_0 be the set of all minimal elements of Q, and define inductively M_α to be the set of all minimal elements in $Q \setminus \bigcup_{\beta<\alpha} M_\beta$. By 4.2 each M_α is finite. Let χ be the least ordinal such that $M_\chi = 0$. Then clearly $\chi \ge \chi(Q)$, and we shall prove that there is a chain $(q_\alpha \mid \alpha \in \chi)$ such that $q_\alpha \in M_\alpha$ for every $\alpha \in \chi$, which will give (i).

If $A = \{\alpha_1 < \ldots < \alpha_n\} \subseteq \chi$ is a finite set then there is a chain $q_{\alpha_1} \le \ldots \le q_{\alpha_n}$ such that $q_{\alpha_i} \in M_{\alpha_i}$ ($i = 1, \ldots, n$); we put $f_A(\alpha_i) = q_{\alpha_i}$. By Rado Selection Lemma (cf. Ore 1962) there is a function $f : \chi \to Q$ such that for every finite set $A \subseteq \chi$ there is a finite set B such that $A \subseteq B \subseteq \chi$ and $f_B \restriction A = f \restriction A$. Hence $(f(\alpha) \mid \alpha \in \chi)$ is the desired chain.

(ii) Clearly $\mathrm{ht}(Q) \ge \chi(Q)$. For the other inequality define $f : \mathrm{Dec}(Q) \to \chi(Q)$ for $q_1 > \ldots > q_n$ by

$$f(q_1, \ldots, q_n) = \chi(\{q \in Q \mid q_n > q\}).$$

By (i) above, f is a character, which gives $\mathrm{ht}(Q) \le \chi(Q)$. □

Remark 4.10. Theorem 4.9 holds under weaker hypothesis than that Q be *wpo*, namely it suffices that for every infinite sequence $q_1, q_2 \ldots$ of elements of Q there are indices i, j such that $i < j$, and either $q_i \le q_j$ or $ht(q_i) \ge ht(q_j)$. See (Kříž), (Milner, Sauer), (Pouzet 1979), or (Schmidt 1981).

4.11 Theorem.
(i) $\overline{\Phi}_{\{\mathrm{Asc}(Q)\}}(\gamma) = \overline{\Phi}_{\{\mathrm{Asc}(U)\}}(\gamma) = \gamma \otimes c(Q)$.
(ii) *If* $\gamma < \kappa^+$ *then* $\Phi_{\mathrm{Asc}(U)}(\gamma) = \gamma \otimes c(Q)$.
(iii) *For* $Q = \omega + 1$ *we have*

$$\Phi_{\mathrm{Asc}(Q)}(\omega + 1) \leq \omega^2 + 2\omega < \omega^2 + 2\omega + 1 = \overline{\Phi}_{\{\mathrm{Asc}(Q)\}}(\omega + 1).$$

Proof. (i) $\gamma \otimes c(Q) \leq \overline{\Phi}_{\{\mathrm{Asc}(Q)\}}(\gamma)$: For $s = ((\alpha_1, q_1), \ldots, (\alpha_m, q_m)) \in \mathrm{Bad}(\gamma \times Q)$ let $a^s = (q_1, \ldots, q_m)$ and let $g^s :\downarrow a^s \to \gamma$ be defined by $g^s(x_1, \ldots, x_p) = \alpha_{i_p}$ if $(q_{i_1}, \ldots, q_{i_p})$ is the first appearance of $(x_1, \ldots, x_p) \subseteq a^s$ in a^s. Then g^s is an $(\mathrm{Asc}(Q), \gamma)$–germ on a^s. Hence $H : \mathrm{Bad}(\gamma \times Q) \to T(\{\mathrm{Asc}(Q)\}; \gamma)$ defined by $H(s) = (a^s, g^s)$ is a tree homomorphism, which gives

$$\gamma \otimes c(Q) = c(\gamma \times Q) \leq \overline{\Phi}_{\{\mathrm{Asc}(Q)\}}(\gamma),$$

using Theorem 4.8.

$\overline{\Phi}_{\{\mathrm{Asc}(Q)\}}(\gamma) \leq \overline{\Phi}_{\{\mathrm{Asc}(U)\}}(\gamma)$: Obvious.

$\overline{\Phi}_{\{\mathrm{Asc}(U)\}}(\gamma) \leq \gamma \otimes c(Q)$: Let $T = T(\{\mathrm{Asc}(U)\}; \gamma)$. We shall define a tree homomorphism $H : T \to \mathrm{Bad}(\gamma \times Q)$, which will give the result. So let $(a, g) \in T$, and let $a = ((\alpha_1, q_1), \ldots, (\alpha_m, q_m))$. We put $b = (q_1, \ldots, q_m)$ and we define $h :\downarrow b \to \gamma$ by

$$h(x_1, \ldots, x_p) = \min g(x_{i_1}, \ldots, x_{i_s}),$$

the min taken over all $i_1 < \ldots < i_s = p$ such that $x_{i_1} \leq x_{i_2} \leq \ldots \leq x_{i_s}$. Now we define

$$H(a, g) = ((h(q_1), q_1), (h(q_1, q_2), q_2), \ldots, (h(q_1, \ldots, q_m), q_m)).$$

It is easily seen that $H(a, g) \in \mathrm{Bad}(\gamma \times Q)$, and hence H is as desired.
(ii) If $\gamma < \kappa^+$ then we may assume that $\gamma \times Q \subseteq U$. By (i) above and 3.7 it suffices to show that $\Phi_{\mathrm{Asc}(U)}(\gamma) \geq \gamma \otimes c(Q)$. So let $g : U^{<\omega} \to \gamma$ be defined by

$$\begin{aligned} g(x_1, \ldots, x_m) &= \gamma_m \quad \text{if } x_i = (\gamma_i, q_i) \in \gamma \times Q \text{ and } q_1 \leq \ldots \leq q_m \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

and let f be the R–character corresponding to $\mathrm{Asc}(U)$ and g. It is easily seen that if a sequence $a \in (\gamma \times Q)^{<\omega}$ is from $\mathrm{Bad}(\gamma \times Q)$, then it is $(\mathrm{Asc}(U), g)$–bad. Hence f is a character on $\mathrm{Bad}(\gamma \times Q)$ and consequently $\Phi_{\mathrm{Asc}(U)}(\gamma) \geq c(\gamma \times Q) = \gamma \otimes c(Q)$, as desired.
(iii) Let $Q = \omega + 1$ and let $g : Q^{<\omega} \to \omega + 1$ be given. Let $g' : \omega^{<\omega} \to \omega$ be the restriction of g to $\omega^{<\omega}$, and let $f' : \omega^{<\omega} \to \omega^2 + \omega$ be the R–character corresponding to $\mathrm{Asc}(\omega)$ and g'. Let $n = g(\omega, \omega)$ if $g(\omega, \omega) \in \omega$, and let $n = 0$ otherwise. We define $f : Q^{<\omega} \to \omega^2 + \omega + n + 3$ by

$$f(q_1, \ldots, q_m) = f'(q_1, \ldots, q_{i_1 - 1}, q_{i_1 + 1}, \ldots, q_{i_p - 1}, q_{i_p + 1}, \ldots, q_m) + n + 2 - p,$$

where $q_{i_1}, \ldots, q_{i_p}$ are all occurences of ω within $(q_1, \ldots, q_m)$. Then it is easily seen that f is an R–character corresponding to $\mathrm{Asc}(Q)$ and g, showing that

$\Phi_{\mathbf{Asc}(Q)}(\omega+1) \le \omega^2 + 2\omega$. On the other hand $\overline{\Phi}_{\{\mathbf{Asc}(Q)\}}(\omega+1) = \omega^2 + 2\omega + 1$ by (i). $\square$

Remark 4.12. Part (ii) of the above theorem is the essential reason for introducing the cardinal κ. For the other results the value of κ is irrelevant. Let us remark that 4.11(iii) can be used to construct a sheaf A for which a universal $(\omega+1)$–testing exists, but $\Phi_{\{A\}}(\omega+1) < \overline{\Phi}_{\{A\}}(\omega+1)$. This shows that the assumptions in 3.9 cannot be weakened.

Theorem 4.13 (The Height-Width Theorem). *Let Q be wpo. Then*

$$c(Q) \le \mathrm{ht}(Q) \otimes \mathrm{wd}(Q).$$

Proof. Let g be a character on $\mathrm{Inc}(Q)$. We define, for $(q_1,\dots,q_n) \in \mathrm{Bad}(Q)$

$$h(q_1,\dots,q_n) = \min\{g(q_{i_1},\dots,q_{i_m}) \mid i_1 < i_2 < \dots < i_m = n,\ \mathrm{ht}(q_{i_1}) \le \dots \le \mathrm{ht}(q_{i_m})\}$$

and

$$f(q_1,\dots,q_n) = ((\mathrm{ht}(q_1), \mathrm{h}(q_1)), (\mathrm{ht}(q_2), \mathrm{h}(q_1,q_2)),\dots \dots(\mathrm{ht}(q_n), h(q_1,\dots,q_n))).$$

It is easily seen that h is well–defined and that $f(g_1,\dots,g_n) \in \mathrm{Bad}(\mathrm{ht}(Q) \times \mathrm{wd}(Q))$. Hence $f : \mathrm{Bad}(Q) \to \mathrm{Bad}(\mathrm{ht}(Q) \times \mathrm{wd}(Q))$ is a tree homomorphism, which gives $c(Q) \le c(\mathrm{ht}(Q) \times \mathrm{wd}(Q)) = \mathrm{ht}(Q) \otimes \mathrm{wd}(Q)$ by 4.8. $\square$

4.14 Remark. The above theorem generalizes the result that a partially ordered set with at least $rs+1$ elements contains either a chain of length $r+1$, or an antichain of $s+1$ elements.

On the contrary to $c(Q)$ and $\mathrm{ht}(Q)$ we did not find any reasonable characterization of $\mathrm{wd}(Q)$. Of course, if $\mathrm{wd}(Q) < \omega$ the Dilworth's decomposition theorem (Dilworth 1950) gives one, but the width behaves much worse, when it is infinite. For example, there is a wpo set Q of width $\omega+1$, which cannot be decomposed into two sets, one of width ω and one of width 1.

5. Erdös-Szekeres Theorem

Definition 5.1. Let $\le$ be a linear ordering on U. We define a 2–sheaf $E^{\le} = (E^{\le}_1, E^{\le}_2)$ by

$$E^{\le}_1 = \{(x_1,\dots,x_m) \mid x_1 \le \dots \le x_m\},$$
$$E^{\le}_2 = \{(x_1,\dots,x_m) \mid x_1 \ge \dots \ge x_m\}.$$

This gives rise to Ramsey 2–system

$$\mathcal{E}_1 := \{E^{\le} \mid \le \text{ is a linear ordering on } U\},$$

which will be called the *Erdös-Szekeres system*.

Definition 5.2. Let $r : U^2 \to \{1,\ldots,k\}$ be given. We define a k–sheaf $S^r = (S_1^r,\ldots,S_k^r)$ by

$$S_i^r := \{(x_1,\ldots,x_m) \in U^{<\omega} \mid r(x_j,x_{j+1}) = i \text{ for } j = 1,\ldots,m-1\}.$$

This defines a Ramsey system $\mathcal{S}_k := \{S^r \mid r : U^2 \to \{1,\ldots k\}\}$, which will be called the *generalized Erdös-Szekeres system.*

A minor modification of this system yields a more general system, which, however, has the same R-ordinals. Recall that

$$R_1^r := \{(x_1,\ldots,x_m) \in U^{<\omega} \mid r(x_i,x_j) = 1 \text{ for all } 1 \le i,\ j \le m\}.$$

We put $Ch^r := (R_1^r, S_2^r,\ldots,S_k^r)$ and

$$Ch_k := \{Ch^r \mid r : U^2 \to \{1,\ldots,k\}\}.$$

Remark 5.3. It is an easy exercise that for ℓ_1,ℓ_2 finite the Erdös–Szekeres theorem is equivalent to the statement $\Phi_{\mathcal{E}_1}(\ell_1,\ell_2) = \ell_1 \cdot \ell_2$.

We have taken the liberty to denote the last k–system of 5.2 by Ch_k, because it corresponds to a weaker version of the Chvátal's Tree–Complete Graph Ramsey Theorem. The systems $\mathcal{E}_1, \mathcal{S}_k$ and Ch_k are closely related and their R-ordinals are easily computed. Later in this section, we introduce the Erdös–Szekeres system $\mathcal{E}_n$ corresponding to n linear orderings. Generally, $\mathcal{E}_n$ is a 2^n–system. The investigation of $\Phi_{\mathcal{E}_n}$ is technically more complicated. It has been done for finite values of the arguments in (Alon, Füredi, Katchalski 1985). We obtain (in the infinite case) a lower bound for $\Phi_{\mathcal{E}_n}$ by a certain ordinal product and an upper bound by the maximal product. In the finite case, of course, these bounds coincide. In the infinite case, however, the upper bound is not generally achieved. The exact form of the function $\Phi_{\mathcal{E}_n}$ seems to be rather profound.

Theorem 5.4.
(i) If $\gamma_1,\ldots,\gamma_k < |U|^+$ *then* $\Phi_{\mathcal{S}_k}(\gamma_1,\ldots,\gamma_k) = \Phi_{Ch_k}(\gamma_1,\ldots,\gamma_k) = \gamma_1 \otimes \ldots \otimes \gamma_k$.
(ii) $\overline{\Phi}_{\mathcal{S}_k}(\gamma_1,\ldots,\gamma_k) = \overline{\Phi}_{Ch_k}(\gamma_1,\ldots,\gamma_k) = \gamma_1 \otimes \ldots \otimes \gamma_k$ *for any ordinals* $\gamma_1,\ldots,\gamma_k$.

Proof. (i) Let $Q = \gamma_1 \times \ldots \times \gamma_k$ be endowed with the product partial ordering. We first prove that $\Phi_{Ch_k}(\gamma_1,\ldots,\gamma_k) \le \gamma_1 \otimes \ldots \otimes \gamma_k$. Let $r : U^2 \to \{1,\ldots,k\}$ and let a $(\gamma_1,\ldots,\gamma_k)$–testing $g = (g_1,\ldots,g_k)$ be given. We define a $(\gamma_1,\ldots,\gamma_k)$–testing $h = (h_1,\ldots,h_k)$ by

$$h_i(a) = \min\{g_i(b) \mid b \in \downarrow a \cap S_i^r,\ a \text{ and } b \text{ have the same last term}\}$$

for $i = 2,\ldots,k$ and by

$$\begin{aligned} h_1(a) = \min\{g_1(x) \mid x = (x_1,\ldots,x_p) \subseteq a,\ x_p = \text{ last term of } a,\\ h_i(x_1) \le h_i(x_1,x_2) \le \ldots \le h_i(x_1,\ldots,x_p) \text{ for } i = 2,\ldots,k\} \end{aligned}$$

for $i = 1$. If we consider h as a function $U^{<\omega} \to Q$, then $H : \mathrm{Bad}(Ch^r, g) \to \mathrm{Bad}(Q)$ defined by

$$H(a_1,\ldots,a_m) = (h(a_1), h(a_1,a_2),\ldots,h(a_1,\ldots,a_m))$$

is a tree homomorphism, showing that

$$\Phi_{Ch^r}(\gamma_1,\ldots,\gamma_k) \leq c(Q) = \gamma_1 \otimes \ldots \otimes \gamma_k,$$

by 4.8. The inequality $\Phi_{\mathcal{S}_k}(\gamma_1,\ldots,\gamma_k) \leq \Phi_{\mathcal{C}h_k}(\gamma_1,\ldots,\gamma_k)$ is obvious.

Finally we prove that $\gamma_1 \otimes \ldots \otimes \gamma_k \leq \Phi_{\mathcal{C}h_k}(\gamma_1,\ldots,\gamma_k)$. We may assume that $Q \subseteq U$. We define $r : U^2 \to \{1,\ldots,k\}$ by

$$\begin{aligned} r(a,b) &= \min(\{i \mid \alpha_i > \beta_i\} \cup \{k\}) \text{ if } a = (\alpha_1,\ldots,\alpha_k) \in Q \text{ and} \\ &\qquad\qquad\qquad\qquad\qquad b = (\beta_1,\ldots,\beta_k) \in Q \\ &= \text{arbitrarily} \qquad\qquad\quad \text{otherwise.} \end{aligned}$$

and $g_i : U^{<\omega} \to \gamma_i$ by

$$\begin{aligned} g_i(a_1,\ldots,a_m) &= \alpha_m^i \text{ if } a_j = (\alpha_j^1,\ldots,\alpha_j^k) \in Q \text{ for } j = 1,\ldots,m \\ &= 0 \quad \text{otherwise} \end{aligned}$$

It follows that the identity is a tree homomorphism $\mathrm{Bad}(Q) \to \mathrm{Bad}(S^r, g)$, and hence $\gamma_1 \otimes \ldots \otimes \gamma_k \leq \Phi_{S^r}(\gamma_1,\ldots,\gamma_k)$.
(ii) This follows from (i) by 3.17, since both $\mathcal{S}_k$ and $\mathcal{C}h_k$ are standard and by Theorem 3.13 we may assume that $\gamma_1,\ldots,\gamma_k < |U|^+$. □

Theorem 5.5.
(i) *If* $\gamma_1,\gamma_2 < |U|^+$ *then* $\Phi_{\mathcal{E}_1}(\gamma_1,\gamma_2) = \gamma_1 \otimes \gamma_2$.
(ii) $\Phi_{\mathcal{E}_1}(\gamma_1,\gamma_2) = \gamma_1 \otimes \gamma_2$ *for any ordinals* γ_1,γ_2.

Proof. Similarly as in 5.4, it suffices to prove (i).
$\leq$: Given a linear ordering $\leq$ on U, we define $r : U^2 \to \{1,2\}$ by

$$\begin{aligned} r(x,y) &= 1 \text{ if} x \leq y \\ &= 2 \text{ otherwise.} \end{aligned}$$

Then the identity is a tree homomorphism $\mathrm{Bad}(E^{\leq}, g) \to \mathrm{Bad}(S^r, g)$ for any (γ_1,γ_2)–testing g, and the inequality follows from 5.4(i).
$\geq$: We may assume that $Q := \gamma_1 \times \gamma_2 \subseteq U$. Let $g_i(a_1,\ldots,a_m)$ be defined to be the i–th coordinate of a_m if $a_m \in Q$ and to be 0 otherwise ($i = 1,2$). Then $g = (g_1,g_2)$ is a (γ_1,γ_2)–testing. For (α,β), $(\alpha',\beta') \in Q$ we define

$$(\alpha,\beta) \leq (\alpha',\beta') \text{ if } \beta < \beta', \text{ or } \beta = \beta' \text{ and } \alpha \geq \alpha'.$$

Then the identity is a tree homomorphism $\mathrm{Bad}(Q) \to \mathrm{Bad}(E^{\leq}, g)$ and the remaining inequality follows from 4.8. □

Definition 5.6. Let $n > 1$ be a natural number and let $\tau = (\leq_1,\ldots,\leq_n)$ be an n–tuple of linear orderings on U. We denote by Σ the set of all mappings $\{1,\ldots,n\} \to \{1,2\}$ and for $\sigma \in \Sigma$ we define $\leq_\sigma$ by

$$x \leq_\sigma y \text{ iff } x \leq_i y \text{ for } i \in \sigma^{-1}(1) \text{ and } y \leq_i x \text{ for } i \in \sigma^{-1}(2)$$

and put

$$E^\tau_\sigma := \{(x_1, \ldots, x_m) \in U^{<\omega} \mid x_1 \leq_\sigma x_2 \leq_\sigma \ldots \leq_\sigma x_m\}.$$

This gives rise to a 2^n–sheaf

$$E^\tau := (E^\tau_\sigma \mid \sigma \in \Sigma)$$

and a 2^n–system

$$\mathcal{E}_n := \{E^\tau \mid \tau \text{ is as above}\}.$$

This system will be called the *Erdös-Szekeres system*, for it clearly generalizes the system $\mathcal{E}_1$ introduced in 5.1.

For the lower bounds to $\Phi_{\mathcal{E}_n}$ we need some more definitions.

Definition 5.7. For $\sigma, \sigma' \in \Sigma$ we define $\sigma \lhd \sigma'$ if there exists $i \in \{1, \ldots, n\}$ such that $\sigma(i) < \sigma'(i)$ and $\sigma(j) = \sigma'(j)$ for all $j = i+1, \ldots, n$. For ordinals γ_σ $(\sigma \in \Sigma)$ we define an irreflexive ordering $\lhd$ on the product $\prod_{\sigma \in \Sigma} \gamma_\sigma$ by

$$(\alpha_\sigma)_{\sigma \in \Sigma} \lhd (\beta_\sigma)_{\sigma \in \Sigma} \text{ if } \begin{array}{l} \text{there exists } \sigma \in \Sigma \text{ such that } \alpha_\sigma < \beta_\sigma \text{ and} \\ \alpha_{\sigma'} = \beta_{\sigma'} \text{ for every } \sigma' \lhd \sigma. \end{array}$$

Let us denote by Q the partially ordered set $(\prod_{\sigma \in \Sigma} \gamma_\sigma, \lhd)$. The set Q is in fact well–ordered, and we denote its type by $\prod_{\sigma \in \Sigma} \gamma_\sigma$. Let us remark that $\prod_{\sigma \in \Sigma} \gamma_\sigma$ is the usual ordinal product of the ordinals γ_σ in the order given by $\lhd$ on Σ.

Lemma 5.8. *Let n be a natural number and let Q be as in 5.7. Then there exists an n–tuple $\tau = (\leq_1, \ldots, \leq_n)$ of linear orderings on Q such that for any $\sigma \in \Sigma$ and for any $x, y \in Q$, $x \rhd y$ and $x \leq_\sigma y$ implies $x_\sigma > y_\sigma$, where x_σ and y_σ are the σ–th coordinates of x and y, respectively.*

Proof. We proceed by induction on n. For $n = 1$ we identify Σ with $\{1, 2\}$ and define $\leq_1$ as $\leq$ in 5.5(i). Now assume the lemma to be proved for $n-1$. Let us prove it for n. For $i = 1, 2$ we put

$$\Sigma^i := \{\sigma \in \Sigma \mid \sigma(n) = i\}$$

and

$$Q^i := (\prod_{\sigma \in \Sigma^i} \gamma_\sigma, \lhd).$$

Let $p^i : Q \to Q^i$ be the projections. By the induction hypothesis, there are linear orderings $\leq^i_1, \ldots, \leq^i_{n-1}$ on Q^i $(i = 1, 2)$ with the desired property. We define for $j = 1, \ldots, n-1$

$$x \leq_j y \quad \text{if } p^1(x) \leq^1_j p^1(y), \text{ and } p^1(x) \geq^1_j p^1(y) \text{ implies } p^2(x) \leq^2_j p^2(y)$$

and

$$x \leq_n y \quad \text{if either } p^1(x) \lhd p^1(y), \text{or } p^1(x) = p^1(y) \text{ and } x \rhd y, \text{or } x = y.$$

Now let $x \triangleright y$ and $x \leq_\sigma y$ for some $\sigma \in \Sigma$. We distinguish two cases.
Case 1: $\sigma \in \Sigma^2$. Then $x \geq_n y$ and it follows that $p^1(x) \triangleright p^1(y)$, since otherwise we would have a contradiction to $x \triangleright y$. It follows that for $j = 1, \ldots, n-1$ we have

$$p^1(x) \leq^1_j p^1(y) \text{ if } \sigma(j) = 1, \text{ and}$$
$$p^1(x) \geq^1_j p^1(y) \text{ if } \sigma(j) = 2.$$

Thus we are done by the induction hypothesis.
Case 2: $\sigma \in \Sigma^1$. Then $x \leq_n y$. Since p^1 is $\triangleleft$–nondecreasing (!), we have $p^1(x) \triangleright p^1(y)$ or $p^1(x) = p^1(y)$. The former case cannot occur and hence the latter one occurs. But then $p^2(x) \triangleright p^2(y)$ and for $j = 1, \ldots, n-1$

$$p^2(x) \leq^2_j p^2(y) \text{ if } \sigma(j) = 1, \text{ and}$$
$$p^2(x) \geq^2_j p^2(y) \text{ if } \sigma(j) = 2.$$

We may again use the induction hypothesis. □

Theorem 5.9. *Let $n > 1$ and let γ_σ $(\sigma \in \Sigma)$ be ordinals. Then*
(i) If all γ_σ are $< |U|^+$, then $\prod_{\sigma \in \Sigma} \gamma_\sigma \leq \Phi_{\mathcal{E}_n}((\gamma_\sigma)_{\sigma \in \Sigma}) \leq \bigotimes_{\sigma \in \Sigma} \gamma_\sigma$, and
(ii) $\prod_{\sigma \in \Sigma} \gamma_\sigma \leq \overline{\Phi}_{\mathcal{E}_n}((\gamma_\sigma)_{\sigma \in \Sigma}) \leq \bigotimes_{\sigma \in \Sigma} \gamma_\sigma$.

Proof. Again, it is suficient to prove (i). We shall skip the expression "$\sigma \in \Sigma$" whenever it will be possible.
$\prod \gamma_\sigma \leq \Phi_{\mathcal{E}_n}((\gamma_\sigma)_\sigma)$: Let Q and $\tau := (\leq_1, \ldots, \leq_n)$ be as in Lemma 5.8. We may assume that $Q \subseteq U$ and we extend the orderings $\leq_i$ to U arbitrarily. Let $g_\sigma : Q \to \gamma_\sigma$ be the projection to the σ–th coordinate. Extend g_σ by 0 outside Q. Then $g = (g_\sigma)_\sigma$ is a $(\gamma_\sigma)_\sigma$–testing and it follows from Lemma 5.8 that the identity is a tree homomorphism $\mathrm{Bad}(Q) \to \gamma_{\mathrm{Bad}(E^\tau, g)}$, which proves the inequality.
$\Phi_{\mathcal{E}_n}((\gamma_\sigma)_\sigma) \leq \bigotimes_{\sigma \in \Sigma} \gamma_\sigma$: Let $\tau = (\leq_1, \ldots, \leq_n)$ and let a $(\gamma_\sigma)_\sigma$–testing g be given. Let $r : U^2 \to \Sigma$ be such that $x \leq_{r(x,y)} y$ for all $x, y \in U$. Then the identity is a tree homomorphism $\mathrm{Bad}(E^\tau, g) \to \mathrm{Bad}(S^r, g)$ and the inequality follows from 5.4. □

Definition 5.10. We are going to show that, for $n \geq 2$, the upper bound from 5.9 is not attained. In the rest of this section we put $n = 2$ so that $|\Sigma| = 4$.

Suppose that $\leq_1, \leq_2$ are linear orderings on U. Then for $a, b \in U$, $a \neq b$, there is a unique $\sigma(a,b) \in \Sigma$ such that $a \leq_{\sigma(a,b)} b$. Let us call σ, σ' *opposite* if $\sigma(i) + \sigma'(i) = 3$ for every $i = 1, 2$.

Lemma 5.11. *Let $a, b, c \in U$ be distinct and let $\sigma(a,b)$, $\sigma(b,c)$, $\sigma(a,c)$ be distinct. Then $\sigma(a,b)$, $\sigma(a,c)$ are not opposite.*

Proof. Suppose the contrary. Because of symetry and possible reversing of the orderings we may assume that $\sigma(a,b)$ is equal to 1 identically. Then we have $a \leq_1 b$, $a \leq_2 b$, $a \geq_1 c$, $a \geq_2 c$. Thus $c \leq_1 b$, and $c \leq_2 b$, and hence $\sigma(a,b) = \sigma(a,c)$, a contradiction. □

Definition 5.12. Let Q be the Cartesian product of $|\Sigma|$ copies of $\omega+1$ equipped with the product partial ordering. For $q = (q_\sigma)_{\sigma\in\Sigma} \in Q$ we define the *pattern* of q by

$$\pi(q) := \{\sigma \in \Sigma \mid q_\sigma < \omega\} \subseteq \Sigma.$$

A sequence $(q_1, \dots, q_m) \in Q^{<\omega}$ is called *jolly* if

(5.12a) for $i \neq j$, $\pi(q_i) \neq \pi(q_j)$,

(5.12b) for $1 \leq i < j \leq n$, $2 \leq |\pi(q_i)| \leq |\pi(q_j)| \leq 3$, and

(5.12c) for $1 \leq i < j \leq n$, if q_i^σ, q_j^σ are the σth coordinates of q_i, q_j, respectively, and if $q_i^\sigma < \omega$, then $q_i^\sigma \leq q_j^\sigma$.

Thus the length of a jolly sequence is at most 10 and every jolly sequence is bad. We put

$$T := \{a \in \mathrm{Bad}(Q) \mid a \text{ contains no jolly subsequence of length } 10\}.$$

Lemma 5.13. $\gamma_T \leq \omega^4 + 4\omega^3 + 5\omega^2 + 4\omega + 1$.

Proof. The idea is straightforward: since T contains no jolly subsequence of maximal possible length, the essential part of some subset of type $\geq \omega^2$ is not included in T. To make it precise we need some more definitions.

We say that $(q_\sigma)_{\sigma\in\Sigma} \in Q$ is *controlled* by $(q'_\sigma)_{\sigma\in\Sigma} \in Q$ if $\omega > q'_\sigma > q_\sigma$ for some $\sigma \in \Sigma$. For $\pi \subseteq \Sigma$ and $q \in Q$ we define

$$\begin{aligned}
Q_\pi &:= \{q \in Q \mid \pi(q) = \pi\},\\
Q_2^q &:= \{q' \in Q \mid q' \text{ is controlled by } q \text{ and } |\pi(q')| = 2\},\\
Q_3^q &:= \{q' \in Q \mid q' \text{ is controlled by } q \text{ and } 2 \leq |\pi(q')| \leq 3\}.
\end{aligned}$$

It is easily seen that $c(Q_\pi) = \omega^{|\pi|}$, $c(Q_2^q) < \omega^2$, $c(Q_3^q) < \omega^3$; let

$$\begin{aligned}
g_\pi &: Q_\pi \to \omega^{|\pi|},\\
g_2^q &: Q_2^q \to c(Q_2^q),\\
g_3^q &: Q_e^q \to c(Q_3^q)
\end{aligned}$$

be characters. In this proof we shall use a convention that g_π of the empty sequence is $\omega^{|\pi|}$ and similarly for g_2^q and g_3^q. If $s \in Q^{<\omega}$ and $\pi \subseteq \Sigma$, we denote by $s \restriction Q_\pi$ the (possibly empty) subsequence of s consisting of all those terms of s which belong to Q_π. It is worth noting that

$$f : \mathrm{Bad}(Q) \to \omega^4 + 4\omega^3 + 6\omega^2 + 4\omega + 1$$

defined by

$$f(s) := \bigoplus_{\pi\subseteq\Sigma} g_\pi(s \restriction Q_\pi)$$

is a character on $\mathrm{Bad}(Q)$.

For some $s \in Q^{<\omega}$ and $q \in Q$ we define $J(s) \in Q^{<\omega}$, $\Pi(s) \subseteq [\Sigma]^2 \cup [\Sigma]^3$, $s^0 \in Q^{<\omega}$ and $s^q \in Q^{<\omega}$ by induction on the length of s. The intended meaning is the following. $J(s)$ will be the "first jolly subsequence", $\Pi(s)$ will be the set of patterns of terms of $J(s)$, s^0 will be the subsequence of elements controlled by no term of $J(s)$ and s^q will be the subsequence of all elements controlled by q.

For s = empty sequence we initialize all these objects to be empty sequences or empty sets. Now let $s = x.(q)$, where x is possibly empty. We distinguish several cases; each one is ment to assume negation of preceding ones.

(i) If $\Pi(s)$ is undefined, or $|\pi(q)| = 3$ and $[\Sigma]^2 \backslash \Pi(x) \neq 0$, then let $J(s)$, $\Pi(s)$, s^0, s^q be undefined.

(ii) If $|\pi(q)| \neq 2$ or $|\pi(q)| \neq 3$ or $\pi(q) \in \Pi(x)$, then let $J(s) := J(x)$, $\Pi(s) := \Pi(x)$, $s^0 := x^0.(q)$, $s^q := x^q$ for any $q \in Q$.

(iii) If q is controlled by no term of $J(x)$, then let $J(s) := J(x).(q)$, $\Pi(s) := \Pi(x) \cup \{\pi(q)\}$, $s^0 := x^0.(q)$, $s^q := x^q$ for any $q \in Q$.

(iv) Let q' be a term of $J(x)$ such that q is controlled by q', we put $J(s) := J(x)$, $\Pi(s) := \Pi(x)$, $s^0 := x^0$, $s^{q'} := x^{q'}.(q)$, $s^q := x^q$ for any $q \in Q \backslash \{q'\}$.

Let $M(s)$ be the set of terms of $J(s)$. We claim the following.

(5.13a) $J(s)$ is a jolly sequence.

(5.13b) If $[\Sigma]^2 \backslash \Pi(s) \neq 0$ then $[\Sigma]^3 \cap \Pi(s) = 0$.

(5.13c) If $s \in T$ and $[\Sigma]^2 \backslash \Pi(s) = 0$ then $[\Sigma]^3 \backslash \Pi(s) \neq 0$.

(5.13d) If s^q is a nonempty sequence, then $s^q \in (Q^q_3)^{<\omega}$ and if moreover $[\Sigma]^2 \backslash \Pi(s) \neq 0$, then $s^q \in (Q^q_2)^{<\omega}$.

(5.13e) If q is a term of s and $\Pi(s)$ is defined, then q is a term either of s^0 or of $s^{q'}$ for some $q' \in M(s)$.

Condition (5.13c) follows from (5.13a) and the definition of T, the other conditions follow from the construction.

Now we define

$$f : T \to \omega^4 + 4\omega^3 + 5\omega^2 + 4\omega + 1$$

by

$$f(s) := \begin{cases} \bigoplus_{\substack{\pi \subseteq \Sigma \\ \pi \neq \kappa}} g_\pi(s^0 \restriction Q_\pi) \oplus \bigoplus_{q \in M(s)} g^q_2(s^q) & \text{if } [\Sigma]^2 \backslash \Pi(s) \neq \emptyset, \text{ where } \kappa \in [\Sigma]^2 \backslash \Pi(s) \\ \bigoplus_{\pi \subseteq \Sigma} q_\pi(s \restriction Q_\pi) & \text{if } \Pi(s) \text{ is undefined} \\ \bigoplus_{\substack{\pi \subseteq \Sigma \\ \pi \neq \kappa}} g_\pi(s^0 \restriction Q_\pi) \oplus \bigoplus_{q \in M(s)} g^q_3(s^q) & \text{if } [\Sigma]^2 \backslash \Pi(s) = \emptyset, \text{ where } \kappa \in [\Sigma]^3 \backslash \Pi(s) \end{cases}$$

It is easily seen that f is a character on T. □

Theorem 5.14. *We have*

$$\overline{\Phi}_{\mathcal{E}_2}(\omega + 1, \omega + 1, \omega + 1, \omega + 1) \leq \omega^4 + 4\omega^3 + 5\omega^2 + 4\omega + 1$$
$$< (\omega + 1) \otimes (\omega + 1) \otimes (\omega + 1) \otimes (\omega + 1).$$

Proof. By standardness, it is sufficient to prove the inequality for $\Phi_{\mathcal{E}_2}$. So let $\tau = (\leq_1, \leq_2)$ be a pair of linear orderings on U and let $g = (g_\sigma)_{\sigma \in \Sigma}$ be an

$(\omega+1,\omega+1,\omega+1,\omega+1)$–testing. We claim that

$$\gamma_{\mathrm{Bad}(E^{\tau},g)} \leq \omega^4 + 4\omega^3 + 5\omega^2 + 4\omega + 1.$$

This will be done by defining a tree homomorphism $H : \mathrm{Bad}(E^{\tau}, g) \to T$. We define $h = (h_{\sigma})_{\sigma \in \Sigma}$ by

$$h_{\sigma}(a) := \min\{g_{\sigma}(b) \mid b \in \downarrow a \cap E^{\tau}_{\sigma},\ a \text{ and } b \text{ have the same last term}\}$$

and put, for $a = (a_1, \dots, a_m) \in \mathrm{Bad}(E^{\tau}, g)$

$$H(a) = (h(a_1), h(a_1, a_2), \dots, h(a_1, \dots, a_m)),$$

where h is considered as a function $\mathrm{Bad}(E^{\tau}, g) \to Q$. It is easily seen that if $\sigma = \sigma(a_i, a_j)$, then

(5.14a) $\quad h_{\sigma}(a_1, \dots, a_i) < h_{\sigma}(a_1, \dots, a_j)$,

and if $a_i = a_j$, then

(5.14b) $\quad h_{\sigma}(a_1, \dots, a_i) > h_{\sigma}(a_1, \dots, a_j)$ for all $\sigma \in \Sigma$.

Hence H is a tree homomorphism $\mathrm{Bad}(E^{\tau}, g) \to \mathrm{Bad}(Q)$ and thus our aim is to show that its range is in fact contained in T. To this end suppose the contrary, namely that

$$\begin{aligned} b^1 &= (b^1_{\sigma})_{\sigma \in \Sigma} := h(a_1, \dots, a_{i_1}), \\ &\vdots \quad \vdots \quad \vdots \\ b^{10} &= (b^{10}_{\sigma})_{\sigma \in \Sigma} := h(a_1, \dots, a_{i_{10}}), \end{aligned}$$

is a jolly sequence for some $1 \leq i_1 < \dots < i_{10} \leq m$. By (5.14b) we have $a_{i_j} \neq a_{i_p}$ for $1 \leq j < p \leq m$ and thus $\sigma_{j,p} := \sigma(a_{i_j}, a_{i_p})$ is well–defined. Now for $j = 7, 8, 9, 10$ let $\pi(b_j) = \{\sigma_j\}$ ($\pi(b_j)$ consists of one element by (5.12b)). There are p, ℓ such that $7 < p < \ell \leq 10$ and σ_p, σ_ℓ are opposite. Now let $j \in \{1, \dots, 6\}$ be such that $\pi(b_j) = \Sigma \setminus \{\sigma_p, \sigma_\ell\}$. By (5.14a)

$$b^7_{\sigma_{7,p}} > b^p_{\sigma_{7,p}}.$$

From this and (5.12c) we deduce that $\sigma_{j,p} = \sigma_\ell$, $\sigma_{j,\ell} = \sigma_p$, $\sigma_{7,p} = \sigma_{7,\ell} = \sigma_7$, $\sigma_{j,7} \in \{\sigma_p, \sigma_\ell\}$ (see fig. 1, where $\star$ denotes a value less then ω).

$$\begin{array}{rl} & \quad\ \sigma_\ell\ \sigma_p\ \sigma_7 \\ b^j &= (\omega, \omega, \star, \star) \\ b^7 &= (\star, \star, \omega, \star) \\ b^p &= (\star, \omega, \star, \star) \\ b^\ell &= (\omega, \star, \star, \star) \end{array}$$

Fig.1

By (5.12a), σ_7, σ_p, σ_ℓ are distinct. Now if $\sigma_{j,7} = \sigma_p$, then $\sigma_{j,7}$ is opposite to $\sigma_{j,p}$, while $\sigma_{j,7}$, $\sigma_{7,p}$, $\sigma_{j,p}$ are distinct. If, on the other hand, $\sigma_{j,7} = \sigma_\ell$, then $\sigma_{j,7}$ is opposite to $\sigma_{j,\ell}$, while $\sigma_{j,7}$, $\sigma_{7,\ell}, \sigma_{j,\ell}$ are distinct. In both cases we obtain a contradiction to Lemma 5.11. □

6. Ramsey Systems

In this section we give bounds for $\Phi_n(\gamma_1,\dots,\gamma_k)$ and $\overline{\Phi}_n(\gamma_1,\dots,\gamma_k)$. Recall that, according to 3.17, $\Phi_n(\gamma_1,\dots,\gamma_k) = \overline{\Phi}_n(\gamma_1,\dots,\gamma_k)$ for $\gamma_1,\dots,\gamma_k < |U|^+$.

Theorem 6.1.

(i) $\Phi_n(\gamma_1,\dots,\gamma_k) = 0$ *if some* $\gamma_i = 0$.

(ii) *If* $\gamma_1,\dots,\gamma_k > 0$ *then* $\overline{\Phi}_1(\gamma_1,\dots,\gamma_k) = \gamma_1 \oplus \dots \oplus \gamma_k$.

(iii) *If* $0 < \gamma_1,\dots,\gamma_k < |U|^+$ *then* $\Phi_1(\gamma_1,\dots,\gamma_k) = \gamma_1 \oplus \dots \oplus \gamma_k$.

(iv) *For* $n > 1$ *and* $\gamma_1,\dots,\gamma_k > 0$

$$\Phi_n(\gamma_1,\dots,\gamma_k) \le \sup_{\gamma_i'<\gamma_i} [\Phi_{n-1}(\Phi_n(\gamma_1',\dots,\gamma_k),\dots,\Phi_n(\gamma_1,\dots,\gamma_i',\dots,\gamma_k),\dots,\Phi_n(\gamma_1,\dots,\gamma_k')) + 1].$$

Proof.

(i) Obvious.

(ii) $\le$: For $(a,g) \in T(\mathcal{R}^1_k;\gamma_1,\dots,\gamma_k)$ we define

$$f(a,g) = \max_r\{\oplus_{i=1}^k \min\{g_i(b_1,\dots,b_m) \mid (b_1,\dots,b_m) \subseteq a, r(b_1) = \dots = r(b_m) = i\}\},$$

the max taken over all colorings $r : U^1 \to \{1,\dots,k\}$ such that g is an $(R^r;\gamma_1,\dots,\gamma_k)$–germ on a.
It is easily seen that f is a character on $T(\mathcal{R}^1_k;\gamma_1,\dots,\gamma_k)$.
$\ge$: Let $\gamma_1,\dots,\gamma_k$ be given, let us choose distinct elements $x_1,\dots,x_k \in U$ and define

$$r(x) = i \quad \text{if } x = x_i$$
$$= \text{arbitrarily otherwise.}$$

Let Q be the set consisting of all pairs (i,α), where $i \in \{1,\dots,k\}$ and $\alpha \in \gamma_i$, partially ordered by the rule $(i,\alpha) \le (j,\beta)$ if $i = j$ and $\alpha \le \beta$. Then $c(Q) = \gamma_1 \oplus \dots \oplus \gamma_k$ by 4.6.

For $s = ((i_1,\alpha_1),\dots,(i_m,\alpha_m)) \in \mathrm{Bad}(Q)$ let $a^s = (a_1,\dots,a_m) = (x_{i_1},\dots,x_{i_m})$ and define $g_i^s :\downarrow a^s \to \gamma_i$ by

$$g_i^s(a_{j_1},\dots,a_{j_p}) = \alpha_{j_p} \text{ if } a_{j_1},\dots a_{j_p} \text{ are the first } p \text{ occurences of } x_i \text{ within } a^s$$
$$= 0 \quad \text{otherwise.}$$

Then $g^s = (g_1^s, \dots, g_k^s)$ is an $(R^r; \gamma_1, \dots, \gamma_k)$–germ on a^s. Hence if we define a tree homomorphism $H : \mathrm{Bad}(Q) \to T(\{R^r\}; \gamma_1, \dots, \gamma_k)$ by $H(s) = (a^s, g^s)$, we see that $\gamma_1 \oplus \dots \oplus \gamma_k = \gamma_{\mathrm{Bad}(Q)} \leq \gamma_{T(\{R^r\}; \gamma_1, \dots, \gamma_k)} \leq \bar{\Phi}_1(\gamma_1, \dots, \gamma_k)$.
(iii) This follows from (ii) above and Theorem 3.17, but we give a direct proof. It suffices to show that if $\gamma_1, \dots, \gamma_k$ are nonzero and $< |U|^+$, then $\gamma_1, \dots, \gamma_k \leq \Phi_1(\gamma_1, \dots, \gamma_k)$. Let Q be as in (ii), we may safely assume that $Q \subseteq U$. We define a $(\gamma_1, \dots, \gamma_k)$–testing $g = (g_1, \dots, g_k)$ by

$$\begin{aligned} g_i(a_1, \dots, a_m) &= \alpha_m \text{ if } a_j = (i, \alpha_j) \in Q \text{ for } j = 1, \dots, m \\ &= 0 \quad \text{otherwise} \end{aligned}$$

and a coloring $r : U^1 \to \{1, \dots, k\}$ by

$$\begin{aligned} r(a) &= i \text{ if } a = (i, \alpha) \in Q \\ &= 1 \text{ otherwise.} \end{aligned}$$

Let $f : U^{<\omega} \to \Phi_1(\gamma_1, \dots, \gamma_k)$ be the R–character corresponding to R^r and g. Since, as easily seen, every sequence from $\mathrm{Bad}(Q)$ is (R^r, g)–bad, it follows that f is a character on $\mathrm{Bad}(Q)$, hence $\Phi_1(\gamma_1, \dots, \gamma_k) \geq \gamma_1 \oplus \dots \oplus \gamma_k$, as desired.
(iv) For $r : U^n \to \{1, \dots, k\}$ and $x \in U^1$ let $r' : U^{n-1} \to \{1, \dots, k\}$ be defined by $r'(a) = r(x.a)$. Let $g = (g_1, \dots, g_k)$ be a $(\gamma_1, \dots, \gamma_k)$–testing and let T be the tree of (R^r, g)–bad sequences. We are going to estimate the type of T_x. Let $\gamma_i' = g_i(x)$, we define $(\gamma_1, \dots, \gamma_i', \dots, \gamma_k)$–testings $g^i = (g_1^i, \dots, g_k^i)$ by

$$g_j^i(a) = \begin{cases} g_j(a) & \text{if } i \neq j \\ g_i(x.a) & \text{if } i = j \text{ and } g_i(x.a) < \gamma_i' \\ 0 & \text{otherwise.} \end{cases}$$

Let h^i be the R–character corresponding to R^r and g^i. Then $h = (h^1, \dots, h^k)$ is a $(\Phi_n(\gamma_1', \dots, \gamma_k), \dots, \Phi_n(\gamma_1, \dots, \gamma_k'))$–testing. It is easy to see that if $a \in U^{<\omega}$ is such that $x.a \in \mathrm{Bad}(R^r, g)$, then $a \in \mathrm{Bad}(R^{r'}, h)$. Thus

$$\gamma_{T_x} \leq \gamma_{\mathrm{Bad}(R^{r'}, h)} \leq \Phi_{n-1}(\Phi_n(\gamma_1', \dots, \gamma_k), \dots, \Phi_n(\gamma_1, \dots, \gamma_k'))$$

and (iv) follows. □

Corollary 6.2. $\Phi_2(\gamma_1, \dots, \gamma_k) \leq (k+1)^{\gamma_1 \oplus \dots \oplus \gamma_k}$.

Remark 6.3. For $n = 2$, Theorem 6.1 gives the same estimate as the one known is the finite case. The estimate contained in Corollary 6.2 is slightly weaker because of the difficulties with limit ordinals. On the other hand, for $n > 2$, the bound from 6.1(iv) is of little interest; in the finite case, for instance, it is not even primitively recursive in n. To obtain sharper estimates for $\Phi_n(\gamma_1, \dots, \gamma_k)$ $(n > 2)$ one has to use different methods. It is convenient to use the strong R–ordinals here.

Definition 6.4. For $n \geq 3$ and colorings $r_i : U^i \to \{1, \dots j_i\}$ $(i = 1, \dots, n)$ put

$$B^{r_1,\dots,r_n} := \{a \in U^{<\omega} \mid r_i(x_1, \dots, x_i) = r(x_1, \dots, x_{i-1}, x_i') \text{ for all } i = 1, \dots, n \text{ and all subsequences } (x_1, \dots, x_i), (x_1, \dots, x_{i-1}, x_i') \subseteq a\}.$$

We define a 1–system $\mathcal{B}_{j_1,\dots,j_n}$ by

$$\mathcal{B}_{j_1,\dots,j_n} := \{B^{r_1,\dots,r_n} \mid r_i : U^i \to \{1, \dots, j_i\}\}$$

and put

$$\tilde{\Phi}_{j_1,\dots,j_n}(\gamma) := \Phi_{\mathcal{B}_{j_1,\dots,j_n}}(\gamma),$$
$$\tilde{\Phi}^n_k(\gamma) = \tilde{\Phi}_{\underbrace{1,\dots,1}_{n-1},k}(\gamma).$$

We denote by **1** the constant mapping $U^{<\omega} \to \{1\}$.

Theorem 6.5.
(i) $\tilde{\Phi}_{j_1,\dots,j_n}(0) = 0$,
and for $\gamma > 0$
(ii) $\tilde{\Phi}_{j_1,j_2,\dots,j_n}(\gamma) \leq j_1 \otimes \tilde{\Phi}_{1,j_2,\dots,j_n}(\gamma)$,
(iii) $\tilde{\Phi}_{1,j_2,\dots,j_n}(\gamma) \leq \sup\{\tilde{\Phi}_{1\cdot j_2, j_2\cdot j_3,\dots,j_{n-1}\cdot j_n, j_n}(\gamma') + 1 \mid \gamma' < \gamma\}$.

Proof.
(i) Obvious.
(ii) Let the colorings $r_i : U^i \to \{1, \dots, j_i\}$ $(i = 1, \dots, n)$, an ordinal $\gamma > 0$ and a (γ)–testing g be given. Let $h : U^{<\omega} \to \tilde{\Phi}_{1,j_2,\dots,j_n}(\gamma)$ be the character corresponding to $B^{\mathbf{1},r_2,\dots,r_n}$ and g, and let $f : U^{<\omega} \to j_1 \otimes \tilde{\Phi}_{1,j_2,\dots,j_n}(\gamma)$ be the R–character corresponding to R^{r_1} and $\underbrace{(h, \dots, h)}_{j_1 \text{ times}}$, which exists by 6.1(ii). Now if a is $(B^{r_1,\dots,r_n}, g)$–bad, then h is a killing on $\downarrow a \cap R^{r_1}_j$ for every $j = 1, \dots, j_1$, and hence f is a killing on $\downarrow a$, which proves (ii).
(iii) Let the colorings $r_i : U^i \to \{1, \dots, j_i\}$ $(i = 2, \dots, n)$, an ordinal $\gamma > 0$ and a (γ)–testing g be given. Let x be a 1–element sequence consisting of an element of U. We put $j_1 = 1$ for definiteness. Let T be the tree of $(B^{\mathbf{1},r_2,\dots,r_n}, g)$–bad sequences. We are going to estimate the type of T_x. We define colorings $\tilde{r}_i : U^i \to \{1, \dots, j_i.j_{i+1}\}$ $(i = 1, \dots, n-1)$ by

$$\tilde{r}_i(a_1, \dots, a_i) := j_{i+1} \cdot (r_i(a_1, \dots, a_i) - 1) + r_{i+1}(x, a_1, \dots, a_i)$$

and put $\tilde{r}_n := r_n$. We define a $(g(x))$–testing h by

$$\begin{aligned} h(b) &= g(x.b) \text{ if } g(x.b) < g(x) \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Now if $b \in B^{\tilde{r}_1,\dots,\tilde{r}_n}$, then $x.b \in B^{\mathbf{1},r_2,\dots,r_n}$. Thus if $x.a$ is $(B^{\mathbf{1},r_2,\dots,r_n}, g)$–bad, then a is $(B^{\tilde{r}_1,\dots,\tilde{r}_n}, h)$–bad. Hence

$$\gamma_{T_x} \leq \Phi_{B^{\tilde{r}_1,\dots,\tilde{r}_n}}(g(x))$$

and (iii) follows. □

Corollary 6.6. *If γ is finite, then $\check{\Phi}^n_k(\gamma)$ is finite. For any γ we have $\check{\Phi}^n_k(\gamma) \leq \omega^\gamma$.*

Theorem 6.7. *For $n \geq 3$ we have*

$$\Phi_n(\gamma_1, \ldots, \gamma_k) \leq \check{\Phi}^n_k(\Phi_{n-1}(\gamma_1, \ldots, \gamma_k)).$$

Proof. Let $r : U^n \to \{1, \ldots, k\}$, let $\gamma_1, \ldots, \gamma_k$ be ordinals and let $g = (g_1, \ldots, g_k)$ be a $(\gamma_1, \ldots, \gamma_k)$–testing. We define $h : U^{<\omega} \to \overline{\Phi}_{n-1}(\gamma_1, \ldots, \gamma_k)$ by

$$\begin{aligned} h(a) &= \Psi_{T(\mathcal{R}^{n-1}_k; \gamma_1, \ldots, \gamma_k)}(a, g \restriction\downarrow a) && \text{if } g \restriction\downarrow a \text{ is an } (R^{\tilde{r}}; \gamma_1, \ldots, \gamma_k)\text{–germ on } a \\ & && \text{for some } \tilde{r} : U^{n-1} \to \{1, \ldots, k\} \\ &= 0 && \text{otherwise.} \end{aligned}$$

We claim that if a is (R^r, g)–bad, then a is $(B^{1,\ldots,1,r}, h)$–bad. Indeed, let $b \in \downarrow a \cap B^{1,\ldots,1,r}$, then for a sequence $(x_1, \ldots, x_n) \subseteq b$ we may define $\tilde{r}(x_1, \ldots, x_{n-1}) = r(x_1, \ldots, x_n)$, since the right hand side does not depend on x_n. Thus $g \restriction\downarrow b$ is an $(R^{\tilde{r}}; \gamma_1, \ldots, \gamma_k)$–germ on b and hence h is a killing on $\downarrow b$. Thus, a is $(B^{1,\ldots,1,r}, h)$–bad. We conclude that the identity is a tree homomorphism $\mathrm{Bad}(R^r, g) \to \mathrm{Bad}(B^{1,\ldots,1,r}, h)$. Hence

$$\Phi_{R^r}(\gamma_1, \ldots, \gamma_k) \leq \Phi_{B^{1,\ldots,1,r}}(\overline{\Phi}_{n-1}(\gamma_1, \ldots, \gamma_k)),$$

which gives the theorem by 3.17. □

Remark 6.8. Note that Theorem 3.17 is used essentially in the proof of 6.7.

Corollary 6.9. *For $n \geq 2$ we have*

$$\Phi_n(\gamma_1, \ldots, \gamma_k) \leq \left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^{(k+1)^{\gamma_1 \oplus \cdots \oplus \gamma_k}}}}}}}\right\}(n-2) \text{ times}.$$

Now we are going to obtain lower bounds for $\Phi_n(\gamma_1, \ldots, \gamma_k)$.

Definition 6.10. We need to consider another set $\overline{U}$ and for every tree $S \subseteq U^{<\omega}$ its dual tree $\overline{S} \subseteq \overline{U}^{<\omega}$. The set $\overline{U}$ is defined as the set of all functions $\varepsilon : U^{<\omega} \to \{0,1\}$ with the property that there exists an $a \in U^{<\omega}$ such that $\varepsilon(b) = 0$ for every $b \in U^{<\omega}$ which is not a segment of a. The letter ε (with or without dashes or suffixes) is reserved to designate elements of $\overline{U}$ or $\overline{U}^{<\omega}$.

For $\varepsilon, \varepsilon' \in \overline{U}$ we define $\varepsilon \lhd \varepsilon'$ if there exists an $a \in U^{<\omega}$ such that $\varepsilon(a) < \varepsilon'(a)$ and for every $b \in U^{<\omega}$, $\varepsilon(b) \neq \varepsilon'(b)$ implies $a = b$ or $a \ll b$. In this case we define $D(\varepsilon, \varepsilon') := a$; this determines $D(\varepsilon, \varepsilon')$ for $\varepsilon \lhd \varepsilon'$ uniquely. The relation $\lhd$ is easily seen to be an ordering. Observe the following properties of $\lhd$ and D :

(6.10a) If $\varepsilon \lhd \varepsilon' \lhd \varepsilon''$, then $D(\varepsilon, \varepsilon') \neq D(\varepsilon', \varepsilon'')$.

(6.10b) If $\varepsilon_1 \lhd \varepsilon_2 \lhd \ldots \lhd \varepsilon_n$, then $D(\varepsilon_1, \varepsilon_n)$ is the $\ll$–minimum element of $\{D(\varepsilon_{i-1}, \varepsilon_i) \mid 1 < i \leq n\}$.

If $S \subseteq U^{<\omega}$ is a tree, we define the dual tree $\overline{S} \subseteq \overline{U}^{<\omega}$ as the set of all sequences $(\varepsilon_1, \dots, \varepsilon_m) \in \overline{U}^{<\omega}$ such that
(6.10c) if $\varepsilon_i(a) = 1$ then $a \in S$ for every $a \in U^{<\omega}$ and $i = 1, \dots, m$,
(6.10d) $\varepsilon_1 \lhd \varepsilon_2 \lhd \dots \lhd \varepsilon_m$.

Lemma 6.11. *Let $S \subseteq U^{<\omega}$ contain no infinite chain and let $\overline{S} \subseteq \overline{U}^{<\omega}$ be the dual tree. Then $\overline{S}$ contains no infinite chain and we have*

$$\gamma_{\overline{S}} \geq 2^{\gamma_S}.$$

Proof. Suppose that

$$(\varepsilon_1),\ (\varepsilon_1, \varepsilon_2), \dots, (\varepsilon_1, \dots, \varepsilon_n), \dots$$

is an infinite chain in $\overline{S}$. We put

$$D_i := \lim_{n \to \infty} D(\varepsilon_i, \varepsilon_n).$$

(By (6.10b) the right hand sequence is eventually constant.) By (6.10b), too, the sequence $D_1, D_2, \dots$ is $\ll$–nondecreasing and by (6.10a) is not eventually constant. Thus it yields an infinite chain in S.

This proves the former statement. To prove the latter one we proceed by induction on γ_S. Let S be fixed and suppose that the lemma holds for every tree $S' \subseteq U^{<\omega}$ such that $\gamma_{S'} < \gamma_S$. Let $\gamma < \gamma_S$ be given. We denote by T the tree of all $(\varepsilon_1, \dots, \varepsilon_m) \in \overline{S}$ such that $\varepsilon_1(x) = \dots = \varepsilon_m(x) = 0$ for every $x \in U^{<\omega}$ such that $|x| = 1$. If $x \in S$ then $\overline{(S_x)}$ (i.e. the dual tree to S_x) is contained in T; hence, by the induction hypothesis
(6.11a) $\gamma_T \leq \gamma_{\overline{(S_x)}} \geq \sup_{x \in S} 2^{\gamma_{S_x}} \geq 2^\gamma$.
We claim that
(6.11b) $\gamma_{\overline{S}_t} \geq 2^\gamma$ for every $t \in T$.
To prove (6.11b) let $x \in S$ be such that $|x| = 1$; we define $H : \overline{(S_x)} \to \overline{S}_t$ by

$$H(\varepsilon_1, \dots, \varepsilon_m) = t.(\varepsilon_1', \dots, \varepsilon_m');$$

where

$$\begin{aligned} \varepsilon_i' &= \varepsilon_1(a) \text{ if } a \neq x \\ &= 1 \qquad \text{if } a = x. \end{aligned}$$

Then H is a tree homomorphism showing that

$$\gamma_{\overline{S}_t} \geq \sup_x \gamma_{\overline{(S_x)}} \geq \sup_x 2^{\gamma_{S_x}} \geq 2^\gamma,$$

which proves (6.11b).

Now (6.11a) and (6.11b) imply

$$\gamma_{\overline{S}} \geq 2^\gamma + 2^\gamma = 2^{\gamma+1}$$

which proves the lemma, since $\gamma < \gamma_S$ was arbitrary. □

6.12 Stepping-Up Lemma. *For $n \geq 3$ and ordinals $\gamma_1, \dots, \gamma_k$ we have*

$$\Phi_{n+1}(\gamma_1', \gamma_2', \dots, \gamma_k') \geq 2^{\Phi_n(\gamma_1, \dots, \gamma_k)},$$

where

$$\begin{aligned} \gamma_i' &= \min(\gamma_1, \omega) + \sup\{\gamma' + n \mid \gamma' < \gamma_1\} && \text{for } i = 1 \\ &= \gamma_2 + \min(\gamma_2 + n - 1, \omega) && \text{for } i = 2 \\ &= \sup\{\gamma' + n \mid \gamma' < \gamma_i\} && \text{for } i = 3, \dots, k \end{aligned}$$

Proof. We may assume that $\gamma_1, \dots, \gamma_k < |U|^+$, for if $\gamma_i \geq |U|^+$ for some i and all γ_i are nozero, then $\Phi_{n+1}(\gamma_1', \dots, \gamma_k') = \Phi_n(\gamma_1, \dots, \gamma_k) = |U|^+ = 2^{|U|^+}$ by 2.6, 6.1(iii) and obvious monotonicity of the R–ordinals. By 3.18 there exists a $(\gamma_1, \dots, \gamma_k)$–testing $g = (g_1, \dots, g_k)$, a coloring $r : U^n \to \{1, \dots, k\}$ and a simple substree S of $\mathrm{Bad}(R^r, g)$ of type $\Phi_n(\gamma_1, \dots, \gamma_k)$. We shall define a coloring $\overline{r} : \overline{U}^{n+1} \to \{1, \dots, k\}$ and a $(\gamma_1', \dots, \gamma_k')$–testing $h = (h_1, \dots, h_k)$ such that $\overline{S} \subseteq \mathrm{Bad}(R^{\overline{r}}, h)$. Then the lemma will follow from 6.11.

For $\delta_1, \delta_2 \in U$ we define $\delta_1 \lhd \delta_2$ if $(\delta_1, \delta_2) \subseteq a$ for some $a \in S$. This is an ordering by simplicity of S and the definition of a subtree. For $\varepsilon, \varepsilon' \in \overline{S}$ let $\delta(\varepsilon, \varepsilon')$ be the last term of $D(\varepsilon, \varepsilon')$. By (6.10a), (6.10b) and simplicity of S we have

(6.12a) if $\varepsilon, \varepsilon', \varepsilon'' \in \overline{S}$ and $\varepsilon \lhd \varepsilon' \lhd \varepsilon''$, then $\delta(\varepsilon, \varepsilon') \neq \delta(\varepsilon', \varepsilon'')$,
(6.12b) if $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m \in \overline{S}$ and $\varepsilon_1 \lhd \varepsilon_2 \lhd \dots \lhd \varepsilon_m$, then
$\delta(\varepsilon_1, \varepsilon_m) = \lhd - \min_{1 < i \leq n} \delta(\varepsilon_{i-1}, \varepsilon_i)$.

Let $E = (\varepsilon_1, \dots, \varepsilon_{n+1}) \in \overline{U}^{n+1}$. We define

$$\begin{aligned} \overline{r}(E) &= r(\delta_1, \dots, \delta_n) && \text{if } \delta_1 \lhd \dots \lhd \delta_n \text{ or } \delta_n \lhd \dots \lhd \delta_1, \\ &= 1 && \text{if } \delta_1 \lhd \delta_2 \rhd \delta_3, \\ &= 2 && \text{if } \delta_1 \rhd \delta_2 \lhd \delta_3, \\ &= \text{arbitrarily} && \text{otherwise.} \end{aligned}$$

Claim 6.13. *Let $\varepsilon_1 \lhd \varepsilon_2 \lhd \dots \lhd \varepsilon_m$ be elements of $\overline{U}$, $m \geq n$ and assume that $(\varepsilon_1, \dots, \varepsilon_m) \in R_i^{\overline{r}}$. Put, for $j = 1, \dots, m-1$, $\delta_j = (\varepsilon_j, \varepsilon_{j+1})$. Then there exists a p such that $1 \leq p \leq m - n + 1$ and*
(6.13a) $(\delta_1, \dots, \delta_p) \in R_i^r$ and $(\delta_p, \dots, \delta_{m-n+1}) \in R_i^r$,
and one of the following possibilities occurs.

(6.13b) $\delta_1 \lhd \delta_2 \dots \lhd \delta_{p-1} \lhd \delta_p \rhd \delta_{p+1} \rhd \dots \rhd \delta_{m-n+1}$, or
(6.13c) $\delta_1 \rhd \delta_2 \rhd \dots \rhd \delta_{p-1} \rhd \delta_p \lhd \delta_{p+1} \lhd \dots \lhd \delta_{m-n+1}$.

Moreover, if $1 < k < m - n + 1$, then either $i = 1$ and (6.13b) holds, or $i = 2$ and (6.13c) holds.

Proof. For $2 \leq j \leq m - n + 1$, let us call j *local max* if $\delta_{j-1} \lhd \delta_j \rhd \delta_{j+1}$ and *local min* if $\delta_{j-1} \lhd \delta_j \rhd \delta_{j+1}$. If $i \neq 1$ then there can be no local max j, since otherwise $(\varepsilon_{j-1}, \varepsilon_j, \varepsilon_{j+1}, \dots, \varepsilon_{n+1}) \in R_1^{\overline{r}}$. Similarly there can be no local min if $i \neq 2$. Since between any two local max's there must be a local min and vice versa, all of the claim except (6.13a) follows.

To prove (6.13a) let $(\delta_{i_1}, \ldots, \delta_{i_n}) \subseteq (\delta_1, \ldots, \delta_{m-n+1})$ and let, say, $\delta_{i_1} \lhd \delta_{i_2} \lhd \ldots \lhd \delta_{i_n}$. The result follows by stepping–up to the sequence

$$E = (\varepsilon_{i_1}, \ldots, \varepsilon_{i_n}, \varepsilon_{i_n+1}).$$

For $1 \leq j < n$

$$\delta(\varepsilon_{i_j}, \varepsilon_{i_{j+1}}) = \lhd - \min\{\delta_\ell \mid i_j \leq \ell < i_{j+1}\} = \delta_{i_j}$$

by monotonicity and

$$\delta(\varepsilon_{i_n}, \varepsilon_{i_n+1}) = \delta_{i_n}.$$

The δ_{i_j} are monotonic so E is colored "by the δ", and $r(\delta_{i_1}, \ldots, \delta_{i_n}) = \overline{r}(E) = i$. If $\delta_{i_1} \rhd \ldots \rhd \delta_{i_n}$ the same argument works with

$$E = (\varepsilon_{i_1}, \varepsilon_{i_1+1}, \ldots, \varepsilon_{i_n+1}). \qquad \square$$

6.14 Proof of 6.12 Continued. =|| For $\delta_m \in U$ we put δ_m $:= m$ if $(\delta_1, \ldots, \delta_m) \in S$ for some $\delta_1, \ldots, \delta_{m-1} \in U$. Note that . is unique, if defined. It remains to define h. So let $(\varepsilon_1, \ldots, \varepsilon_m) \subseteq \varepsilon \in \overline{S}$, let $(\varepsilon_1, \ldots, \varepsilon_m) \in R_i^{\overline{r}}$ and let p be as in 6.13. We define $\delta_j := \delta(\varepsilon_j, \varepsilon_{j+1})$ for $j = 1, \ldots, m-1$ and observe the following.

(6.14a) If $\delta_j \rhd \delta_k$ then $\delta_j \quad < \quad \delta_k$.

(6.14b) If $\delta_{i_1} \lhd \ldots \lhd \delta_{i_s}$ then $g_i(\delta_{i_1}, \ldots, \delta_{i_{s-1}}) < g_i(\delta_{i_1}, \ldots, \delta_{i_s})$.

Now we define for $i = 1$

$$\begin{aligned} h_1(\varepsilon_1, \ldots, \varepsilon_m) &= \min(\gamma_1, \omega) + g_1(\delta_1) + n - m && \text{if } m \geq n \\ &= \min(\gamma_1, \omega) + g_1(\delta_1, \ldots, \delta_{m-n+1}) && \text{if } m \geq n \ p = m - n + 1 \\ &= \quad \delta_{m-1} && \text{if } m \geq n,\ p < m - n + 1, \\ & && \text{and } \gamma_1 \geq \omega \\ &= \gamma_1 - m && \text{if } m \geq n,\ p < m - n + 1, \\ & && \text{and } \gamma_1 < \omega \end{aligned}$$

for $i = 2$

$$\begin{aligned} h_2(\varepsilon_1, \ldots, \varepsilon_m) &= \gamma_2 + \quad \delta_1 \quad + n - m \text{ if } m \leq n,\ \gamma_2 \geq \omega \\ &= 2\gamma_2 + n - m - 1) \quad \text{if } m \leq n \text{ or } p = m - n + 1, \text{ and } \gamma_2 < \omega \\ &= \gamma_2 + \quad \delta_{m-n+1} \qquad \text{if } m \geq n,\ p = m - n + 1,\ \gamma_2 \geq \omega \\ &= g_2(\delta_p, \ldots, \delta_{m-n+1}) \text{ if } m \geq n,\ p < m - n + 1 \end{aligned}$$

and for $i = 3, \ldots, k$

$$\begin{aligned} h_i(\varepsilon_1, \ldots, \varepsilon_m) &= \max(\ \delta_1 \ , g_i(\delta_1)) + n - m \text{ if } m \leq n,\ \gamma_i \geq \omega \\ &= g_i(\delta_1, \ldots, \delta_{m+n-1}) && \text{if } \delta_1 \lhd \ldots \lhd \delta_{m+n-1},\ \gamma_i \geq \omega \\ &= \quad \delta_{m-n+1} && \text{if } \delta_1 \rhd \ldots \rhd \delta_{m+n-1},\ \gamma_i \geq \omega \\ &= \gamma_i + n - m + 1 && \text{if } \gamma_i < \omega \end{aligned}$$

$\square$

Corollary 6.15. *For p, ℓ finite we have*

$$\Phi_n(2p+n-1, 2\ell+n-1, \omega, \dots, \omega) \geq 2^{\Phi_n(p,\ell,\omega,\dots,\omega)}.$$

Using the trivial lower bound $\Phi_3(\omega, \dots, \omega) \geq \omega^k$ (k repetitions of ω), which follows from 5.4 we obtain

Theorem 6.16. *For $k \geq 3$ we have*

$$\left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^k}}}}}\right\}{\scriptstyle (n-1)\ \text{times}} \geq \Phi_n\underbrace{(\omega, \dots, \omega)}_{k\,\text{times}} \geq \left.\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^k}}}}}\right\}{\scriptstyle (n-2)\ \text{times}}.$$

7. Canonical Ramsey Theorem

In this section $n \geq 1$ will be a fixed integer.

Definition 7.1. Let $r : U^n \to U^n$ and $\kappa \subseteq \{1, \dots, n\}$. We say that a sequence $a \in U^{<\omega}$ is (r,κ)-*canonical* if for every two sequences $(x_1, \dots, x_n), (y_1, \dots, y_n) \subseteq a$ we have $r(x_1, \dots, x_n) = r(y_1, \dots, y_n)$ if and only if $x_i = y_i$ for every $i \in \kappa$. Now we define a 2^n-sheaf $C^r = (C^r_\kappa)_{\kappa\subseteq\{1,\dots,n\}}$ by

$$C^r_\kappa := \{a \in U^{<\omega} \mid a \text{ is } (r,\kappa)\text{-canonical}\}$$

and put

$$\mathcal{C}_n := \{C^r \mid r : U^n \to U^n\}.$$

It is easily seen that $\mathcal{C}^n$ is a standard 2^n-system. It has the R-property by the canonical Ramsey theorem of Erdös and Rado.

Definition 7.2. Let $b = (b_1, \dots, b_m) \in U^{<\omega}$, and let $x = (b_{\alpha_1}, \dots, b_{\alpha_n})$, $y = (b_{\beta_1}, \dots, b_{\beta_n})$, $u = (b_{\gamma_1}, \dots, b_{\gamma_n})$ and $v = (b_{\delta_1}, \dots, b_{\delta_n})$ be subsequences of b. We put $x : y = u : v$ if we have

$$\alpha_i \leq \beta_i \text{ iff } \gamma_i \leq \delta_i, \text{ and } \alpha_i \geq \beta_i \text{ iff } \gamma_i \geq \delta_i.$$

Let $r : U^n \to U^n$. We say that $b \in U^{<\omega}$ is r-*invariant* if for all subsequences x, y, u, v of b such that $x : y = u : v$ and $r(x) = r(y)$ we have $r(u) = r(v)$.

Lemma 7.3. *Let $r : U^n \to U^n$ and $b \in U^{<\omega}$. If b is r-invariant and contains at least $2n+1$ distinct elements, then it is (r,κ)-canonical for some $\kappa \subseteq \{1, \dots, n\}$.*

Proof. This is a standard argument. See e.g. Rado's paper in this volume. □

Theorem 7.4. *Let k be the number of equivalence relations on the set $[\{1, \dots, 2n\}]^n$. Then*

$$\Phi_{\mathcal{C}^n}((\gamma_\kappa)_{\kappa\subseteq\{1,\dots,n\}}) \leq \Phi_{2n}((\gamma_\kappa + 2n - 1)_{\kappa\subseteq\{1,\dots,n\}}, 3n, \dots, 3n),$$

where the argument $3n$ is repeated $(k - 2^n)$ times.

Proof. Let T be the germ tree $T(C^n; (\gamma_\kappa)_\kappa)$, and let $\mathcal{E}$ designate the set of all equivalence relations on $[\{1, ..., 2n\}]^n$. For $\kappa \subseteq \{1, ..., n\}$ let $E_\kappa \in \mathcal{E}$ be defined by

$$\{\alpha_1 < ... < \alpha_n\} E_\kappa \{\beta_1 < ... < \beta_n\} \text{ iff } \alpha_i = \beta_i \text{ for every } i \in \kappa.$$

Let $(a, g) \in T$ and let $g = (g_\kappa)_\kappa$ be a $(C^r, (\gamma_\kappa)_\kappa)$-germ on a, where $C^r = (C^r_\kappa)_\kappa$ and $r : U^n \to U^n$. We define

$$\overline{r} : U^{2n} \to \mathcal{E}$$

so that $\overline{r}(z_1, ..., z_{2n}) = E$ if and only if it holds

$$\{\alpha_1 < ... < \alpha_n\} E \{\beta_1 < ... < \beta_n\} \text{ iff } r(z_{\alpha_1}, ..., z_{\alpha_n}) = r(z_{\beta_1}, ..., z_{\beta_n}).$$

Finally, we define $h = (h_E)_{E \in \mathcal{E}}$ by the rule

$$\begin{aligned} h_E(b) &= \gamma_\kappa + 2n - 1 - |b| && \text{if } E = E_\kappa \text{ for some } \kappa \text{ and } |b| < 2n \\ &= g_\kappa(b) && \text{if } E = E_\kappa \text{ for some } \kappa \text{ and } |b| \geq 2n \\ &= \max(3n - |b|, 0) && \text{otherwise .} \end{aligned}$$

We are going to show that $H : T \to T(\mathcal{R}^{2n}_k; (\gamma_\kappa + 2n - 1)_\kappa, 3n, ..., 3n)$ defined by $H(a, g) = (a, h)$ is a tree homomorphism, which will give the theorem. To this end we must show that h is an $(R^{\overline{r}}; (\gamma_\kappa + 2n - 1)_\kappa, 3n, ..., 3n)$-germ on a. To see this it is sufficient to show that

(7.4a) if $b \in R^{\overline{r}}_{E_\kappa} \cap \downarrow a$ and $|b| \geq 2n$ then $b \in C^r_\kappa$, and

(7.4b) if $b \in R^{\overline{r}}_E \cap \downarrow a$ and $|b| \geq 3n + 1$, then $E = E_\kappa$ for some κ.

To prove (7.4a) let $b \in R^{\overline{r}}_{E_\kappa} \cap \downarrow a$ and let $|b| \geq 2n$. Let $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$, $(z_1, \ldots, z_{2n}) \subseteq b$ be such that $x_i = z_{\alpha_i}$, $y_i = z_{\beta_i}$, for some $\alpha_1 < ... < \alpha_n, \beta_1 < ... < \beta_n$. Now we have $r(x_1, \ldots, x_n) = r(y_1, \ldots, y_n) \Leftrightarrow r(z_{\alpha_1}, \ldots, z_{\alpha_n}) = r(z_{\beta_1}, ..., z_{\beta_n}) \Leftrightarrow \{\alpha_1, ..., \alpha_n\} E_\kappa \{\beta_1, ..., \beta_n\} \Leftrightarrow \alpha_i = \beta_i$ for every $i \in \kappa \Leftrightarrow x_i = y_i$ for every $i \in \kappa$, which shows that $b \in C^r_\kappa$.

To prove (7.4b) let $b \in R^{\overline{r}}_E \cap \downarrow a$ and let $|b| \geq 3n + 1$. Let us choose $(b_{-n}, \ldots, b_{-1}, b_0, b_1, \ldots, b_{2n}) \subseteq b$ and let $c = (b_0, \ldots, b_{2n})$. We claim that c is r-invariant. So let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_n)$ be subsequences of c, let $x : y = u : v$ and let $r(x) = r(y)$. It follows from $x : y = u : v$ that we can find subsequences $(z_1, \ldots, z_{2n}), (w_1, \ldots, w_{2n})$ of b (here we need that $|b| \geq 3n + 1$) and sets $\{\alpha_1 < ... < \alpha_n\} \subseteq \{1, ..., 2n\}$, $\{\beta_1 < ... < \beta_n\} \subseteq \{1, ..., 2n\}$ such that $x_i = z_{\alpha_i}, y_i = z_{\beta_i}, u_i = w_{\alpha_i}$ and $v_i = w_{\beta_i}$. Since $\overline{r}(z_1, \ldots, z_{2n}) = \overline{r}(w_1, ..., w_{2n}) = E$ we have $r(x_1, \ldots, x_n) = r(y_1, ..., y_n) \Rightarrow r(z_{\alpha_1}, \ldots, z_{\alpha_n}) = r(z_{\beta_1}, \ldots, z_{\beta_n}) \Rightarrow \{\alpha_1, ..., \alpha_n\} E \{\beta_1, ..., \beta_n\} \Rightarrow r(w_{\alpha_1}, ..., w_{\alpha_n}) = r(w_{\beta_1}, ..., w_{\beta_n}) \Rightarrow r(u_1, ..., u_n) = r(v_1, ..., v_n)$, which proves that c is r-invariant. Now by Lemma 7.3 there exists a set $\kappa \subseteq \{1, ..., n\}$ such that c is (r, κ)-canonical. Hence $E = E_\kappa$, which proves (7.4b) and thus completes the proof of the theorem. □

Corollary 7.5. *We have for $n \geq 1$*

$$\Phi_{C^n}((\gamma_\kappa)_{\kappa \subseteq \{1, ..., n\}}) \leq \left.\omega^{\omega^{\cdot^{\cdot^{\omega \oplus \{\gamma_\kappa | \kappa \subseteq \{1, \ldots, n\}\}}} \oplus \omega}}\right\}(2n-1) \textit{times}.$$

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Part III

Structural Theory

Partite Construction and Ramsey Space Systems

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Abstract

We prove several Ramsey type theorems for parameter sets, affine and vector spaces by an amalgamation technique known as Partite Construction. This approach yields solution of several open problems and uniform treatment of several strongest results in the area. Particularly we prove Ramsey theorem for systems of spaces.

1. Introduction

The following result is one of the most useful and fundamental combinatorial statements

Finite Ramsey Theorem *(Ramsey 1930). For every choice of positive integers t, a, b there exists a positive integer c such that $c \longrightarrow (b)_t^a$.*

Here the undefined symbol $c \longrightarrow (b)_t^a$ is a shorthand notation (due to Erdös and Rado) for the following statement:
For every partition of all a–element subsets of a set X of size c into t classes there exists a b–element subset B of X such that all a–element subsets of B belong to the same class of the partition.

This theorem has been generalized many times and several of these generalizations are both profound and difficult to prove. One of the most useful and celebrated theorems is due to Graham and Rotschild (1971) which we state after introducing a few standard notions. Let $A = \{a_1, \ldots, a_q\}$ be fixed finite set and let $B \subseteq A$ be non–empty. For non–negative integers $k \leq n$, we will define special subsets P_k, called *k–parameter sets*, of the cartesian product A^n, in the following way (cf. Graham, Rothschild 1971, Graham, Rothschild, Spencer 1980):

For disjoint, nonempty subsets $\omega_1, \omega_2, \ldots, \omega_k$ of $[n] = \{1, 2, \ldots, n\}$ define P_k to consist of all those $(x_1, \ldots, x_n) \in A^n$ such that:

(i) If $u, v \in \omega_j$ for some j then $x_u = x_v$

(ii) If $u \in [n] - \bigcup_j \omega_j$ then $x_u = a_u$ – a fixed element of B.

The elements of $\bigcup_j \omega_j$ are usualy called the *moving coordinates* of P_k and the elements of the (possibly empty) subset $[n] - \bigcup_j \omega_j$ are called the *fixed coordinates* of P_k. In a certain sence, P_k is the combinatorial analogue of a k–dimensional affine space over a q–element field (at least, when q is a prime power). Observe that $|P_k| = q^k$ for $k \geq 0$. A set $X \subseteq A^n$ is said to be an i–parameter subset of P_k if X is an i–parameter set in A^n and $x \subseteq P_k$. A discussion of various properties of k–parameter sets can be found in Graham, Rothschild (1971) and in the paper by Prömel and Voigt in this volume.

When q is a prime power and $A = GF(q)$, a more common substructure of A^n is that of a k–dimensional affine (or vector) space over $GF(q)$. Since we will be treating both k–parameter sets and k–dimensional spaces in A^n simultaneously, we will call them both *k–spaces* in A^n (although when we use the term we will always have one particular interpretation in mind). We will denote the set of k–spaces in A^n by $\binom{A^n}{k}$ and their number by $\left[{n \atop k}\right]$. X will be called subspace of A^n if $X \in \binom{A^n}{k}$ for some k, in which case k is called the *dimension* of X, denoted by $\dim X$.

The following statement expresses the basic Ramsey theorem for k–spaces.

Theorem *(Ramsey theorem for spaces)(Graham, Rothschild 1971, Graham, Leeb, Rothschild 1972). For all integers t, a, b with $0 \leq a \leq b$ there exists integer $N_0(t, a, b, A, B)$ such that if $N \geq N_0(t, a, b, A, B)$ and $\binom{A^N}{a} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_t$ is an arbitrary partition of the a–subspaces of A^N into t classes, then there is always a b–subspace $X \in \binom{A^N}{b}$ with $\binom{X}{a} \subseteq \mathcal{A}_i$ for some i.*

We should remark that the case of this theorem for k–parameter sets with $b = 1$, $a = 0$ is known as the Hales–Jewett theorem (Hales, Jewett 1963), and will be needed below in Section 3 in the proof of Partite Lemma.

Putting $\mathbf{A} = A^a$, $\mathbf{B} = A^b$, $\mathbf{C} = A^c$ we shall denote the validity of the above statement by $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_t$.

Ramsey theorem has been generalized by the authors to

- induced theorem (Nešetřil, Rödl 1977, 1982)
- to Ramsey theorem for classes of set systems of given type (Nešetřil, Rödl 1977)
- to Ramsey theorem for systems not containing a given configuration (Nešetřil Rödl 1977, 1981) and finally to
- Ramsey theorem for sparse systems (Nešetřil, Rödl 1984, 1987) (i.e. systems not containing short cycles).

These results are easy to state (at least on this level of generality) but the proofs were quite involved and several of these results presented a challenging problems for several years. See Nešetřil, Rödl (1979) for an outline of this development.

In Nešetřil, Rödl (1978) the authors posed as a problem to carry out this program for spaces and space system. This perhaps too optimistic project remained an open problem and only fragmentary results were obtained (see e.g. Nešetřil, Rödl 1979). The first major breakthrough has been the proof of induced space system theorem by Prömel (1985). One can say that this proof is an elaborate refinement of the original Graham Rothschild ideas.

Meanwhile, the authors found a new method for proving theorems of this type. This is based on a particular *Amalgamation Technique* and the heart of it is so called *Partite Construction*, see e.g. Nešetřil, Rödl (1981, 1982). This construction has been applied to space systems in Frankl, Graham, Rödl (1987) and Nešetřil, Rödl (1987) where a simple proof of a generalization of Prömel's theorem has been given and a problem concerning space systems related to Rado-Folkman-Sanders has been solved. It was apparent that once a proper amalgamation pattern has been realized the methods for set systems may be carried over to space systems. This has been sketched in Nešetřil, Rödl (1987) and it is here where we carry out in full the program of Ramsey theorems for space systems. Somehow surprisingly the methods are *formally* very similar to those methods used for set systems and the formal similarity with the companison paper (Nešetřil, Rödl (1989)) is at certain places striking. Let us remark that recently Prömel and Voigt (1988) claimed independently several results of this paper (on parameter sets). Their attempt is also based on the Partite Construction however the proof is defective.

The paper is organized as follows: The first part contains the statement of results, the second one Partite Lemma which is a key place in our proof and the third one Partite Construction. In the last part we present generalizations and strengthenings which follow from our method.

2. Statement of Results

Our theorems deal with *space–systems*. It will be irrelevant whether we deal with vector or affine spaces or with parameter sets. Thus, for brevity, we use short term *space* and we shall assume that we have a particular interpretation in mind.

For a space V we denote by $\dim(V)$ the dimension of V. Denote by $\binom{V}{k}$ the set of all subspaces of V with dimension k. Thus $\left|\binom{V}{k}\right| = \left[{\dim V \atop k}\right]$ – the Gaussian coefficient.

A *type* is a sequence $(n_\delta;\ \delta \in \Delta)$ of non–negative integers. A type will be fixed throughout this paper.

A *space system* $\mathbf{A}$ (briefly *system*) is a pair $(V, \mathcal{S})$ where V is a space, $\mathcal{S} = (\mathcal{S}_\delta; \delta \in \Delta)$ and $\mathcal{S}_\delta \subseteq \binom{V}{n_\delta}$. Elements of V are called *points*, elements of $\bigcup_{\delta \in \Delta} \mathcal{S}_\delta$ *edges* of $\mathbf{A}$. Sometimes we write $V = V(\mathbf{A})$, $\mathcal{S}_\delta = \mathcal{S}_\delta(\mathbf{A})$. Without loss of generality we will assume that the edge sets $\mathcal{S}_\delta$ are mutually disjoint.

A system will be always considered with a linear ordering of its points. This needs a bit careful consideration:

As an easy consequence of Graham, Rothschild (1971) and Graham, Leeb, Rothschild (1972) theorems there exists an ordering of points of A^X with the property that for any two subspaces V, V' of A^X with the same dimension the monotonne mapping $f : V \to V'$ is linear. Note that the structure of such orderings, called canonical orderings was characterized in Nešetřil, Prömel, Rödl, Voigt (1985) From now on when we speak about systems we understand that these are (canonically) ordered, and the corresponding linear mappings are order preserving.

We say that system $\mathbf{A}$ is a subsystem of system $\mathbf{B} = (U, \mathcal{T})$ if V is an (ordered) subspace of U and $\mathcal{T}_\delta \cap \binom{V}{n_\delta} = \mathcal{S}_\delta$ for every $\delta \in \Delta$. Denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all subspaces of $\mathbf{B}$ which are isomorphic to $\mathbf{A}$.

Using these concepts we may formulate the main result of this paper:

2.1 Theorem *(Ramsey theorem for space system). Let t be a positive integer, $\mathbf{A}$, $\mathbf{B}$ systems. Then there exists a system $\mathbf{C}$ with the following properties:*
(i) $\mathbf{C} \to (\mathbf{B})_t^{\mathbf{A}}$
(ii) $\mathbf{C}$ *contains an irreducible system* $\mathbf{F}$ *if* $\mathbf{B}$ *contains* $\mathbf{F}$ *as a subsystem*

Here the undefined notions have the following meaning: $\mathbf{C} \to (\mathbf{B})_t^{\mathbf{A}}$ is the classical *Erdös–Rado partition arrow* which is a shorthand notation for the following statement: For every partition $\binom{\mathbf{C}}{\mathbf{A}} \subseteq \mathcal{A}_i \cup \ldots \cup \mathcal{A}_t$ there exists $\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\mathbf{B}'}{\mathbf{A}} \subseteq \mathcal{A}_i$ for some i.

A system $\mathbf{F}$ is *irreducible* if every pair x, y of points of $\mathbf{F}$ belongs to an edge of $\mathbf{F}$.

We prove this result in the Section 4. Let us remark that below in Section 5 we prove several stronger statements. Yet the proof of the above theorem mirrors all the essential features of our method. The above Theorem perhaps provides a good balance between generality and clarity.

3. Partite Lemma

In this section we prove the key step in our proof – the Partite Lemma (cf. Nešetřil, Rödl 1982, 1989, Frankl, Graham, Rödl 1987 for analogous results). Before proceeding to this result, we first need several additional notions and results.

Advanced Preliminaries

Let A be a fixed finite set (alphabet).
For a finite set X, A^X will denote the set of all $|X|$ - tuples $(a_x; a_x \in A, x \in X)$. A map $f : A^X \to A^Y$ is called *linear* if $f(V) = \{f(v); v \in V\}$ is a subspace of A^Y for each subspace V of A^X. If $f : V \to V'$ is linear and $\mathbf{A} = (V, \mathcal{S})$ is a system then we denote by $f(\mathbf{A})$ the system with points $f(V)$ and edges of the form $f(E)$; explicitely $\mathcal{S}_\delta(f(\mathbf{A})) = \{f(E); E \in \mathcal{S}_\delta\}$, $\delta \in \Delta$. For $\emptyset \neq Y \subseteq X$, the *projection map* p_Y is defined by setting $p_Y((a_x; x \in X)) = (a_y; y \in Y)$. It is easy to see that the projection p_Y is linear. For $\emptyset \neq Y \subseteq X$, a subspace V of A^X is called *Y–transverse* if $\dim V = \dim p_Y(V)$. (Note that in this case the projection map p_Y is $1-1$ on V). The canonical ordering $<$ will be chosen in such a way that for any points $v, v' \in A^X$, $p_Y(v) < p_Y(v')$ implies $v < v'$.

A system $\mathbf{A}$ is *Y–transverse* system if every edge of $\mathbf{A}$ is Y–transverse.

For sets $X^{(1)}, \ldots, X^{(m)}$ we define the *amalgamated direct product* $\bigoplus_i A^{X^{(i)}}$ to be the set of all tuples $(a_x; x \in \bigcup X^{(i)})$. We start with the following easy proposition (cf. Frankl, Graham, Rödl 1987):

3.1 Proposition. *Suppose for sets $X^{(1)}, \ldots, X^{(m)}$ we have $X^{(i)} \cap X^{(j)} = Y \neq \emptyset$ for $1 \leq i < j \leq m$. Let $\mathbf{A}^{(i)} = (A^{X^{(i)}}, \mathcal{S}^{(i)})$ be Y–transverse system, let $p_Y(\mathbf{A}^{(i)}) = \mathbf{B}$ for all $i = 1, \ldots, m$. Then there exists a unique Y–transverse system $\mathbf{A} = (\bigoplus_i A^{X^{(i)}}, \mathcal{S})$ such that*

$$p_Y(\mathbf{A}) = \mathbf{B}$$

$$p_{X^{(i)}}(\mathbf{A}) = \mathbf{A}^{(i)}.$$

Symbolically, we write $\mathbf{A} = \bigoplus_i \mathbf{A}^{(i)} = \mathbf{A}^{(1)} \ldots \mathbf{A}^{(m)}$, $\mathcal{S} = \mathcal{S}^{(1)} \ldots \mathcal{S}^{(m)}$.

Proof. We define $E \in \mathcal{S}_\delta(\mathbf{A})$ iff $p_{X^{(i)}}(E) \in \mathcal{S}_\delta(\mathbf{A}^{(i)})$. □

A central notion used in this paper is the following: Suppose $X \supseteq Y$, $Z \supseteq Y$ and $f : A^Z \to A^X$. Then f is called *Y–linear* if f is linear and for all $v \in A^Z$, $f(v)$ has the same Y–part as v.
Further for Y–transverse systems $\mathbf{A}$ in A^Z, $\mathbf{B}$ in A^X we say that $\mathbf{A}$ and $\mathbf{B}$ are *Y–isomorphic systems* if there exists a Y–linear map $A^Z \to A^X$ which when restricted to $V(\mathbf{A})$ is an isomorphism $\mathbf{A} \to \mathbf{B}$.

Finally, let $Y \subseteq X$ and let $\mathbf{A}$ be a system in A^Y and let $\mathbf{B}$ be a system in A^X satisfying $p_Y(\mathbf{B}) = \mathbf{A}$. Denote by $\binom{\mathbf{B}}{\mathbf{A}}_Y$ the family of all subsystems $\mathbf{A}' \in \binom{\mathbf{B}}{\mathbf{A}}$ which satisfy $p_Y(\mathbf{A}') = \mathbf{A}$. Elements of the set $\binom{\mathbf{B}}{\mathbf{A}}_Y$ are called Y–copies of $\mathbf{A}$ in $\mathbf{B}$.

Remark. The notions Y–copies, Y–transverse system are space analogies of partite - set systems, Nešetřil, Rödl (1981), (1982), cf. Frankl, Graham, Rödl (1987).

Now we come to the key construction of this section:
Let $\mathbf{B}$ be a system with points A^X, let $\emptyset \neq Y \subseteq X$ and let $\mathbf{A}$ be a system in A^Y satisfying $p_Y(\mathbf{B}) = \mathbf{A}$. Let $\mathbf{B}$ be Y–transverse. Let the edge set of $\mathbf{A}$ be

denoted by $\mathcal{S}$ and the edge set of $\mathbf{B}$ by $\mathcal{T}$. We assume that $\mathbf{A}$ is a complete system (i.e. $\bigcup_{\delta\in\Delta} \mathcal{S}_\delta(\mathbf{A})$ is the set of all subspaces of $\mathbf{A}$). (It will turn out that we may assume that without loss of generality). Fix a positive integer m and denote by $\bigoplus^{\mathbf{A}}_{1\leq i\leq m} \mathbf{B}$ the system $\mathbf{C}$ defined in the following way:

Let $X^{(1)},\ldots,X^{(m)}$ be copies of X satisfying $X^{(i)} \cap X^{(j)} = Y$ for all $1 \leq i < j \leq m$. For an edge M of $\mathbf{B}$ (in A^X) denote by $M^{(i)}$ the corresponding edge in $A^{X^{(i)}}$.

For the edge set $\mathcal{T}_\delta$ of $\mathbf{B}$ we put $\mathcal{T}_\delta = \mathcal{T}^1_\delta \cup \mathcal{T}^2_\delta$ where $\mathcal{T}^1_\delta$ consists of all edges of $\mathbf{B}$ which belong to a copy $\mathbf{A}' \in \binom{\mathbf{B}}{\mathbf{A}}_Y$; $\mathcal{T}^2_\delta = \mathcal{T}_\delta - \mathcal{T}^1_\delta$. We put $\mathbf{C} = (\bigoplus_{1\leq i\leq m} A^{X^{(i)}}, \mathcal{U})$, $\mathcal{U} = (\mathcal{U}_\delta;\ \delta \in \Delta)$, where $M^{(1)}_{k(1)} \ldots M^{(m)}_{k(m)} \in \mathcal{U}_\delta$ iff

$M_{k(i)} \in \mathcal{T}_\delta$ for $i = 1,\ldots,m$ and either one of the two possibilities occurs:

(i) $p_Y(M^{(1)}_{k(1)}) = \ldots = p_Y(M^{(m)}_{k(m)}) \in \mathcal{S}_\delta$

(ii) There exists a non-empty set $\omega \subseteq \{1,\ldots,m\}$ such that $M^{(j)}_{k(j)} = M^{(j')}_{k(j')} = M \in \mathcal{T}^2_\delta$ for $j,j' \in \omega$, $p_Y(M^{(i)}_{k(i)}) = p_Y(M_{k(i)})$, $M_{k(i)} \in \mathcal{T}^1_\delta$ (thus also $p_Y(M^{(i)}_{k(i)}) \in \mathcal{S}_\delta$).

This completes the definition of $\mathbf{C} = \bigoplus^{\mathbf{A}}_{1\leq i\leq m} \mathbf{B}$. ($\mathbf{C}$ is a subsystem of $\bigoplus_{1\leq i\leq m}$ $\mathbf{B}$.)
We derive several properties of this construction:

Fact 1. $\mathbf{A}' \in \binom{\mathbf{C}}{\mathbf{A}}_Y$ *iff* $\mathbf{A}' = \mathbf{A}^{(1)}_{j(1)} \ldots \mathbf{A}^{(m)}_{j(m)}$ *where* $\mathbf{A}_{j(i)} \in \binom{\mathbf{B}}{\mathbf{A}}_Y$, $i = 1,\ldots,m$.

Proof. Check the definition of $\mathbf{C}$. □

Put $s = \left|\binom{\mathbf{B}}{\mathbf{A}}_Y\right|$ and let L be a line in Hales-Jewett cube $[s]^m$. By virtue of Fact 1 to each line L of $[s]^m$ associate the corresponding m-tuple of $\mathbf{A}$-systems in $\mathbf{C}$ $\{\mathbf{A}^{(1)}_{i(1)} \ldots \mathbf{A}^{(m)}_{i(m)};\ (i(1),\ldots,i(m)) \in L\}$.

Let the line L be fixed. Denote by ω the set of moving coordinates of L and $[m] - \omega = \kappa$ the set of all constant coordinates.

We also define a map $f_L : A^X \to \bigoplus_i A^{X^{(i)}}$. For $v \in A^X$ let $v' = p_Y(v)$ denote the Y-part of v and $v'' = p_{X-Y}(v)$ denote the $X-Y$-part of v. The map f_L is then defined as follows: the Y-part of $f_L(v)$ is v'; for $j \in \omega$ the $(X^{(j)}-Y)$-part of $f_L(v)$ is v'' (i.e. a copy of v'' in $X^{(j)}$); for $j \in \kappa$ the $(X^{(j)} - Y)$-part of $f_L(v)$ is $v^{(j)}$ — the $(X^{(j)} - Y)$-part of the unique element in $\mathbf{A}^{(j)}_{i(j)}$ which has Y-part v' (note that $\mathbf{A}^{(j)}_{i(j)}$ for $j \in \kappa$ is the "constant" system of the line L). It is easy toe see that f_L is Y-linear map $A^X \to \bigoplus_i A^{X^{(i)}}$.
Now we can formulate next

Fact 2. *Let* $\mathbf{A} = (V,\mathcal{S})$ *be a system in* A^Y, $\mathbf{B} = (U,\mathcal{T})$ *be a system in* A^X *with* $p_Y(\mathbf{B}) = \mathbf{A}$. *Let* $\mathbf{B}$ *be* Y*-transverse. Put* $s = \left|\binom{\mathbf{B}}{\mathbf{A}}\right|$ *and let* L *be a line in*

Hales-Jewett cube $[s]^m$. *Consider system* $\mathbf{C} = \bigoplus_{1\le i\le m}^{\mathbf{A}} \mathbf{B}$ *defined above. Put* $U' = f_L(U)$. *Then* U' *induces a* Y*-copy of* $\mathbf{B}$ *in* $\mathbf{C}$.

Explicitely, the system $(f_L(U), (\mathcal{T}_\delta \cap \binom{f_L(U)}{n_\delta}); \delta \in \Delta)$ is Y-isomorphic to $\mathbf{B}$.

Notation. The copy of $\mathbf{B}$ determined by $f_L(U)$ will be denoted as $f_L(\mathbf{B})$.

Proof of Fact 2. Let κ, ω be (as above) sets of the constant and moving coordinates of L.

For $i \in \kappa$ let $\mathbf{K}(i)$ be the value of "constant" system $\mathbf{A} \in \binom{\mathbf{B}}{\mathbf{A}}_Y$ and let $\mathbf{K}^{(i)}$ be the copy of $\mathbf{K}(i)$ in $A^{X^{(i)}}$. Thus, using Fact 1, the family of copies of $\mathbf{A}$ corresponding to L is the set $\{\mathbf{A}^{(1)}\dots\mathbf{A}^{(m)};\ \mathbf{A}^{(i)} = \mathbf{K}^{(i)}$ for $i \in \kappa$, $\mathbf{A}^{(i)} = \mathbf{A}^{(j)}$ for $i,j \in \omega\}$. Let $\overline{\mathbf{A}} \in \binom{\mathbf{B}}{\mathbf{A}}_Y$ and let M be an edge of $\overline{\mathbf{A}}$. Then $f_L(M) = M^{(1)}\dots M^{(m)}$ where $M^{(j)}$ is the edge of $\mathbf{A}^{(j)}$ uniquelly determined by $p_Y(M^{(j)}) = p_Y(M)$.
If $M \in \mathcal{T}_\delta^1$ and hence also $M^{(j)}$ belongs to the copy of $\mathcal{T}_\delta^1$ in $\mathbf{A}^{(j)}$, $j = 1,\dots,m$ then according to the definition of $\mathbf{C}$ we infer that $f_L(M) \in U_\delta$.

Let now $M \in \mathcal{T}_\delta^2$ then $f(M) = M^{(1)}\dots M^{(m)}$ where for $j \in \kappa$, $M^{(j)}$ is the edge of "constant" system $\mathbf{A}^{(j)}$ determined by $p_Y(M^{(j)}) = p_Y(M)$, and for $j \in \omega$, $M^{(j)}$ is a copy of M. According to the definition of $\mathcal{U}_\delta$ we infer that again $f_L(M) \in \mathcal{U}_\delta$ holds. Thus for each $\delta \in \Delta$

$$\{f_L(M);\ M \in \mathcal{T}_\delta\} \subseteq \mathcal{U}_\delta \cap \binom{U'}{n_\delta}$$

To conclude the proof we must show that there are no other edges in $\mathcal{U}_\delta \cap \binom{U'}{n_\delta}$, $\delta \in \Delta$.
Set $\mathcal{T}_\delta^1 = \{M_1,\dots,M_r\}$, $\mathcal{T}_\delta^2 = \{M_{r+1},\dots,M_s\}$ and for contradiction suppose that $M_{\mu(1)}^{(1)}\cdots M_{\mu(m)}^{(m)}$ is an edge not belonging to $\mathcal{U}_\delta \cap \binom{U'}{n_\delta}$. Note that this edge has to satisfy $p_Y(M_{\mu(i)}^{(i)}) = M$, $i = 1,\dots,m$, for a fixed edge M of $\mathbf{A}$. We distinguish two possibilities which may occur:

(i) there are $j, j' \in \omega$ with $\mu(j) \neq \mu(j')$.

Since $M_{\mu(j)} \neq M_{\mu(j')}$ there must exist some point $\overline{v} \in p_Y(M_{\mu(j)}^{(j)}) = p_Y(M_{\mu(j)}) = p_Y(M_{\mu(j')}^{(j')}) = p_Y(M_{\mu(j')}) = M$ such that the unique elements v_j and $v_{j'}$ in edges $M_{\mu(j)}^{(j)}$ and $M_{\mu(j')}^{(j')}$ which have Y-part $\overline{v}$ are different. Thus according to the definition of f_L the point

$$v \in M_{\mu(1)}^{(1)}\cdots M_{\mu(m)}^{(m)}$$

which has Y-part $\overline{v}$ is not in U', a contradiction.

(ii) For some $j \in \kappa$, $M^{(j)}_{\mu(j)}$ is different from $M^{(j)}$ — the unique edge of $\mathbf{A}^{(j)}$ with $p_Y(M^{(j)}) = M$.

Again (as in (i)) there must be $\overline{v} \in M$ such that the points of $M^{(j)}_{\mu(j)}$ and $M^{(j)}$ with Y–part $\overline{v}$ are different. This also implies that the point $v \in M^{(1)}_{\mu(1)} \dots M^{(m)}_{\mu(m)}$ with Y–part $\overline{v}$ is not in U', a contradiction. This proves the Fact 2. $\square$

Summarizing the above properties we infer the following:

3.2 Theorem *(Partite Lemma for Transverse Systems). Let t be a positive integer. Let $\emptyset \neq Y \subseteq X$ be sets and let $\mathbf{A}$ be a system in A^Y, $\mathbf{B}$ a system in A^X satisfying $p_Y(\mathbf{B}) = \mathbf{A}$. Let $\mathbf{B}$ be Y–transverse. Then there exists an m such that the system $\mathbf{C} = \bigoplus^{\mathbf{A}}_{1\leq i\leq m} \mathbf{B}$ has the following property:*
for every partition $\binom{\mathbf{C}}{\mathbf{A}}_Y = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_t$ one of the classes contains $f_L(\mathbf{B})$ for a line (in the corresponding Hales–Jewett cube).

Proof. follows immediately from the above Facts 1 and 2 if we set $s = \left|\binom{\mathbf{B}}{\mathbf{A}}_Y\right|$, m equal to number guaranteed by Hales-Jewett theorem (t classes, alphabet $|A| = s$, monochromatic line). Put $\mathbf{C} = \bigoplus^{\mathbf{A}}_{1\leq i\leq m} \mathbf{B}$. $\square$

The next two properties of the above construction of $\mathbf{C}$ will be used effectively bellow.

Fact 3. *Let systems $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, sets $Y \subseteq X$ be as above in Fact 2. Let r be a positive integer, $r < \dim V(\mathbf{A})$.*
Let any two copies of $\mathbf{A}$ in $\mathbf{B}$ are either disjoint or intersect in an edge with dimension $\leq r$. Let moreover L_1, L_2 be two different lines and $U_i = f_{L_i}(U)$. Moreover suppose that there exists a transverse $(r+1)$–space V satisfying $V \subseteq U_1 \cap U_2$. Let $\mathbf{B}_1, \mathbf{B}_2$ be copies of $\mathbf{B}$ determined by U_1, U_2, $\mathbf{B}_i = (U_i, \mathcal{T}_i)$, $i = 1, 2$. Then V is a subset of an edge M belonging both to $\mathbf{B}_1$ and $\mathbf{B}_2$.

Proof. Suppose for contradiction that M does not exist. Thus there exists $i_0 \leq m$ such that $p_{X^{(i_0)}}(V) \subseteq p_{X^{(i_0)}}(M)$ for all edges M belonging to both $\mathbf{B}_1$ and $\mathbf{B}_2$. Using definition of $\mathbf{C}$ again, for every $\delta \in \Delta$ the set $p_{X^{(i_0)}}(\mathcal{T}_{j\delta}) = \{p_{X^{(i_0)}}(M); M \in \mathcal{T}_{j\delta}\}$, $j = 1, 2$, is either one element set or set $p_{X^{(i_0)}}(\mathcal{T}_{j\delta}) = p_{X^{(i_0)}}(f_{L_j}(\mathcal{T}_\delta))$ forming a family Y–isomorphic to $\mathcal{T}_\delta$ (depending whether i_0 belongs to the constant or moving part of L_j). Therefore we get one of the following cases:

α) Either $p_{X^{(i_0)}}(\mathcal{T}_{1\delta})$ or $p_{X^{(i_0)}}(\mathcal{T}_{2\delta})$ is one element set;
β) Both $p_{X^{(i_0)}}(\mathcal{T}_{1\delta})$ and $p_{X^{(i_0)}}(\mathcal{T}_{2\delta})$ form a family Y–isomorphic to $\mathcal{T}_\delta$.

Both these cases lead to a contradiction. $\square$

Before presenting the final auxiliar result let us introduce the following notion: Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be systems. Let $\mathcal{B}$ be a family of copies of $\mathbf{B}$ in $\mathbf{C}$. We denote by $H^{\mathbf{A}}_{\mathcal{B}}$ the hypergraph defined as follows: $H^{\mathbf{A}}_{\mathcal{B}} = (V, E)$ where $V = \binom{\mathbf{C}}{\mathbf{A}}$, $E =$

$\{\binom{\mathbf{B}'}{\mathbf{A}};\ \mathbf{B}' \in \mathcal{B}\}$. **A**–*cycle* in $\mathcal{B}$ is every cycle of the hypergraph $H^{\mathbf{A}}_{\mathcal{B}}$. Particularly, if **A** is the point then we speak about 0–cycle in $\mathcal{B}$.

Fact 4. *Let systems* **A**, **B**, **C** *sets* X, Y *have the same meaning as above in Fact 2. Let* $l \geq 2$ *be an integer. Put* $s = \left|\binom{\mathbf{B}}{\mathbf{A}}_Y\right|$ *and let* $\mathcal{L}$ *be a family of lines in Hales-Jewett cube* $[s]^m$ *which does not contain 0–cycles of length* $\leq l$*. Then every* **A**–*cycle in* $f_{\mathcal{L}}(\mathbf{B}) = \{f_L(\mathbf{B});\ L \in \mathcal{L}\}$ *has length* $> l$.

Proof. Let $f_{L_1}(\mathbf{B})$, $\mathbf{A}_1$, $f_{L_2}(\mathbf{B})$, $\mathbf{A}_2, \ldots, \mathbf{A}_{r-1}$, $f_{L_r}(\mathbf{B})$, $\mathbf{A}_r$ be an **A**–cycle in $f_{\mathcal{L}}(\mathbf{B})$. We suppose that all L_i and $\mathbf{A}_i$ are pairwise different. Using 1–1 correspondence between copies of **A** in **C** and points in the Hales–Jewett cube we get the statement of Fact 4. □

It is very convenient and a bit surprising that for the **A**–sparse Partite Lemma one needs only sparse form of Hales–Jewett theorem. This follows from our proof of Partite Lemma and from Fact 4. Note that the sparse form of Hales–Jewett theorem was established by probabilistic means in Rödl (1981) and bellow we give a construction of such a family (Theorem 5.2).

4. Partite Construction

In this part we prove Ramsey Theorem for Systems by means of the Partite Construction. We follow the Partite Construction introduced in Nešetřil, Rödl (1982, 1984, 1987), (cf. Frankl, Graham, Rödl (1987)).

Let t be a positive integer, let **A**, **B** be systems. Let a be the dimension of **A**, b the dimension of **B**. Without loss of generality let every point of **B** belong to an edge of **B**. Also, without loss of generality let us assume that **A** is irreducibile (in fact we may assume that $\bigcup_{\delta \in \Delta} S_\delta(\mathbf{A})$ is the set of all subspaces in $V(\mathbf{A})$; we possibly add to **A** and **B** some classes of dummy edges).
Let p be an integer sufficiently large to guarantee that if a–spaces of p–space are arbitrarily partitioned into t classes then some b–space has all its a–spaces monochromatic. Such a p exists by virtue of Graham-Leeb-Rothschild theorem, cf. Graham, Leeb, Rothschild (1972).

Let z_0 be a large integer (to be specified later) and consider a z_0–element set Z_0 and p–element set $Y \subseteq Z_0$. Let the set of all b–spaces in A^Y be denoted by $\{M_1, \ldots, M_r\}$, $r = \left[{p \atop b}\right]$. Let the set all a–spaces in A^Y be denoted by $\{\mathbf{A}_1, \ldots, \mathbf{A}_q\}$, $q = \left[{p \atop a}\right]$.

We shall construct systems $\mathbf{P}^0, \ldots, \mathbf{P}^k, \ldots, \mathbf{P}^q$ by induction on k. Picture $\mathbf{P}^q$ will be the desired system **C**.

Choose a collection of b–spaces in A^{Z_0} (say $V_1, \ldots, V_r$) which are as disjoint as possible; i.e. pairwise disjoint or having pairwise intersection zero, and furthermore so that $p_Y(V_i) = M_i$, $1 \leq i \leq r$. It follows that $p_Y : A^{Z_0} \to A^Y$ is $1-1$ on each V_i. This is certainly possible if z_0 is taken sufficiently large. For each V_i let $\mathbf{B}^0_i$ be system isomorphic to **B** in V_i.
We define the system $\mathbf{P}^0 = (A^{Z_0}, \mathcal{U}^0)$ where an edge belongs to $\mathcal{U}^0_\delta$ iff it belongs to one of the edge–sets of $\mathbf{B}^0_i$, $1 \leq i \leq r$. Suppose now that for some $k < q$ we

defined $\mathbf{P}^k = (A^{Z_k}, \mathcal{U}^k)$ in which every edge of $\mathbf{P}^k$ is Y–transversal (this is obviously satisfied for $\mathbf{P}^0$): We will describe the construction of $\mathbf{P}^{k+1}$. To start, first let $\mathbf{D}^{k+1}$ be the system determined by all edges E of $\mathbf{P}^k$ which satisfy $p_Y(E) \subseteq V(\mathbf{A}_{k+1})$.
By Partite Lemma there exists a set $Z \supseteq Y$ and system $\mathbf{E}^{k+1}$ in A^Z with the following properties:
(1) each edge of $\mathbf{E}^{k+1}$ is Y–transverse
(2) for each edge E of $\mathbf{E}^{k+1}$ the set $p_Y(E)$ is an edge of $\mathbf{A}_{k+1}$ (of the same type);
(3) $\mathbf{E}^{k+1} \underset{Y}{\longrightarrow} (\mathbf{D}^{k+1})^{\mathbf{A}}_t$
(4) Recall (from the proof of the Partite Lemma) that in fact $\mathbf{E}^{k+1} \overset{\mathcal{D}^{k+1}}{\underset{Y}{\longrightarrow}} (\mathbf{D}^{k+1})^{\mathbf{A}}_t$

where $\mathcal{D}^{k+1}$ is the family of copies of $\mathbf{D}^{k+1}$ which correspond to Hales-Jewett lines of a cube.

For each $\mathbf{D} \in \mathcal{D}^{k+1}$ let $V_{\mathbf{D}}$ be the minimal subspace of A^{Z-Y} such that $\mathbf{D}$ is a subspace of $V(\mathbf{A}_{k+1}) \oplus V_{\mathbf{D}}$.

Now let $h: A^Y \to V(\mathbf{A}_{j+1})$ be arbitrary but fixed retract i.e. h is a fixed linear map which is the identity when restricted to $V(\mathbf{A}_{j+1})$.

For each copy $\mathbf{D} \in \mathcal{D}^{k+1} \subseteq \binom{\mathbf{E}^{k+1}}{\mathbf{D}^{k+1}}$ take set $Y_{\mathbf{D}}$ which is a copy Y, and where all sets $Y_{\mathbf{D}}$ are mutually disjoint and disjoint from Z.
Also, for each $\mathbf{D}$, extend the copy $\mathbf{D}$ of $\mathbf{D}^{k+1}$ to full Y–isomorphic copy $\mathbf{P}_{\mathbf{D}}$ of $\mathbf{P}^k$ in $A^Y \oplus V_{\mathbf{D}}$.

$$\text{Put } Z^* = Z \cup \bigcup_{\mathbf{D} \in \mathcal{D}^{k+1}} Y_{\mathbf{D}}.$$

Finally, for each $\mathbf{D} \in \mathcal{D}^{k+1}$ define a map $g_{\mathbf{D}}: A^Y \oplus V_{\mathbf{D}} \to A^{Z^*}$ as follows:
For $x \in A^Y \oplus V_{\mathbf{D}}$ let

$g_{\mathbf{D}}(x)$ has the same Z–part as x
$g_{\mathbf{D}}(x)$ has the same $Y_{\mathbf{D}}$–part as x (we think about $Y_{\mathbf{D}}$ as a copy of Y)
$g_{\mathbf{D}}(x)$ has $Y'_{\mathbf{D}}$–part equal to $h(Y$–part of $x)$ for $\mathbf{D}' \neq \mathbf{D}$.

Finally define $\mathbf{P}^{k+1} = (Z^{k+1}, \mathcal{U}^{k+1})$ by:

$$\begin{aligned} Z^{k+1} &= Z^* \\ E \in \mathcal{U}^{k+1}_\delta \text{if } E &= g_{\mathbf{D}}(E'), \\ E' &\in \mathcal{S}_\delta(\mathbf{P}_{\mathbf{D}}), \\ \mathbf{D} &\in \mathcal{D}^{k+1}. \end{aligned}$$

If we need to stress the dependence of P^{k+1} on $\mathcal{D}^{k+1}$ and P^k we shall write $P^{k+1} = \mathcal{D}^{k+1} * P^k$. Denote by $g_{\mathbf{D}}(P_{\mathbf{D}})$ the system induced by the set of points $\{g(x);\ x \in P_{\mathbf{D}}\}$. Note that for distinct $\mathbf{D}$, $\mathbf{D}' \in \mathcal{D}^{k+1}$ we have $g_{\mathbf{D}}(\mathbf{P}_{\mathbf{D}}) \cap$

$g_{\mathbf{D}'}(\mathbf{P}_{\mathbf{D}'}) \subseteq g_{\mathbf{D}}(E^{k+1}) = g_{\mathbf{D}'}(E^{k+1})$ and thus $g_{\mathbf{D}}(\mathbf{P}_{\mathbf{D}}) \cap g_{\mathbf{D}'}(\mathbf{P}_{\mathbf{D}'}) = g_{\mathbf{D}}(\mathbf{D}) \cap g_{\mathbf{D}'}(\mathbf{D}')$. As $b > a$ this implies that $\mathbf{P}^{k+1}$ does not contain any new irreducible system (which would not be contained in $\mathbf{P}^k$). It is also easy to see that every copy of $\mathbf{A}$ in $\mathbf{P}^k$ is Y-transverse, thus $\binom{\mathbf{P}^k}{\mathbf{A}} = \binom{\mathbf{P}}{\mathbf{A}}_Y$.
Finally consider picture $\mathbf{P}^q$. Put $\mathbf{P}^q = \mathbf{C}$. We claim that $\mathbf{C}$ has the desired properties.

4.1 Claim. *Every irreducible subsystem in* $\mathbf{C}$ *is a subsystem of* $\mathbf{B}$.

Proof. Induction on k. This beeing trivial for $k = 0$ the inductive step follows by the above remark. □

4.2 Claim. $\mathbf{C} \to (\mathbf{B})_t^{\mathbf{A}}$.

Proof. Backward induction on $k = q, q-1, \ldots, 1$: To see this, suppose that $\binom{\mathbf{C}}{\mathbf{A}}_Y$ is partitioned into classes $\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_t$. Consider the subsystem of $\mathbf{P}^q$ induced by all edges E of $\mathbf{P}^q$ which satisfy $p_Y(E) \subseteq V(\mathbf{A}_q)$. By the construction of $\mathbf{P}$ this system is isomorphic to $\mathbf{E}^q$ and thus, by the construction and by the Partite Lemma there exists $\mathbf{D} \in \mathcal{D}^q$ such that the set belongs to one of the classes of partition. Moreover, $\mathbf{D}$ is a subspace of a copy $\mathbf{P}$ of $\mathbf{P}^{k-1}$.

Now we can repeat the same argument applied to $\mathbf{P}$ and all edges of $\mathbf{P}$ with Y-projection in $V(\mathbf{A}_{q-1})$.
Continuing this process, we eventually reach a set $\overline{Z_0} \supseteq Y$ and a Y-copy $\mathbf{P}$ of $\mathbf{P}^0$ in $A^{\overline{Z_0}}$ such that the color of $\mathbf{A}' \in \binom{\mathbf{P}}{\mathbf{A}}_Y = \binom{\mathbf{P}}{\mathbf{A}}$ depends on $p_Y(\mathbf{A}')$ only. By the choice of $p = |Y|$ there exists b-subspace V of $\binom{A^Y}{b}$ such that all $\mathbf{A}' \in \binom{\mathbf{P}}{\mathbf{A}}$ with $p_Y(\mathbf{A}) \in \binom{V}{a}$ have the same color.
However, again by construction of $\mathbf{P}^0$, there exists a subsystem $\mathbf{B}'$ of $\mathbf{P}^0$ such that $p_Y(V(\mathbf{B}')) \subseteq V$.
This completes the proof of Theorem. □

5. Applications

The method of the above proof yields in the spirit of Nešetřil, Rödl (1984, 1987, 1988) several stronger results than Ramsey theorem for space systems. We list some of them:

A. Hom-Connected Systems

First we give two auxiliary definitions:
Let $\mathbf{B} = (V, \mathcal{T})$ be a system. A space R is called a *cut* of $\mathbf{B}$ if there is a partition of $V - R$ into two disjoint sets V_1 and V_2 such that no pair $\{v_1, v_2\}$, $v_1 \in V_1$, $v_2 \in V_2$ is contained in an edge of $\mathbf{B}$.
We shall consider R together with all edges of B contained in R; this will be denoted by $\mathbf{R}$ — thus $\mathbf{R}$ is a system. $\mathbf{B}$ is said to be *lin* $\mathbf{A}$*-connected* if there is no cut $\mathbf{R}$ of $\mathbf{B}$ for which there is a linear map $\mathbf{R} \to \mathbf{A}$.

It is also convenient to recall the following notion (Nešetřil, Rödl 1977). Given a (possible infinite) set $\mathcal{F}$ of systems denote by *Forb* $(\mathcal{F})$ the set of all those systems $\mathbf{A}$ which do not contain any system $\mathbf{F} \in \mathcal{F}$ as a weak subsystem. ($\mathbf{A}$ is a *weak subsystem* of $\mathbf{B}$ if every point (edge) of $\mathbf{A}$ is a point (edge) of $\mathbf{B}$.) Now we have

5.1 Theorem. *Let $\mathcal{F}$ be a set lin $\mathbf{A}$–connected systems. Then for every positive t and every $\mathbf{B} \in \mathrm{Forb}(\mathcal{F})$ there exists $\mathbf{C} \in \mathrm{Forb}(\mathcal{F})$ such that $\mathbf{C} \to (\mathbf{B})_t^{\mathbf{A}}$.*

Proof. First fix system $\mathbf{P}$ in A^Y such that $\mathbf{P} \to (\mathbf{B})_t^{\mathbf{A}}$ (thus we apply Ramsey theorem for spaces). Put $\binom{\mathbf{P}}{\mathbf{B}} = \{\mathbf{B}_1, \ldots, \mathbf{B}_r\}$. Define picture $\mathbf{P}^0$ as follows: Choose a set Z_0 and a collection of systems $\mathbf{B}'_1, \ldots, \mathbf{B}'_r$ in A^{Z_0} which are as disjoint as possible (i.e. pairwise disjoint or having pairwise intersection zero) so that $\mathbf{B}'_i$ and $\mathbf{B}_i$ are Y–isomorphic and $p_Y(\mathbf{B}'_i) = \mathbf{B}_i$ for $1 \leq i \leq r$. It follows that $p_Y : A^{Z_0} \to A^Y$ is $1-1$ on each $\mathbf{B}'_i$. This is certainly possible if $z_0 = |Z_0|$ is taken sufficiently large.
We define the system $\mathbf{P}_0 = (A^{Z_0}, \mathcal{U}^0)$ where an edge belongs to $\mathcal{U}^0_\delta$ iff it belongs to one of the edge sets of $\mathbf{B}'_i$. (Thus $\mathbf{P}^0$ is a "dijoint union" of copies $\mathbf{B}_1, \ldots, \mathbf{B}_r$ with Y–parts induced by $\binom{\mathbf{P}}{\mathbf{B}}$.)
Clearly $\mathbf{P}^0 \in \mathrm{Forb}(\mathcal{F})$ and the amalgamation does not create any weak lin $\mathbf{A}$–connected subsystem of $\mathbf{P}^{k+1}$. (Note that each of the systems $\mathbf{D}^1, \ldots, \mathbf{D}^q$ may be linearly mapped into $\mathbf{A}$.) □

The Partite Construction is very convenient for construction of sparse Ramsey systems. This is not surprising as one of the byproducts of the partite construction is a new easy construction of highly chromatic graphs without short cycles (Nešetřil, Rödl 1979), c.f. Lovász (1968). There are various ways how to define sparseness and we list them in the order of increasing difficulty.

B. Sparse Ramsey Theorems - Ramsey Families

We say that $\mathcal{B} \subseteq \binom{\mathbf{C}}{\mathbf{B}}$ is a *Ramsey family* if for every partition $\binom{\mathbf{C}}{\mathbf{A}} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_t$ there exists $\mathbf{B}' \in \mathcal{B}$ such that $\binom{\mathbf{B}'}{\mathbf{A}} \subseteq \mathcal{A}_i$ for some i. We denote this by $\mathbf{C} \xrightarrow{\mathcal{B}} (\mathbf{B})_t^{\mathbf{A}}$.
Recall the definition of the hypergraph $H_{\mathcal{B}}^{\mathbf{A}}$ and the $\mathbf{A}$-cycle in $\mathcal{B}$ introduced in Section 3 (before Fact 4).
We have:

5.2 Theorem *(Sparse Ramsey Families). For every system $\mathbf{A}$, $\mathbf{B}$ and positive integers t, l there exist system $\mathbf{C}$ and family $\mathcal{B} \subseteq \binom{\mathbf{C}}{\mathbf{B}}$ such that*

1) $\mathbf{C} \xrightarrow{\mathcal{B}} (\mathbf{B})_t^{\mathbf{A}}$
2) *$\mathcal{B}$ contains no $\mathbf{A}$-cycle of length $\leq l$*
3) *If $\mathbf{A}, \mathbf{B}$ are T–transverse, $p_T(\mathbf{B}) = \mathbf{A}$, then $\mathbf{C}$ may be chosen T–transverse as well.*

Proof. We proceed by induction on l and construct $\mathcal{B}$ by means of Partite Construction. We adopt the notation introduced in Section 3 and stress the key points of the proof only. We use picture $\mathbf{P}^0$ as defined above in the proof of Theorem 4.1. Moreover if $\mathbf{A}$, $\mathbf{B}$ are T–transverse then let $\mathbf{P}^0$ be chosen T–transverse as well (we apply then the Partite Lemma for Transverse Spaces). Put $\mathcal{B}^0 = \binom{\mathbf{P}^0}{\mathbf{A}}$. In the inductive step (i.e. in the definition of $\mathbf{P}^{k+1}$), we assume, by induction on l, that there exists a system $\mathcal{D}^{k+1} \subseteq \binom{\mathbf{E}^{k+1}}{\mathbf{D}^{k+1}}_Y$ such that $\mathcal{D}^{k+1}$ does not contain $\mathbf{A}$-cycles of length $\leq l-1$ and $\mathbf{E}^{k+1} \xrightarrow[Y]{\mathcal{D}^{k+1}} (\mathbf{D}^{k+1})^{\mathbf{A}}_t$.

We form picture $\mathbf{P}^{k+1} = \mathcal{D}^{k+1} * \mathbf{P}^k$. Assuming that in $\mathbf{P}^k$ we have a system $\mathcal{B}^k \subseteq \binom{\mathbf{P}^k}{\mathbf{B}}_Y$ then in $\mathbf{P}^{k+1}$ we may define a system $\mathcal{B}^{k+1}$ as $\mathcal{D}^{k+1} * \mathcal{B}^k$ of Y–copies of $\mathbf{B}$ (in the same way as above in Section 3). In fact we may put $\mathcal{B}^{k+1} = \{g_{\mathbf{D}}(\mathbf{B}');\ \mathbf{B}' \in \mathcal{B}^k,\ \mathbf{D} \in \mathcal{D}^{k+1}\}$. The fact that $\mathcal{B}^k$ does not contain $\mathbf{A}$-cycles of length $\leq l$ and $\mathcal{D}^{k+1}$ does not contain $\mathbf{A}$-cycles of length $\leq l-1$ together implies that $\mathcal{B}^{k+1}$ does not contain cycles of length $\leq l$. This may be seen as follows (compare Lovász 1968, Nešetřil, Rödl 1979 for similar set–system argument):

Let $\mathbf{A}(1), \mathbf{B}(1), \mathbf{A}(2), \ldots, \mathbf{A}(w), \mathbf{B}(w)$ be an $\mathbf{A}$-cycle in $\mathcal{B}^{k+1}$.

Observe that each $\mathbf{B}(i)$ is of the form $\mathbf{B}(i) = D(i) * B'(i)$ where $\mathbf{D}(i) \in \mathcal{D}^{k+1}$, $\mathbf{B}'(i) \in \mathcal{B}^k$.

As $g_{\mathbf{D}}(\mathbf{P}_{\mathbf{D}})$ and $g_{\mathbf{D}'}(\mathbf{P}_{\mathbf{D}'})$ intersect in a copy of $\mathbf{A}$ it follows that the set $\{\mathbf{D}(i);\ i = 1, \ldots, w\}$ either contains an $\mathbf{A}$-cycle of length $w_{\mathbf{D}}$ or $\mathbf{D}(1) = \mathbf{D}(2) = \ldots = \mathbf{D}(w) = \mathbf{D}$. The second alternative is not possible as in this case the cycle $\mathbf{A}(1), \mathbf{B}(1), \ldots, \mathbf{A}(w), \mathbf{B}(w)$ belongs to the copy $\mathbf{P}_{\mathbf{D}}$ of $\mathbf{P}^k$ a contradiction. However, observe that if $p_Y(\mathbf{A}(i)) = \mathbf{A}_{k+1}$ (i.e. if $\mathbf{A}(i)$ is a subsystem of $\mathbf{E}^{k+1}$) then both $\mathbf{A}(i-1)$ and $\mathbf{A}(i+1)$ fail to be subsystems of $\mathbf{E}^{k+1}$. It follows that $w_{\mathbf{D}} > l-1$ and thus $w > l$ (in fact $\geq 2w_{\mathbf{D}}$).

This proves 2); 1) follows by Partite Construction; 3) follows by observing that if picture $\mathbf{P}^0$ is T–transverse then all pictures $\mathbf{P}^k$ are T-transverse as well. □

C. Sparse Ramsey Theorem - Cycles in Copies

We prove the following:

5.3 Theorem. *Let t, l be positive integers, $p \leq a \leq b$ non negative integers. Exclude possibility $a = 0$, $b = 1$.*

Then there exists a set $\mathcal{S} \subseteq \binom{A^X}{p}$ with the following properties:

1) $(A^X, \mathcal{S}) \to (A^b)^{A^a}_t$

2) There are no a–cycles in $\mathcal{S}$ of length $\leq l$.

The Theorem fails to be true for $a = 0$, $b = 1$ as in this case we deal with a perfect graph.

Proof. Without loss of generality let $a < b$. We apply the Partite Construction and we proceed by induction on k. Put $\mathbf{A} = (A^a, \binom{A^a}{p})$, $\mathbf{B} = (A^b, \binom{A^b}{p})$. We use picture $\mathbf{P}^0$ defined as in the above proof of Theorem 4.1. In the inductive step we use Theorem 4.2 to obtain a Ramsey system $\mathcal{D}^{k+1} \subseteq \binom{\mathbf{E}^{k+1}}{\mathbf{D}^{k+1}}$ without cycles of length $\leq l$. Putting $\mathbf{P}^{k+1} = \mathcal{D}^{k+1} * \mathbf{P}^k$ one can check as above that the hypergraph $H^{\mathbf{A}}_{\binom{\mathbf{P}^{k+1}}{\mathbf{A}}}$ has no cycles of length $\leq l$. □

Remark. We can prove the more general theorem for a system **B** which is lin **A**–connected. We omit the details.

Note also that our construction readily implies the Sparse Form of Graham, Rothschild theorem from Theorem 4.2. Moreover, Theorem 4.2 may be easily derived (by our proof of Partite Lemma) from Theorem 4.2 specialized to singletons (i.e. Hales Jewett theorem). See Section 2, Fact 4 and the remark following it.

D. Linearity

We say that a system **B** is *A–r–linear* if every two copies of **A** in **B** intersect in at most r–dimensional space. In Nešetřil, Rödl (1987) we proved Ramsey Theorem for Steiner systems which is a set analogy of this notion. Here we prove the following:

5.4 Theorem. *Let* **A** *be a system, t positive integer. Then for every* **A**–*r–linear* **B** *there exists* **A**–*linear* **C** *such that*

$$\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_t$$

Proof. Check the construction in the Proof of Partite Lemma. Use the fact that Hales–Jewett lines form a 0–linear system. This implies that any distinct copies $f_L(\mathbf{B})$, $f_{L'}(\mathbf{B})$ intersect either in a copy of **A** or the dimension of $p_Y(f_L(\mathbf{B}) \cap f_{L'}(\mathbf{B}))$ is at most r. Use this to check that Partite construction preserves **A**–r–linearity. □

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Graham–Rothschild Parameter Sets

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Abstract

In their, by now classical, paper 'Ramsey's theorem for n–parameter sets' (Trans. Amer. Math. Soc. 159 (1971), 257–291) Graham and Rothschild introduced a combinatorial structure which turned out be central in Ramsey theory. In this paper we survey the development related to the structure of Graham–Rothschild parameter sets.

1. Introduction

Besides Ramsey's theorem, van der Waerden's theorem on arithmetic progressions is commonly considered to be one of the main roots of Ramsey theory. Both results are partition theorems of the same type, however they remained quite unrelated for a long time. Compare Graham, Rothschild, Spencer (1980) for historical remarks concerning these theorems.

Years after van der Waerden obtained his 'Beweis einer Baudetschen Vermutung', Hales and Jewett revealed the combinatorial part of van der Waerden's theorem. Basically, Hales–Jewett's theorem says the following :
for any finite set A there exists a positive integer n with the property that for every partition $A^n = \mathcal{A} \bigcup \mathcal{B}$ there exists a combinatorial line $\mathcal{L} \subseteq A^n$ with $\mathcal{L} \subseteq \mathcal{A}$ or $\mathcal{L} \subseteq \mathcal{B}$. Via the mapping $\psi : A^n \to \mathbb{N}$, $\psi(a_0, \ldots, a_{n-1}) = \sum_{i=1}^{n-1} a_i$, Hales–Jewett's theorem then implies immediately van der Waerden's theorem.

Graham and Rothschild extended Hales–Jewett's result in a remarkable way. Using the notion of k–parameter sets in A^n , where, basically, 0–parameter sets are just the elements of A^n (n–tupels) and 1–parameter sets are the combinatorial lines occuring in Hales–Jewett's theorem, they showed that, choosing n large enough, for every partition of the k–parameter subsets of A^n into two parts, one of the parts contains all k–parameter subsets of some m–parameter set.

This is a complete analogue to Ramsey's theorem carried over to the structures of parameter sets and, as it turns out, Ramsey's theorem itself is an immediate consequence of the Graham–Rothschild theorem. But the concept of parameter sets does not only glue arithmetic progressions and finite sets together. Also, it provides a natural framework for seemingly different structures like Boolean lattices, partition lattices, hypergraphs and Deuber's (m,p,c)–sets, just to mention a few. So, the Graham–Rothschild theorem can be viewed as a starting point of *Ramsey Theory*.

Besides the various applications, several ramifications and generalizations of the original Graham–Rothschild theorem have been discovered. In this paper we try to survey the development based on and related to the structure of Graham–Rothschild parameter sets. For more information compare Graham, Rothschild, Spencer (1980) or Prömel, Voigt (to appear).

Some Conventions

1. Small letters $k, \ell, m, \ldots$ denote nonnegative integers, resp., finite ordinals. As usual, the number k is identified with the set of its predecessors, i.e., $k = \{0, \ldots, k-1\}$.
2. ω is the smallest infinite ordinal, i.e., $\omega = \{0, 1, 2 \ldots\}$. $\mathbb{N}$ denotes the set of positive integers.
3. 'A' always denotes a finite set.

2. Parameter Sets and Parameter Words (Definition and Basic Examples)

We are concerned with A^n, the set of n–tuples over A, and certain subsets (parameter sets).

0–parameter sets are simply singleton elements of A^n. In general, an m–parameter subset $M \subseteq A^n$ is given by an *m-parameter word* $f = (f_0, \ldots, f_{n-1}) \in (A \cup \{\lambda_0, \ldots, \lambda_{m-1}\})^n$. We require that each parameter λ_i, $i < m$, occurs at least once in f. In order to avoid ambiguities we assume that $A \cap \{\lambda_i \mid i < m\} = \emptyset$; the set of constants $a \in A$ should be distinguished from the set of parameters λ_i, $i < m$. If $f \in (A \cup \{\lambda_0, \ldots, \lambda_{m-1}\})^n$ is an m–parameter word in A^n and $g \in (A \cup \{\lambda_0, \ldots, \lambda_{k-1}\})^m$ is a k–parameter word in A^m, the *composition* $f \cdot g \in (A \cup \{\lambda_0, \ldots, \lambda_{k-1}\})^n$ is the k–parameter word in A^n resulting from replacing the parameter λ_i in f by g_i, the ith component of g. In particular, for $k = 0$, we obtain a set $M = \{f \cdot (a_0, \ldots, a_{m-1}) \mid (a_0, \ldots, a_{m-1}) \in A^m\} \subseteq A^n$. This is the *m–parameter set* related to f. Clearly, two parameter words yield the same parameter set iff they differ only by a permutation of their parameters. We get a *rigid representation* requiring the first occurrences of different parameters to be in increasing order, first λ_0, then λ_1, etc.

Let us summarize these ideas in a formal definition. The concept of m–parameter sets is due to Graham and Rothschild (1971), the formal calculus of parameter words has been introduced by Leeb (1973).

Definition. For nonnegative integers $m \leq n$ we denote by $[A]\binom{n}{m}$ the set of all words (mappings) $f : n \to A \cup \{\lambda_0, \ldots, \lambda_{m-1}\}$ satisfying

for every $j < m$ there exists $i < n$ with $f(i) = \lambda_j$, and
$\min f^{-1}(\lambda_i) < \min f^{-1}(\lambda_j)$ for all $i < j < m$.

Mappings $f \in [A]\binom{n}{m}$ are called *m–parameter words* of length n over A . For $f \in [A]\binom{n}{m}$ and $g \in [A]\binom{m}{k}$ the composition $f \cdot g \in [A]\binom{n}{k}$ is defined by $(f \cdot g)(i) = f(i)$ if $f(i) \in A$ and $(f \cdot g)(i) = g(j)$ if $f(i) = \lambda_j$.

Observe that $[A]\binom{n}{0} = A^n$. For $f \in [A]\binom{n}{m}$ the set $M = \{f \cdot g \mid g \in [A]\binom{m}{0}\} = f \cdot [A]\binom{m}{0}$ is the m–parameter subset of A^n described by f .

Note, however, that we have defined parameter words also with respect to the empty resp., one–element alphabet. Corresponding to different alphabets A, parameter words admit the following interpretations:

2.1 Parameter Words over the Empty Alphabet

Parameter words $f \in [\emptyset]\binom{n}{k}$ represent equivalence relations on $\{0, \ldots, n-1\}$ with precisely k equivalence classes, and vice versa. The ith equivalence class is given by $f^{-1}(\lambda_i)$. So, $[\emptyset](n) = \bigcup_{k<n}[\emptyset]\binom{n}{k}$ is the set of all equivalence relations on $\{0, \ldots, n-1\}$. For $f \in [\emptyset]\binom{n}{m}$ and $g \in [\emptyset]\binom{n}{k}$ put $f \leq g$ iff there exists $h \in [\emptyset]\binom{m}{k}$ such that $g = f \cdot h$. Then $([\emptyset](n), \leq)$ becomes the lattice of equivalence relations on $\{0, \ldots, n-1\}$.

2.2 Parameter Words over the One–Element Alphabet

Parameter words $f \in [\{0\}]\binom{n}{k}$ represent families of k nonempty and disjoint subsets of $\{0, \ldots, n-1\}$, viz., $f^{-1}(\lambda_i)$, $i < k$. Then $f \cdot [\{0\}]\binom{k}{1}$ is the set of all unions of these k sets. Using the language of extremal problems, $[\{0\}]\binom{n}{k}$ is the set of strong Δ–systems with k terms.

2.3 Parameter Words over the Two–Element Alphabet

Let $A = 2 = \{0, 1\}$. Every $f \in [2]\binom{n}{0}$ can be interpreted as the characteristic function of a subset of $\{0, \ldots, n-1\}$ (where the letter 1 indicates the occurrence of an element of this subset). The inclusion of subsets imposes a lattice structure $\leq$ on $[2]\binom{n}{0}$. Provided with this order $([2]\binom{n}{0}, \leq)$ is isomorphic to the Boolean lattice $\mathcal{B}(n)$ of rank n . Parameter words $f \in [2]\binom{n}{k}$ represent $\mathcal{B}(k)$–sublattices in $\mathcal{B}(n)$, and vice versa. The composition $f \cdot g$ corresponds to taking a sublattice inside a sublattice. The partial order of Boolean sublattices of $\mathcal{B}(n)$ by set inclusion can be defined using the composition of parameter words. For $f \in [2]\binom{n}{k}$ and $g \in [2]\binom{n}{m}$ put $f \le g$ if there exists $h \in [2]\binom{m}{k}$ such that $f = g \cdot h$.

2.4 Parameter Words over $GF(q)$

Let us denote by $GF(q)$, q a prime power, the Galois field with q elements. Every k–parameter word $f \in [GF(q)]\binom{n}{k}$, resp., the corresponding k–parameter subset in $GF(q)^n$, is a k–dimensional affine subspace. However, in general there exist affine subspaces which are not parameter subsets.

The notion of parameter words, resp., parameter sets can be slightly generalized allowing a finite group G to act on A .
So let G be a finite group with unit element e operating on A , i.e., there exists an operation $G \times A \to A$ such that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b \in G$ and $c \in A$.

Definition. $[A, G]\binom{n}{m}$ consists of all mappings $f : n \to A \cup (G \times \{\lambda_0, \dots, \lambda_{m-1}\})$ such that

$f^{-1}(G \times \{\lambda_i\}) \ne \emptyset$ for every $i < m$,
$f(\min f^{-1}(G \times \{\lambda_i\})) = (e, \lambda_i)$ for every $i < m$ and
$\min f^{-1}(G \times \{\lambda_i\}) < \min f^{-1}(G \times \{\lambda_j\})$ for all $i < j < k$.

For $f \in [A, G]\binom{n}{m}$ and $g \in [A, G]\binom{m}{k}$ the composition $f \cdot g \in [A, G]\binom{n}{k}$ is defined by

$$\begin{aligned} (f \cdot g)(i) &= f(i) && \text{if } f(i) \in A \\ &= a \cdot b && \text{if } f(i) = (a, \lambda_j) \text{ and } g(j) = b \in A \\ &= (a \cdot b, \lambda_\ell) && \text{if } f(i) = (a, \lambda_j) \text{ and } g(j) = (b, \lambda_\ell) \ . \end{aligned}$$

What has changed is that parameters λ_i are labelled by group elements. In order to make these parameter words rigid, the first occurence of λ_i is labelled with the unit element e . Composition then is defined via group multiplication, resp., via the group action on A . With respect to these more general parameter words further interpretations are possible:

2.5 Parameter Words in $[\{0\}, GF(q)^*]\binom{n}{m}$

Consider the multiplicative group $GF(q)^*$ operating on $\{0\}$, where 0 is the zero element of the Galois field $GF(q)$. Every $f \in [\{0\}, GF(q)^*]\binom{n}{m}$ represents an m–dimensional (homogeneous) linear subspace of the n–dimensional vector space over $GF(q)$. In general there exist additional m–dimensional linear subspaces, except for $m = 1$, where we have bijective correspondence.

2.6 Parameter Words in $[\{a\},G]\binom{n}{m}$

Using a different terminology, Dowling (1973) investigates parameter words $f \in [\{a\},G]\binom{n}{m}$. The finite group G operates trivially on the singleton set $\{a\}, a$ is a kind of annihilator. Put $[\{a\},G](n) = \bigcup_{k<n}[\{a\},G]\binom{n}{k}$. For $f \in [\{a\},G]\binom{n}{m}$ and $g \in [\{a\},G]\binom{n}{k}$ put $f \le g$ iff there exists $h \in [\{a\},G]\binom{m}{k}$ such that $g = f \cdot h$. For the trivial group $G = \{e\}$ one easily observes that $([\{a\},\{e\}](n), \le)$ is the lattice of equivalence relations of an $(n+1)$–element set. Dowling shows that, in general, $([\{a\},G](n),\le)$ is a geometric lattice of rank $n+1$. Also, nonisomorphic groups yield nonisomorphic geometric lattices. Dowling also considers the problem to what extent $([\{a\},G](n),\le)$ is representable over a field k . He shows that this is the case if and only if G is isomorphic to a subgroup of the multiplicative group of k (necessity requires $n \ge 3$) . The reader should compare Dowling's results with the example $[\{0\}, GF(q)^*]$.

3. Hales–Jewett's Theorem

Hales–Jewett's theorem is concerned with partitions of 0–parameter words, i.e., of A^n .

Theorem *(*Hales–Jewett 1963*).*
Given A, m *and* r *there exists a number* $n = HJ(|A|, m, r)$ *such that for every mapping* $\Delta : [A]\binom{n}{0} \to \{0,\ldots,r-1\}$ *there exists a monochromatic* m*–parameter word* $f \in [A]\binom{n}{m}$, *i.e.* $\Delta(f\cdot g) = \Delta(f \cdot h)$ *for all* $g,h \in [A]\binom{m}{0}$.

We give two proofs of Hales Jewett's theorem. The first one is the original argument of Hales and Jewett. The second one is due to S. Shelah (1988) and has the additional advantage of providing a primitive recursive upper bound for $HJ(|A|,m,r)$.

Proof. (Hales and Jewett) One shows

(1) $HJ(t,m+1,r) \le HJ(t,1,r) + HJ(t,m,r^{t^{HJ(t,1,r)}})$,

(2) $HJ(t+1,1,r+1) \le HJ(t, 1 + HJ(t+1,1,r), r+1)$.

Together with the trivial assertion '$HJ(1,m,r) = m$' observations (1) and (2) yield a proof of Hales–Jewett's theorem by induction on $t = |A|$, m and r .

ad (1) let $n'' = HJ(t,1,r)$ and $n' = HJ(t,m,r^{t^{HJ(t,1,r)}})$ and consider $\Delta : [A]\binom{n''+n'}{0} \to r$. This induces $\Delta' : [A]\binom{n'}{0} \to r^{t^{n''}}$ by $\Delta'(a_0,\ldots,a_{n'-1}) = \; < \Delta(a_0,\ldots,a_{n'-1},b_0,\ldots,b_{n''-1}) \mid (b_0,\ldots,b_{n''-1}) \in [A]\binom{n''}{0} >$.
By choice of n' there exists a monochromatic m–parameter word, finally then working on $[A]\binom{n''}{0}$ yields the $(m+1)$st parameter.

ad (2) let $\Delta : [A \cup \{b\}]\binom{n}{0} \to \{0,\ldots,r\}$, where $|A| = t$, $b \notin A$ an $n = HJ(t, 1 + HJ(t+1,1,r), r+1)$.

Consider $\Delta_A = \Delta\rceil[A]\binom{n}{0}$. Let $g \in [A]\binom{n}{m+1}$, where $m = HJ(t+1,1,r)$, such that $\Delta_A\rceil g \cdot [A]\binom{m+1}{0}$ is constant, say, in color r . If $\Delta(g\cdot(b,a_0,\ldots,a_{m-1})) = r$

for some $(a_0, \ldots, a_{m-1}) \in (A \cup \{b\})^m$, replace all b's in $g \cdot (b, a_0, \ldots, a_{m-1})$ by λ_0 and call the resulting 1–parameter word f .
Clearly, $\Delta \rceil f \cdot [A \cup \{b\}]\binom{1}{0} =$ constant. If no such $(a_0, \ldots, a_{m-1})$ exists, consider $\Delta' : [A \cup \{b\}]\binom{m}{0} \to \{0, \ldots, r-1\}$ which is defined by $\Delta'(a_0, \ldots, a_{m-1}) = \Delta(g \cdot (b, a_0, \ldots, a_{m-1}))$ and apply the properties of $m = HJ(t+1, r)$. □

For the second proof of Hales Jewett's theorem we need a lemma.

Shelah's cube lemma. *Let m and r be positive integers. Then there exists a positive integer $n = Sh(m, r)$ with the following property: Let Δ_i for $i < m$ be colorings*

$$\Delta_i : \underbrace{\binom{n}{2} \times \ldots \times \binom{n}{2}}_{i\ -\text{times}} \times \binom{n}{1} \times \underbrace{\binom{n}{2} \times \ldots \times \binom{n}{2}}_{(m-1-i)\ -\text{times}} \longrightarrow r$$

where $\binom{n}{2}$ denotes the set of two-element subsets of $\{0, \ldots, n-1\}$. Then there exist two–element subsets $\{a_i, b_i\} \in \binom{n}{2}$ for $i < m$ such that

$$\begin{aligned} &\Delta_i(\{a_0, b_0\}, \ldots, \{a_{i-1}, b_{i-1}\}, a_i, \{a_{i+1}, b_{i+1}\}, \ldots, \{a_{m-1}, b_{m-1}\}) \\ = &\Delta_i(\{a_0, b_0\}, \ldots, \{a_{i-1}, b_{i-1}\}, b_i, \{a_{i+1}, b_{i+1}\}, \ldots, \{a_{m-1}, b_{m-1}\}) \end{aligned}$$

for every $i < m$. The function $Sh(m, r)$ satisfies the inequalities:
$Sh(1, r) = r + 1$ *and* $Sh(m+1, r) \leq 1 + r^{\binom{Sh(m,r)}{2}^m} \leq r^{(Sh(m,r))^{2m}}$.

Proof of Shelah's cube lemma.
The assertion $Sh(1, r) = r + 1$ is just the pigeon hole principle and so we turn to the second inequality.

Let $n = Sh(m, r)$, let $n^* = 1 + r^{\binom{n}{2}^m}$ and consider colorings

$$\Delta_i : \underbrace{\binom{n}{2} \times \ldots \times \binom{n}{2}}_{i\ -\text{times}} \times \binom{n}{1} \times \underbrace{\binom{n}{2} \times \ldots \times \binom{n}{2}}_{(m-1-i)\ -\text{times}} \times \binom{n^*}{2} \longrightarrow r, \quad i < m,$$

as well as the mth coloring

$$\Delta_m : \underbrace{\binom{n}{2} \times \ldots \times \binom{n}{2}}_{m\ -\text{times}} \times \binom{n^*}{1} \longrightarrow r.$$

By choice of n^* there exists a two–element subset $\{a_m, b_m\} \in \binom{n^*}{2}$ such that

$$\begin{aligned} &\Delta_m(\{a_0, b_0\}, \ldots, \{a_{m-1}, b_{m-1}\}, a_m) \\ = &\Delta_m(\{a_0, b_0\}, \ldots, \{a_{m-1}, b_{m-1}\}, b_m) \end{aligned}$$

for all choices of two-element subsets $\{a_i, b_i\} \in \binom{n}{2}$, $i < m$. By projection the pair $\{a_m, b_m\}$ induces colorings

$$\Delta_i^* : \underbrace{\binom{n}{2} \times \ldots \times \binom{n}{2}}_{i\text{ -times}} \times \binom{n}{1} \times \underbrace{\binom{n}{2} \times \ldots \times \binom{n}{2}}_{(m-1-i)\text{ -times}} \longrightarrow r, \quad i < m,$$

namely $\Delta_i^*(\ldots) = \Delta_i(\ldots, \{a_m, b_m\})$. To these colorings the inductive assumption on n may be applied yielding $\{a_0, b_0\}, \ldots, \{a_{m-1}, b_{m-1}\} \in \binom{n}{2}$. By construction, then, $\{a_0, b_0\}, \ldots, \{a_{m-1}, b_{m-1}\}$, $\{a_m, b_m\}$ have the desired properties. □

Note that, obviously, the function $Sh(m, r)$ is primitive recursive, e.g.,

$$Sh(m,r) \leq r^{r^{2\cdot(m-1)\cdot r^{2\cdot(m-2)\cdot r^{\cdot^{\cdot^{\cdot 6\cdot r^{4\cdot r^2}}}}}}}.$$

Next we show how Shelah's cube lemma can be used to prove Hales Jewett's theorem.

Second proof of Hales-Jewett's theorem. (Shelah)
With the aid of Shelah's Cube Lemma the following recursion is derived:

$$(3) \qquad HJ(t+1,1,r) \leq HJ(t,1,r) \cdot Sh(HJ(t,1,r), r^{(t+1)^{HJ(t,1,r)}}).$$

Together with the trivial observation $HJ(1,1,r) = 1$ this provides an inductive proof for the existence of $HJ(t,1,r)$ in general. The numbers $HJ(t,m,r)$ are then estimated by

$$(4) \qquad HJ(t,m,r) \leq HJ(t^m,1,r).$$

Inequality (4) may be derived by forgetting brackets, i.e., $(A^m)^n$ is identified with $A^{m\cdot n}$: $((a_0, \ldots, a_{m-1}), (b_0, \ldots, b_{m-1}), \ldots, (c_0, \ldots, c_{m-1}))$ is viewed as $(a_0, \ldots, a_{m-1}, b_0, \ldots, b_{m-1}, \ldots\ldots, c_0, \ldots, c_{m-1})$.
Using this interpretation one-parameter words $f \in [A^m]\binom{n}{1}$ correspond to m-parameter words $\tilde{f} \in [A]\binom{m\cdot n}{m}$, but not necessarily vice versa.

So we concentrate on proving inequality (3). For convenience we write $m = HJ(t,1,r)$ and $n = Sh(m, r^{(t+1)^m})$, and we consider the alphabet $B = \{0, \ldots, t\}$. Let $\Delta : B^{m\cdot n} \longrightarrow r$ be an r-coloring. Using the properties of m and n we derive the existence of a monochromatic one-parameter word $f \in [B]\binom{m\cdot n}{1}$.
For every number $a < n$ we define $h_a \in B^n$ by

$$h_a = (\underbrace{t-1, \ldots, t-1}_{a\text{-times}}, \underbrace{t, \ldots \ldots \ldots, t}_{(n-a)\text{-times}}).$$

For every two-element subset $\{a, b\} \in \binom{n}{2}$, $a < b$, we define a one-parameter word $g_{\{a,b\}} \in [B]\binom{n}{1}$ by

$$g_{\{a,b\}} = (\underbrace{t-1,\ldots,t-1}_{a-times}, \underbrace{\lambda_0,\ldots,\lambda_0}_{(b-a)-times}, \underbrace{t,\ldots\ldots\ldots,t}_{(n-b)-times}).$$

Note that

$$g_{\{a,b\}} \cdot (t-1) = h_b \text{ and } g_{\{a,b\}} \cdot (t) = h_a.$$

For each $i < m$ we induce a coloring

$$\Delta_i : \underbrace{\binom{n}{2} \times \ldots \times \binom{n}{2}}_{i-times} \times \binom{n}{1} \times \underbrace{\binom{n}{2} \times \ldots \times \binom{n}{2}}_{(m-1-i)-times} \times B^m \longrightarrow r$$

as follows: let $\{a_j, b_j\} \in \binom{n}{2}$ for $j < m$ with $j \neq i$, let $a < n$, and let $(\alpha_0, \ldots, \alpha_{m-1}) \in B^m$, then put

$$\begin{aligned} (5) \qquad & \Delta_i(\{a_0,b_0\},\ldots,\{a_{i-1},b_{i-1}\},a,\{a_{i+1},b_{i+1}\},\ldots, \\ & \{a_{m-1},b_{m-1}\},(\alpha_0,\ldots,\alpha_{m-1})) \\ & = \\ & \Delta(g_{\{a_0,b_0\}} \cdot (\alpha_0) \times \ldots \times g_{\{a_{i-1},b_{i-1}\}}(\alpha_{i-1}) \times \\ & h_a \times g_{\{a_{i+1},b_{i+1}\}} \cdot (\alpha_{i+1}) \times \ldots \\ & \times g_{\{a_{m-1},b_{m-1}\}} \cdot (\alpha_{m-1})). \end{aligned}$$

By choice of n there exist two–element subsets $\{a_0,b_0\},\ldots,\{a_{m-1},b_{m-1}\} \in \binom{n}{2}$ such that for all $i < m$ and all $(\alpha_0,\ldots,\alpha_{m-1}) \in B^m$ it follows that

$$\begin{aligned} (6) \qquad & \Delta_i(\{a_0,b_0\},\ldots,\{a_{i-1},b_{i-1}\},a_i,\{a_{i+1},b_{i+1}\},\ldots, \\ & \{a_{m-1},b_{m-1}\},(\alpha_0,\ldots,\alpha_{m-1})) \\ & = \\ & \Delta_i(\{a_0,b_0\},\ldots,\{a_{i-1},b_{i-1}\},b_i,\{a_{i+1},b_{i+1}\},\ldots, \\ & \{a_{m-1},b_{m-1}\},(\alpha_0,\ldots,\alpha_{m-1})). \end{aligned}$$

Consider the m–parameter word

$$g = g_{\{a_0,b_0\}} \times g_{\{a_1,b_1\}} \times \ldots \times g_{\{a_{m-1},b_{m-1}\}} \in [B]\binom{m \cdot n}{m}$$

and observe that the property of the colorings Δ_i implies that the color $\Delta(g \cdot (\alpha_0,\ldots,\alpha_{m-1}))$ for $(\alpha_0,\ldots,\alpha_{m-1}) \in B^m$ remains unchanged whenever some $\alpha_i = t$ is replaced by $\alpha_i = t-1$. So we may consider the restriction $\Delta^* : (B\backslash\{t\})^m \longrightarrow r$ which is given by $\Delta^*(\alpha_0,\ldots,\alpha_{m-1}) = \Delta(g \cdot (\alpha_0,\ldots,\alpha_{m-1}))$. By choice of m there exists a monochromatic one-parameter word $h \in [B\backslash\{t\}]\binom{m}{1}$. Then the one-parameter word $g \cdot h \in [B]\binom{m \cdot n}{1}$ is monochromatic with respect to Δ. □

Inequalities (1) and (2) involve a nested double recursion which leads to a non-primitive upper bound for the Hales-Jewett function $HJ(t,m,r)$. In contrast to that, as the function $Sh(m,r)$ coming from Shelah's cube lemma is primitive recursive, inequality (3) shows that $HJ(t,m,r)$ is primitive recursive. This was an open problem for about 60 years.

Originally, Hales-Jewett's theorem was used to analyse higher dimensional versions of the well–known game tic tac toe. Let us mention two typical applications:

3.1 Arithmetic Progressions

Consider the alphabet $A = \{0,\ldots,k-1\}$; the mapping $\psi : A^n \to \{0,\ldots,(k-1)\cdot n\}$ with $\psi(a_0,\ldots,a_{n-1}) = \sum_{i<n} a_i$ has the property that it maps every one–parameter set onto a k–term arithmetic progression. Hence, Hales–Jewett's theorem implies van der Waerden's theorem on arithmetic progressions:

3.1 Theorem (van der Waerden (1927))
For every pair k and r of positive integers there exists a number $n = vdW(k,r)$ such that for every mapping $\Delta : \{0,\ldots,n-1\} \to \{0,\ldots,r-1\}$ there exists a monochromatic k–term arithmetic progression. □

Hales–Jewett's theorem can also be used to establish partition theorems for multiple arithmetic progressions and for Deuber's partition theorem on (m,p,c)–sets (Deuber (1973), Leeb (1975), Deuber, Rothschild, Voigt (1982)).

3.2 Idempotents in Finite Algebras

Let $\mathcal{A}$ be a class of finite algebras which is closed under finite products and such that every $A \in \mathcal{A}$ consists of idempotent elements only. Observe that for every algebra $A \in \mathcal{A}$ (by abuse of language, A denotes also the underlying set where we omit the algebraic operations) each one–parameter set $M \subseteq A^n$ is an A–subalgebra. Hence, by Hales–Jewett's theorem one gets the following result:

3.2 Theorem (Prömel, Voigt (1981), Ježek, Nešetřil (1983))
For every $A \in \mathcal{A}$ and every positive integer r there exists a number n such that for every mapping $\Delta : A^n \to \{0,\ldots,r-1\}$ there exists a monochromatic A–subalgebra. □

Typical classes $\mathcal{A}$ are, e.g., distributive lattices, modular lattices, general lattices and so forth.
Several generalizations and ramifications of Hales–Jewett's theorem have been considered:

3.3 A $*$–Version

Here not only 0-parameter words of one fixed length are partitioned, as in Hales–Jewett's theorem, but words of variable length (were a "$*$" indicates the end of a word). Such $*$–parameter words were introduced in (Voigt 1980) to prove a partition theorem for finite abelian groups.

Let "$*$" be some symbol not contained in $A\cup\{\lambda_0,\dots\lambda_{m-1}\}$ and let $[A]^*\binom{n}{m}$ denote the set of all m–parameter words f of length n over $A\cup\{*\}$ satisfying the condition

$$\text{if } f(i)=* \text{ for some } i<n \text{ then } f(j)=* \text{ for all } i\le j<n\ .$$

Hence $[A]^*\binom{n}{m}$ is the set of m–parameter words of length at most n over A . Note that $[A]\binom{n}{m}\subset[A]^*\binom{n}{m}$. For $f\in[A]^*\binom{n}{m}$ and $g\in[A]^*\binom{m}{k}$ the composition $f\cdot g\in[A]^*\binom{n}{k}$ is defined by

$$\begin{aligned}
(f\cdot g)(i) &= * , && \text{if there exists } j<i \text{ such that } (f\cdot g)(j)=* ,\\
(f\cdot g)(i) &= f(i) , && \text{if } f(i)\in A\cup\{*\} \text{ and } (f\cdot g)(j)\neq * \text{ for all } j<i ,\\
(f\cdot g)(i) &= g(j) , && \text{if } f(i)=\lambda_j \text{ and } (f\cdot g)(j)\neq * \text{ for all } j<i .
\end{aligned}$$

Intuitively, the composition $f\cdot g$, interpreted as the insertion of g into the parameters of f is performed as long as possible, eventually $*$'s are filled in.

3.3 Theorem (Voigt (1980))
Given A,m and r there exists a number $n=HJ^*(|A|,m,r)$ such that for every mapping $\Delta:[A]^*\binom{n}{0}\to\{0,\dots,r-1\}$ there exists a monochromatic $f\in[A]^*\binom{n}{m}$, i.e. $\Delta(f\cdot g)=\Delta(f\cdot h)$ for all $g,h\in[A]^*\binom{m}{0}$.

Proof. Let $n_{mr}=mr$ and $n_{mr-j}=HJ(|A|,n_{mr-j+1}-mr+j,r)+mr-j$. Choose $n=n_0$ and let $\Delta:[A]^*\binom{n}{0}\to\{0,\dots,r-1\}$ be given. For $g\in[A]^*\binom{k}{0}$ let $*(g)=k-\max\{i\mid g(i)\in A\}$ and for $i=0,\dots,k$ put $[A]^i\binom{k}{0}=\{g\in[A]\binom{k}{0}\mid *(g)=i\}$.

In particular, $\displaystyle\bigcup_{i=1}^{k}[A]^i\binom{k}{0}=[A]^*\binom{k}{0}$.

First we prove inductively that for every $j\le mr$ there exists $f_j\in[A]\binom{n}{n_j+1}$ such that for every $g,h\in\bigcup_{i=0}^{j}[A]^i\binom{n_j+1}{0}$ with $*(g)=*(h)$ we have $\Delta(f_j\cdot g)=\Delta(f_j\cdot h)$.

For $j=0$ this is Hales–Jewett's theorem. So assume that the assertion is true for some $j<mr$ and let $\Delta^{j+1}:[A]^{j+1}\binom{n_j+1}{0}\to\{0,\dots,r-1\}$ be given by $\Delta^{j+1}(g)=\Delta(f_j\cdot g)$. By choice of $n_{j+1}=HJ(|A|,n_{j+2}-j-1,r)+j+1$ and Hales–Jewett's theorem there exists $f'\in[A]\binom{n_j+1-j-1}{n_j+2-j-1}$ monochromatic. Then $f_{j+1}=f_j\cdot(f'(0),\dots,f'(n_{j+1}-j-2),\lambda_{n_j+2-j-1},\dots,\lambda_{n_j+2-1})$ fulfills the requirement of the induction.

Choosing $j=mr$ we get $f_{mr}\in[A]\binom{n}{mr}$ such that all $g,h\in[A]^*\binom{mr}{0}$ with $*(g)=*(h)$ have the same color with respect to Δ . This defines an r–coloring Δ' of the integers $\{0,\dots,mr\}$ by $\Delta'(i)=\Delta(f_{mr}\cdot g)$ for any g with $*(g)=i$. By the pigeon–hole principle we get $0\le i_0<\dots<i_m\le m\cdot r$ in one color. Now let $f''\in[A]\binom{mr}{m}$ be given by $f''(i)=a$ for some $a\in A$, if $i<i_0$, $f''(i)=\lambda_j$, if $i_j\le i<i_{j+1}$ and $f''(i)=*$ for $i_m\le i$. Obviously $f=f_{mr}\cdot f''$ has the desired properties. □

3.4 An Induced Version

Let $\Gamma : A^m \to \{0,1\}$ be a structural mapping, i.e., the points $(a_0, \ldots, a_{m-1}) \in A^m$ are split into two classes, blue points and red points. An induced partition theorem respects the color of the points:

3.4 Theorem (Deuber, Rothschild, Voigt (1982))
Given A, m, r and a structural mapping $\Gamma : [A]\binom{m}{0} \to \{0,1\}$ there exists a number $n = HJ_{\text{ind}}(|A|, m, r)$ and there exists a structural mapping $\Gamma^* : [A]\binom{n}{0} \to \{0,1\}$ such that for every mapping $\Delta : [A]\binom{n}{0} \to \{0, \ldots, r-1\}$ there exists an m–parameter word $f \in [A]\binom{n}{m}$ which is induced, i.e., $\Gamma^*(f \cdot g) = \Gamma(g)$ for every $g \in [A]\binom{m}{0}$, and monochromatic, i.e., $\Delta(f \cdot g) = \Delta(f \cdot h)$ for all $g, h \in [A]\binom{m}{0}$ with $\Gamma(g) = \Gamma(h)$. □

As a corollary one obtains for example an induced version of van der Waerden's theorem on arithmetic progressions. This has been established independently by Spencer (1975) and Nešetřil and Rödl (1976).

3.5 A Restricted Version

The interest in restricted versions has been initiated by Erdös (1975) who asked whether there exists a set S of positive integers which does not contain any $(k+1)$–term arithmetic progession, however, for every partition $S = S_0 \cup S_1$ one of the parts S_0 or S_1 contains a k–term arithmetic progression.

This question has been answered affirmatively by Spencer (1975) and Nešetřil and Rödl (1976). More generally, Hales–Jewett's theorem admits a restricted version:

3.5 Theorem (Deuber, Prömel, Rothschild, Voigt (1981))
Given A, m and r there exist a number $n = HJ_{res}(|A|, m, r)$ and a set $S \subseteq [A]\binom{n}{0}$ satisfying the following two conditions:

(1) for every $f \in [A]\binom{n}{m+1}$ there exists $g \in [A]\binom{m+1}{0}$ such that $f \cdot g \notin S$, i.e., S does not contain any $(m+1)$–parameter set, however,
(2) for every mapping $\Delta : S \to \{0, \ldots, r-1\}$ there exists $f \in [A]\binom{n}{m}$ such that $f \cdot [A]\binom{m}{0} \subseteq S$ and $\Delta] f \cdot [A]\binom{m}{0} = \text{constant}$. □

3.6 Forbidding Short Cycles

Apparently Tutte (Descartes (1948)) was the first to ask for graphs which have a large chromatic number, and, simultaneously, do not possess short cycles. In 1948 he showed that graphs without triangles may have arbitrary large girth. Eventually Erdös (1959) resolved the general problem by showing that for every pair r and g of positive integers there exists a graph with chromatic number larger than r and girth larger than g. Erdös' proof used a counting argument. Incidentally, this proof laid down the source for the nowadays so-called probabilistic method in Discrete Mathematics. J. Spencer (1975) considered graphs and hypergraphs which are defined on combinatorial structures and then asked how to refine the structure to obtain a large girth, but to maintain a large chromatic number. E.g., let $AP_{n,k}$ be the set of k-term arithmetic progressions $A \subseteq n$. We may view $AP_{n,k}$ as the edge set of a k-regular hypergraph on n. Notice that van der Waerden's Theorem 3.1 may be formulated by saying that the chromatic numbers $\chi(AP_{n,k})$ tend to infinity with n tending to infinity, and k fixed. Also using the probabilistic method Spencer showed how to forbid short cycles in $AP_{n,k}$:

3.6 Theorem *(*Spencer (1975)*)*
Let k, r and g be positive integers. Then there exists a family $\mathcal{A} \subseteq AP_{n,k}$ of k-term arithmetic progressions such that $\chi(\mathcal{A}) > r$ and girth$(\mathcal{A}) > g$.

The corresponding result in connection with Hales-Jewett's theorem has been established by Rödl (1981).

Let $\mathcal{F} \subseteq [A]\binom{n}{m}$ be a set of m-parameter words. $\mathcal{F}$ defines a hypergraph $\mathcal{H}(\mathcal{F})$ on $[A]\binom{n}{0}$ with edges $f \cdot [A]\binom{m}{0}$, $f \in \mathcal{F}$, i.e., edges are the corresponding m-parameter sets. Hales-Jewett's theorem can be formulated by saying that for m fixed and n tending to infinity the chromatic numbers $\chi(\mathcal{H}([A]\binom{n}{m}))$ tend to infinity. Using probabilistic means, Rödl shows the following strengthening of Hales-Jewett's theorem:

3.6a Theorem *(*Rödl (1981)*)*
Given A, m, r and g there exists a number $n = HJ_{forb}(A, m, r, g)$ and there exists a family $\mathcal{F} \subseteq [A]\binom{n}{m}$ such that $\chi(\mathcal{H}(\mathcal{F})) > r$ and girth$(\mathcal{H}(\mathcal{F})) > g$, i.e. $\mathcal{H}(\mathcal{F})$ does not contain any cycle of length smaller or equal than g.

3.7 Sparse Versions

It was, then, an open question to find constructive proofs for the results in 3.6. But not only this. As it turns out, the probabilistic method allows to select a family of edges, viz., k-term arithmetic progressions, in $AP_{n,k}$ which do not form short cycles. However, these edges almost surely cover nearly all the vertices in n. Spencer (1975) suggested to look for constructions of sets $S \subseteq n$ such that the set $AP_{S,k}$ of all k-term arithmetic progressions $A \subseteq S$ has chromatic number larger than r and girth larger than g. So rather than selecting edges we select vertices and consider all edges which are spanned (induced) by these vertices. Accordingly, with respect to Hales-Jewett's theorem, for sets $S \subseteq [A]\binom{n}{0}$ we let $\mathcal{H}_m(S)$ be the set of all $f \in [A]\binom{n}{m}$ with $f \cdot [A]\binom{m}{0} \subseteq S$. One easily observes that for $|A| = 2$ and for every set $S \subseteq [A]\binom{n}{0}$ such that $\chi(\mathcal{H}_1(S)) > 2$ the hypergraph $\mathcal{H}_1(S)$ must contain a triangle. However, this case (viz., $|A| = 2$ and $m = 1$) is the only exception which does not allow a sparse version of Hales-Jewett's theorem.

3.7 Theorem *(*Prömel, Voigt (1988)*)*
Let A be a finite set and let m, r and g be positive integers such that $|A| \geq 2$ and

$|A| + m \geq 4$. *Then there exists a positive integer* $n = HJ_{sparse}(A, m, r, g)$ *and there exists a family* $S \subseteq [A]\binom{n}{0}$ *such that* $\chi(\mathcal{H}_m(S)) > r$ *and* $girth(\mathcal{H}_m(S)) > g$.

This result has been proved by a constructive approach relying on an amalgamation techniques for parameter sets. This amalgamation techniques extends the approach of Frankl, Graham and Rödl (1987) and previous amalgamation procedures of Nešetřil and Rödl for graphs and hypergraphs, as well as the authors' attempts in Prömel, Voigt (1987, 1989). More elaborate versions also allow to establish a sparse version of van der Waerden's theorem, thus resolving the conjecture of Spencer (1975).

3.7a Theorem (Prömel, Voigt (1988))
Let k, r *and* g *be positive integers. Then there exists a positive integer* $n = vdW_{sparse}(k, m, g)$ *and there exists a set* $S \subseteq n$ *such that* $\chi(AP_{S,k}) > r$ *and* $girth(AP_{S,k}) > g$.

3.8 Density Versions

Again it started with a problem about arithmetic progressions. Szemeredi (1975), proving a famous conjecture of Erdös and Turan, showed that for every $\epsilon > 0$ and every positive integer k there exists $n = Sze(\epsilon, k)$ such that every set $S \subseteq \{0, \ldots, n-1\}$ with $|S| > \epsilon \cdot n$ contains a k–term arithmetic progression. Rödl (1982) proved a density version of Hales–Jewett's theorem with respect to two–element alphabets:

3.8 Theorem (Rödl (1982))
For every $\epsilon > 0$ *and positive integer* m *there exists a number* $Rö(\epsilon, m)$ *such that for every* $n \geq Rö(\epsilon, k)$ *and every* $S \subseteq [\{0,1\}]\binom{n}{0}$ *with* $|S| > \epsilon \cdot 2^n$ *there exists* $f \in [\{0,1\}]\binom{n}{m}$ *satisfying* $f \cdot [\{0,1\}]\binom{m}{0} \subseteq S$. □

For $m = 1$ this result follows from Sperner's theorem (Sperner 1928) saying that a maximal antichain in 2^n has cardinality $\binom{n}{n/2} \approx 2^n/\sqrt{n}$. The case $m = 2$ has been proved by Erdös and Kleitman (1971). Rödl's argument basically proceeds by induction on m.

Somewhat more generally, Brown and Buhler (1984) show that, for any finite alphabet A, a density version for $m = 1$ implies a density version for arbitrary m.

However, it is a challenging problem whether such a density result holds for arbitrary A. Even the case $|A| = 3$ is still open. We should mention that Graham (1983) offers some money for a solution. Recently, Fürstenberg and Katznelson proved the following (slightly weaker) density theorem:

3.8a Theorem (Fürstenberg–Katznelson (1985))
Let G *be a cyclic group. Then for every* $\epsilon > 0$ *there exists a number* $FK(|G|, \epsilon)$ *such that for every* $n \geq FK(G, \epsilon)$ *and every* $S \subseteq [G, G]\binom{n}{0}$ $(= [G]\binom{n}{0})$, *where* G *acts on itself, with* $|S| > \epsilon \cdot |G|^n$ *there exists* $f \in [G, G]\binom{n}{1}$ *such that* $f \cdot [G, G]\binom{1}{0} \subseteq S$. □

The Fürstenberg–Katznelson theorem immediately implies a density result for affine points in affine spaces over finite fields. For the ternary field $GF(3)$ this has been established before by Brown and Buhler (1982).

3.9 A Canonizing Version

The Erdös–Graham canonical version of van der Waerden's theorem on arithmetic progressions (cf. Deuber, Graham, Prömel, Voigt (1983)) asserts that for every mapping $\Delta : \{0,\ldots,n-1\} \to \mathbb{N}$, where $n \geq EG(k)$ is sufficiently large, there exists a k–term arithmetic progression $\mathcal{A} = \{a + \lambda \cdot d \mid \lambda = 0,\ldots,k-1\}$ such that $\Delta\rceil\mathcal{A}$ is constant or one–to–one. Analogously, the canonizing (resp., canonical – the terminology seems to be confusing) version of Hales–Jewett's theorem states the following:

3.9 Theorem (Prömel, Voigt (1983))
Given A and m there exists a number $n = PV(|A|,m)$ such that for every mapping $\Delta : [A]\binom{n}{0} \to \mathbb{N}$ there exists an equivalence relation $\approx$ on A and there exists $f \in [A]\binom{n}{m}$ such that for all $g = (g_0,\ldots,g_{m-1})$, $h = (h_0,\ldots,h_{m-1}) \in [A]\binom{m}{0}$ it follows that $\Delta(f \cdot g) = \Delta(f \cdot h)$ iff $g_i \approx h_i$ for all $i < m$. □

Schmerl (preprint 1985) applies this result in order to show that for every countable nonstandard model $\mathcal{M}$ of Peano Arithmetic and every positive integer $k \geq 2$ there exists a cofinal extension $\mathcal{N}$ of $\mathcal{M}$ such that the lattice $\mathcal{L}(\mathcal{N}/\mathcal{M})$ of intermediate models is isomorphic to $\Pi(k)$, the lattice of equivalence relations of a k–element set (cf. also Schmerl (1985)).

The special case $|A| = 2$ of theorem 3.8 admits the following formulation (cf. 2.3):

Corollary. *Let $n = PV(2,m)$. For every mapping $\Delta : \mathcal{B}(n) \to \mathbb{N}$ there exists a $\mathcal{B}(m)$–sublattice $\mathcal{L} \subseteq \mathcal{B}(n)$ such that $\Delta\rceil\mathcal{L}$ is constant or one–to–one.* □

Nešetřil and Rödl call this phenomenon 'selectivity' (cf. Nešetřil, Rödl (1978)).

In Nešetřil, Rödl (1984) a proof of this corollary is indicated which relies on the Erdös–Graham canonical version of van der Waerden's theorem. A short proof has been given in Prömel, Voigt (preprint 1985).

There is a common generalization of the canonical version and the restricted version of van der Waerden's theorem (and even its induced version) (Prömel, Rothschild (1987)). But we do not know whether such a common generalization of the canonical and the restricted version of Hales–Jewett's theorem is also valid. Even the case $|A| = 2$ is unsolved.

3.10 Canonical (Natural) Orders

A total order on A^n is called a canonical (or a natural) order if this order is invariant with respect to parameter sets in A^n . This is to say that the restrictions of a canonical total order to any two parameter sets are of the same type. In particular, the definition of a canonical order in A^n is independent of n.

A typical example for a canonical order is the lexicographic order on A^n (with respect to some fixed order on A). If A^n is provided with the lexicographic order then also every parameter set of A^n is of this kind.

A complete characterization of all canonical orders on A^n is contained in Nešetřil et al. (1985). For the description of these orders see also Nešetřil (1984). Here we follow the approach of Prömel (1989).

Let $A = \{a_0, \dots, a_t\}$ be provided with a total order $\leq$, i.e $a_0 < a_1 < \dots < a_t$ and let $\mathcal{B}(t)$ be the Boolean lattice on the t (ordered) atoms $0, \dots, t-1$. A 0–1 chain $\mathcal{D}$ in $\mathcal{B}(t)$ is a family $\mathcal{D} = (D_0, \dots, D_k)$ of pairwise distinct subsets of $\{0, \dots, t-1\}$ such that $D_i \subseteq D_{i+1}$ for every $i < k$ and $D_0 = \emptyset$, $D_k = \{0, \dots, t-1\}$. For $g \in A^n$ let $I(g(m))$ be the index of $g(m)$, i.e., $I(g(m)) = i$ if $g(m) = a_i$. Now we associate to every $0-1$ chain $\mathcal{D}$ in $B(t)$ a total order $\leq$ on A^n as follows:
Let $g < h$ with respect to $\mathcal{D} = (D_0, \dots, D_k)$ if and only if

(1) there exist $i \leq k$, $j \in D_i$ and $m \leq n$ such that $I(g(m)) \leq j$ and $I(h(m)) > j$
(2) for all $m' < m$ and all $j \in D_i : I(g(m')) \leq j$ iff $I(h(m')) \leq j$
(3) for all $i' < i$, all $j \in D_{i'}$ and all $m \leq n : I(g(m)) \leq j$ iff $I(h(m)) \leq j$.

We denote this order on A^n by $\leq_{\mathcal{D}}$.

Obviously, if $\mathcal{D} = (D_0, D_1)$ (where $D_0 = \emptyset$ and $D_1 = \{0, \dots, t-1\}$) we get the usual lexicographic order with respect to the given total order $\leq$ on A. It should be mentioned that different pairs $(\leq, \mathcal{D})$ and $(\leq', \mathcal{D}')$ yield different orders $\leq_{\mathcal{D}}, \leq'_{\mathcal{D}'}$ resp.
There is the following characterization theorem for canonical total orders on A^n:

3.10 Theorem (Nešetřil, Prömel, Rödl, Voigt (1985), Prömel (1989))
Given A and m there exists a number $n = NPRV(|A|, m)$ such that for every one–to–one mapping $\mathcal{O} : [A]\binom{n}{0} \to \mathbb{N}$ there exist a total order $\leq$ on A, a 0–1 chain $\mathcal{D}$ in $\mathcal{B}(|A|-1)$ and there exists $f \in [A]\binom{n}{m}$ such that for all $g, h \in [A]\binom{m}{0}$ it follows that

$$\mathcal{O}(f \cdot g) \leq \mathcal{O}(f \cdot h), \text{ where } \leq \text{ is the order on } \mathbb{N},$$
$$\text{if and only if} \quad f \cdot g \leq_{\mathcal{D}} f \cdot h. \qquad \square$$

Obviously, theorem 3.10 can be viewed as an asymmetric analogue to the canonizing version of Hales–Jewett's theorem (i.e. theorem 3.9).

The special case $|A| = 2$, which is already proved in Nešetřil et. al. (1982), admits the following formulation:

Corollary. (Nešetřil, Prömel, Rödl, Voigt (1982))
Let $n = NPRV(2, m)$. For every one–to–one mapping $\mathcal{O} : \mathcal{B}(n) \to \mathbb{N}$ there exists a $\mathcal{B}(m)$–sublattice $\mathcal{L} \subseteq \mathcal{B}(n)$ such that $\Delta \rceil \mathcal{L}$ is either lexicographic with respect to $0 < 1$ or lexicographic with respect to $1 < 0$. $\square$

We should mention that if we impose an arbitrary *partial* order $\leq$ on 2^n (n sufficiently large) then there exists a 2^m–sublattice $\mathcal{L} \subseteq 2^n$ such that $\mathcal{L}$ is ordered as 2^m (i.e. $g \leq h$ if $g(i) \leq h(i)$ for every $i < m$) or lexicographic or $\mathcal{L}$ is an antichain. Of course we can interchange the role of 0 and 1, and so we get five canonical partial orders in this case (cf. Prömel (1989)). It is an open question, how the canonical partial orders look like for general alphabets A.

4. Graham–Rothschild's Theorem

Basically speaking, Graham–Rothschild's theorem is concerned with partitions of k–parameter words in A^n. It generalizes Hales–Jewett's theorem to higher dimension.

Theorem. (Graham, Rothschild (1971))
Let A be a finite set, let G be a finite group operating on A and let k, m and r be positive integers. Then there exists $n = GR(|A|, |G|, k, m, r)$ such that for every mapping $\Delta : [A, G]\binom{n}{k} \to \{0, \ldots, r-1\}$ there exists a monochromatic $f \in [A, G]\binom{n}{m}$, i.e., $\Delta(f \cdot g) = \Delta(f \cdot h)$ for all $g, h \in [A, G]\binom{m}{k}$. □

In particular, for the trivial group $G = \{e\}$ we have the following corollary:

Corollary.
Let A be a finite set and let k, m and r be positive integers. Then there exists $n = GR(|A|, k, n, r)$ such that for every mapping $\Delta : [A]\binom{n}{k} \to \{0, \ldots, r-1\}$ there exists a monochromatic $f \in [A]\binom{n}{m}$. □

Proof of Graham–Rothschild's theorem:
By induction on k. The case $k = 0$ is settled by Hales–Jewett's theorem. Hence we can assume the theorem for some $k - 1 \geq 0$ (and every finite set B and every color–number r'). We use the $*$–version of Hales–Jewett's theorem, viz., theorem 3.3.

Let $x = HJ^*(|A|, m, r)$ and $n_x = x + k$. For $0 < j \leq x$ let

$$n_{x-j} = GR(|A| + |G|, |G|, k-1, n_{x-j+1} - x + j - 1, r^{|A|^{(x-j)}}) + x - j + 1 .$$

Choose $n = n_0$ and let $\Delta : [A, G]\binom{n}{k} \to \{0, \ldots, r-1\}$ be an arbitrary mapping. For $g \in [A, G]\binom{n}{k}$ let $In(g) \in [A]^*\binom{n}{0}$ be given by $In(g)(i) = g(i)$ for $i < \min g^{-1}(\lambda_0, e)$ and $In(g)(i) = *$ otherwise. Moreover let $|In(g)| = \min g^{-1}(\lambda_0, e)$. First we prove inductively that for every $0 \leq j \leq x$ there exists $f_j \in [A, G]\binom{n}{n_j}$ such that for every $g, h \in [A, G]\binom{n_j}{k}$ with $In(g) = In(h)$ and $|In(g)| < j$ we have $\Delta(f_j \cdot g) = \Delta(f_j \cdot h)$.

For $j = 0$ the assertion becomes vacuous. So assume the claim is true for some $0 \leq j < x$ and let $\Delta^j : [A \cup (G \times \lambda_0), G]\binom{n_j - j - 1}{k-1} \to r^{|A|^j}$ be given by $\Delta^j(g) = (\Delta(f_j \cdot g') \mid (g'(0), \ldots, g'(j-1)) \in A^j, g'(j) = (e, \lambda_0), g'(j+k) = g(k-1)$ for $1 \leq k \leq n_j - j - 1)$.

By choice of $n_j = GR(|A| + |G|, |G|, k-1, n_{j+1} - j - 1, r^{|A|^j}) + j + 1$ and by Graham–Rothschild's theorem for $k-1$ there exists $f' \in [A \cup (G \times \lambda_0), G]\binom{n_j - j - 1}{n_{j+1} - j - 1}$ monochromatic with respect to Δ_j. Let $f'' \in [A, G]\binom{n_j}{n_{j+1}}$ be given by $f''(i) = (\lambda_i, e)$, if $i \leq j, f''(i) = f'(i - j - 1)$ if $j < i < n_j$ and $f'(i-j-1) \in A$ and $f''(i) = (\alpha, \lambda_{j+k})$ if $j < i < n_j$ and $f'(i-j-1) = (\alpha, \lambda_k)$. Then $f_{j+1} = f_j \cdot f''$ fulfills the requirement of the j–induction.

Choosing $j = x$ we get $f_x \in [A, G]\binom{n}{x+k}$ such that all $g, h \in [A, G]\binom{x+k}{k}$ satisfying $In(g) = In(h)$ have obtained the same color with respect to Δ. This defines an r–coloring Δ' of $[A]^*\binom{x}{0}$ by $\Delta'(g) = \Delta(g')$ for any g' satisfying $\min g'^{-1}(e, \lambda_0) = \min g^{-1}(*)$ and $g'(i) = g(i)$ for all $i < \min g^{-1}(*)$. By choice of $x = HJ^*(|A|, m, r)$ and theorem 3.3 we get some $f^0 \in [A]^*\binom{x}{m}$ in one color. Define $f^{00} \in [A, G]\binom{x+k}{m}$ by $f^{00}(i) = f^0(i)$, if $f^0(i) \in A$, $f^{00}(i) = (e, \lambda_j)$, if $f^0(i) = \lambda_j$ and $f^{00}(i) = a$ for some $a \in A$ otherwise. Now let $f = f_x \cdot f^{00}$. Then f has the desired properties. □

The Graham–Rothschild theorem has many applications:

4.1 Ramsey's Theorem

For nonnegative integers k, m let us denote by $\binom{\mathrm{m}}{k}$ the set of strictly ascending injections $f : \{0, \ldots, k-1\} \to \{0, \ldots, m-1\}$. Clearly, every $f \in \binom{\mathrm{m}}{k}$ describes a k–element subsets of $\{0, \ldots, m-1\}$, and vice versa. Hence, $\binom{\mathrm{m}}{k}$ is the set of k–element subset of $\{0, \ldots, m-1\}$. For $f \in \binom{\mathrm{n}}{m}$ and $g \in \binom{\mathrm{m}}{k}$ the usual composition of mappings yields $f \cdot g \in \binom{\mathrm{n}}{k}$.

Parameter words very naturally admit a surjective functor onto finite subsets, i.e., consider $\Phi : [A]\binom{m}{k} \to \binom{\mathrm{m}}{k}$ which is given by $\Phi \cdot f : \{0, \ldots, k-1\} \to \{0, \ldots, m-1\}$, where $(\Phi \cdot f)(i) = \min f^{-1}(\lambda_i)$. The functorial property of Φ is that for $f \in [A]\binom{n}{m}$ and $g \in [A]\binom{m}{k}$ it follows that $\Phi(f \cdot g) = (\Phi \cdot f) \cdot (\Phi \cdot g)$, where the left hand side uses composition of parameter words, the right hand side refers to ordinary composition of mappings.

Now, given $\Delta : \binom{\mathrm{n}}{k} \to \{0, \ldots, r-1\}$ define $\Delta_\Phi : [A]\binom{n}{k} \to \{0, \ldots, r-1\}$ by $\Delta_\Phi(f) = \Delta(\Phi \cdot f)$. Using this idea, the Graham–Rothschild theorem (in fact, already the case $G = \{e\}$) implies Ramsey's theorem:

4.1 Theorem *(*Ramsey (1930)*)*
Let k, m and r be positive integers. Then there exists $n = Ram(k, m, r)$ such that for every mapping $\Delta : \binom{\mathrm{n}}{k} \to \{0, \ldots, r-1\}$ there exists $f \in \binom{\mathrm{n}}{m}$ with $\Delta \rceil f \cdot \binom{\mathrm{m}}{k} =$ constant. □

Ramsey's theorem may be visualized using the picture of figure 1, where $\mathcal{B}(n)$, as before, denotes the lattice of subsets of an n–element set.

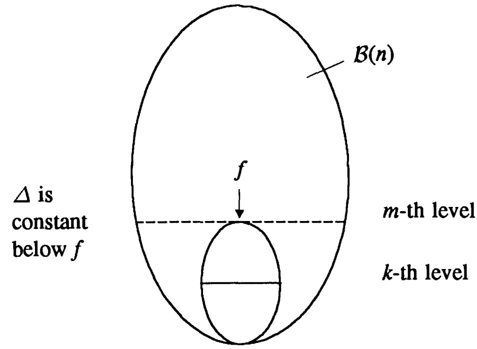

Fig. 1

4.2 The Dual Ramsey Theorem

Ramsey's theorem deals with strictly ascending *(rigid)* injections, i.e., subsets. The *dual Ramsey theorem* deals with *rigid surjections,* i.e., partitions. Consider $\Pi(n)$, the lattice of equivalence relations (partitions) on an n–element set, say $\{0,\ldots,n-1\}$. Let $\Pi\binom{n}{k}$ denote all those equivalence relations on $\{0,\ldots,$ $n-1\}$ which have precisely k classes. Recall that these equivalence relations are represented by parameter words (rigid surjections) $f \in [\emptyset]\binom{n}{k}$ and vice versa (cf. 2.1). Hence, composition is well–defined. Using this interpretation we derive immediately from the Graham–Rothschild theorem:

4.2 Corollary *(Dual Ramsey theorem)*
Let k,m and r be positive integers. Then there exists $n = DR(k,m,r)$ such that for every mapping $\Delta : \Pi\binom{n}{k} \to \{0,\ldots,r-1\}$ there exists a monochromatic $f \in \Pi\binom{n}{m}$. □

The dual Ramsey theorem can be visualized using figure 2:

4.3 Finite Unions, Finite Sums

The particular case $A = \{0\}$ and $k = 1$ of the Graham–Rothschild theorem can be stated as follows (cf. 2.2).

4.3 Corollary *(Finite union theorem)*
Let m and r be positive integers. Then there exists $n = FU(m,r)$ such that for every mapping $\Delta : \mathcal{B}(n) \to \{0,\ldots,r-1\}$ there exist m mutually disjoint and nonempty subsets $A_0,\ldots,A_{m-1} \in \mathcal{B}(n)$ such that for all nonempty $I,J \subseteq \{0,\ldots,m-1\}$ it follows that $\Delta(\bigcup_{i\in I}A_i) = \Delta(\bigcup_{j\in J}A_j)$. □

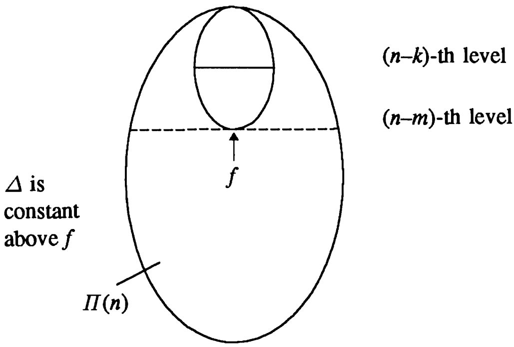

Fig. 2

Considering subsets as binary expansions of positive integers, one gets the following result, known as Rado–Folkman–Sanders' theorem (cf. Graham, Rothschild, Spencer (1980)).

4.3a Corollary
Let m and r be positive integers. Then there exists $n = FS(m,r)$ such that for every mapping $\Delta : \{1,\ldots,n\} \to \{0,\ldots,r-1\}$ there exist mutually distinct integers $a_0,\ldots a_{m-1}$ such that for all nonempty $I, J \subseteq \{0,\ldots,m-1\}$ it follows that $\Delta(\sum_{i\in I} a_i) = \Delta(\sum_{j\in J} a_j)$. □

4.4 A Partition Theorem for Finite Graphs

Recently, Nešetřil and Rödl (1985) observed that the Graham–Rothschild theorem can also be used to prove a partition theorem for finite graphs, which originally is due independently to Deuber (1975) and Nešetřil and Rödl (1975).

Recall that a finite graph G is a pair (V,E) , where V is a finite (not necessarily ordered) set and E is a subset of $\mathcal{P}_2(V)$, the 2–element subsets of V . Thereby V is the set of vertices and E is the set of edges of G . Graphs G and G' are isomorphic if there exists a bijection between V and V' such that images and preimages of edges are edges again. $G' = (V',E')$ is an induced subgraph of some graph $H = (W,F)$ iff $V' \subseteq W$ and $E' = F \cap \mathcal{P}_2(V')$. If G' is an induced subgraph of H and G' is isomorphic to G we say that G' is an induced G–subgraph of H .

4.4 Theorem (Deuber (1975), Nešetřil, Rödl (1975))
Let k,r be positive integers and G be a finite graph. Then there exists a finite graph H such that for every partition of the k–cliques (i.e., the complete subgraphs on k vertices) into r classes there exists an induced G–subgraph G' of H with all its k–cliques in the same class.

Proof. (Nešetřil, Rödl (1985))
Define a graph on $\mathcal{B}(n)$ joining $X, Y \in \mathcal{B}(n)$ by an edge iff $X \cap Y = \emptyset$. Observe that every finite graph G can be embedded into some $\mathcal{B}(n)$ provided with this structure. Now apply the Graham–Rothschild theorem with $A = \{0\}$. □

Next we discuss some extensions of the Graham–Rothschild theorem:

4.5 A ∗–Version

First we consider a partition theorem for k–parameter words of length at most n , i.e., for k–parameter ∗–words. This extends the Graham–Rothschild theorem in the same way as Theorem 3.3 extends Hales–Jewett's theorem.

4.5 Theorem *(*Deuber, Voigt (1982)*)*
Let A, k, m and r be given. Then there exists $n = GR^*(|A|, k, m, r)$ such that for every mapping $\Delta : [A]^*\binom{n}{k} \to \{0, \ldots, r-1\}$ there exists a monochromatic $f \in [A]^*\binom{n}{m}$, i.e. $\Delta(f \cdot g) = \Delta(f \cdot h)$ for all $g, h \in [A]^*\binom{m}{k}$. □

4.6 An Induced Version

Here a higher dimensional analogue to theorem 3.4 is considered. Let $[A](n) = \bigcup_{k \leq n} [A]\binom{n}{k}$ and let $\Gamma : [A](n) \to \{0, 1\}$ be a structural mapping, i.e., all parameter sets in A^n are split into two classes. One can view this as a (hyper–) graph imposed on the parameter sets in A^n .

The following partition theorem shows that it is possible to respect the additionally imposed structure:

4.6 Theorem *(*Prömel (1985)*)*
Let A, k, m, r and a structure mapping $\Gamma : [A](m) \to \{0, 1\}$ be given. Then there exists $n = GR_{ind}(|A|, m, r)$ and there exists a structure mapping $\Gamma^* : [A](n) \to \{0, 1\}$ such that for every mapping $\Delta : [A]\binom{n}{k} \to \{0, \ldots, r-1\}$ there exists $f \in [A]\binom{n}{m}$ which is induced, i.e., $\Gamma^*(f \cdot g) = \Gamma(g)$ for every $g \in [A](m)$, and which is monochromatic, i.e., $\Delta(f \cdot g) = \Delta(f \cdot h)$ for all $g, h \in [A]\binom{m}{k}$ with $\Gamma(g \cdot g') = \Gamma(h \cdot g')$ for all $g' \in [A]$. □

We should note that the applications and corollaries of the Graham–Rothschild theorem described above become induced versions using Theorem 4.6 instead of the original Graham–Rothschild theorem. Let us mention one consequence of the induced Graham–Rothschild theorem in more detail.

A finite ordered hypergraph G is a pair $(V, \mathcal{E})$ where V is a finite ordered set, the vertex–set of G , and $\mathcal{E}$, the set of edges of G , is a subset of $\mathcal{P}(V) \backslash V$. Subhypergraph means induced subhypergraph. G is said to be isomorphic to a hypergraph G' if there exists an orderpreserving bijection between G and G' such that images and preimages of edges are edges again. If, additionally, G' is a subhypergraph of a hypergraph $\mathcal{H}$ then G' is a G–subhypergraph of $\mathcal{H}$. Partition theory for finite graphs and hypergraphs culminates in the following theorem due to Nešetřil and Rödl (1977, 1983) and, independently, Abramson and Harrington (1978).

4.6a Theorem (Nešetřil, Rödl (1977, 1983), Abramson, Harrington (1978))
Let k be a positive integer an G be a finite ordered hypergraph. Then there exists a finite ordered hypergraph $\mathcal{H}$ such that for every partition of all subhypergraphs of $\mathcal{H}$ on k vertices into r classes there exists a G–subhypergraph G' of $\mathcal{H}$ such that any two subhypergraphs of G' which are isomorphic and have k vertices are in the same class.

Proof. Assume the atoms of $([2]\binom{m}{0}, \leq)$ to be the vertices of G. Define $\Gamma \cdot [2](m) \to \{0,1\}$ by $\Gamma(g) = 1$ iff $g \cdot [2]\binom{k}{0} \cap V(G) \in \mathcal{E}(G)$, where $g \in [2]\binom{m}{k}$. Hence $(2^m, \Gamma)$ bears the structure of G. Apply the induced Graham–Rothschild theorem and get n and $\Gamma^* : [2](n) \to \{0,1\}$. Define $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ by $V(\mathcal{H}) = [2]\binom{n}{0}$ and $[2]\binom{n}{0} \supseteq e \in \mathcal{E}(\mathcal{H})$ if and only if e is the set of atoms of a Boolean sublattice $L \subseteq ([2]\binom{n}{0}, \leq)$ and $\Gamma^*(L) = 1$. A straightforward calculation shows that $\mathcal{H}$ is the desired hypergraph. □

Compare also Prömel, Voigt (1989) for another proof relying on a direct amalgamation procedure.

4.7 A Sparse Version

Let $S \subseteq [A]\binom{n}{0}$ be a set of points in A^n and let $k \leq m$ be positive integers. Extending the notation from section 3.7 we let $\mathcal{H}_{k,m}(S)$ be the hypergraph whose vertices are $[A]\binom{n}{k}$ and whose edges are $f \cdot [A]\binom{n}{k}$ for $f \in \mathcal{H}_m(S)$. In other words, every m-parameter set f which is spanned by S in the sense that $f \cdot [A]\binom{m}{0} \subseteq S$ gives rise to an edge, and this edge consists of all k-parameter subsets of $f \cdot [A]\binom{m}{0}$. The sparse Graham-Rothschild theorem now says the following:

4.7 Theorem (Prömel, Voigt (1988))
Let A be a finite set with $|A| \geq 2$ and let k, m, r and g be positive integers. Then there exists a positive integer $n = GR_{sparse}(A, k, m, r, g)$ and there exists a set $S \subseteq [A]\binom{n}{0}$ such that $\chi(\mathcal{H}_{k,m}(S)) > r$ and girth $(\mathcal{H}_{k,m}(S)) > g$.

A similar result may be established with respect to one-element sets $A = \{0\}$. In view of the interpretation given in section 4.3 this implies a sparse finite union theorem, viz.,

4.7a Theorem (Prömel, Voigt (1988))
Let m, r and g be positive integers. Then there exists a family $\mathcal{S}$ of nonempty and finite sets such that $\chi(\{\mathcal{S}\}_m) > r$ and girth $\{\mathcal{S}\}_m > g$, where $\{\mathcal{S}\} <_m$ is the hypergraph having $\mathcal{S}$ as vertices and edges $\{\bigcup_{i \in I} S_i \mid I \subseteq m,\ I \neq \emptyset\}$ for each choice of pairwise disjoint sets $S_0, ldots, S_{m-1} \in \mathcal{S}$ such that also all unions $\bigcup_{i \in I} S_i$ still belong to $\mathcal{S}$.

For $g \geq 2$ it particularly follows that $\mathcal{S}$ does not contain $(m+1)$ mutually disjoint sets together with all their unions. This restricted result has been proved by Nešetřil and Rödl (1986).

For more general restricted and sparse induced versions of the Graham-Rothschild theorem compare Frankl, Graham, Rödl (1987) and Prömel, Voigt (1988).

4.8 A Canonizing Version

Next we consider an extension of the canonizing version of Hales–Jewett's theorem to higher dimensions. First we describe the canonical equivalence relations of k–parameter words.

Let $\zeta \subseteq \{0, \ldots, k-1\}$ be any subset and let $|\zeta|$ denote its cardinality. We view ζ as a strictly increasing mapping $\zeta : \{0, \ldots, |\zeta|-1\} \to \{0, \ldots, k-1\}$. Thus ζ is identified with its enumerating function. Moreover, we put $\zeta(|\zeta|) = k$. Consider a family of equivalence relations $\{\approx_i\}_{i \leq |\zeta|}$, where $\approx_i$ is defined on $A \cup \{\lambda_\mu \mid \mu < \zeta(i)\}$ and satisfies (1) $a \approx_i b$ implies $a \approx_{i+1} b$ for all $a, b \in A \cup \{\lambda_\mu \mid \mu < \zeta(i)\}$ and (2) if $a \approx_{i+1} b$ implies $a \approx_i b$ for all $a, b \in A \cup \{\lambda_\mu \mid \mu < \zeta(i)\}$ then there exists $c \in A \cup \{\lambda_\mu \mid \mu < \zeta(i)\}$ such that $c \approx_{i+1} \lambda_{\zeta(i)}$. Such a pair $(\zeta, \{\approx_i\}_{i \leq |\zeta|})$ is called a *canonical* (A,k)*–pair.*

We associate to every canonical (A,k)–pair $\Pi = (\zeta, \{\approx_i\}_{i \leq |\zeta|})$ an equivalence relation $\approx_\Pi$ on $[A]\binom{n}{k}$ by putting $g \approx_\Pi h$ iff for every $i \leq |\zeta|$ we have that $g(\mu) \approx_i h(\mu)$ for every $\mu < \min g^{-1}(\lambda_{\zeta(i)})$. Equivalence relations which are defined from canonical (A,k)–pairs in this way are called *canonical* (A,k)*–equivalence relations.*

Note that the definition of a canonical (A,k)–equivalence relation does not depend on the dimension of the parameter words where it is imposed on. Moreover, as shown in Prömel, Voigt (1983), these equivalence relations are hereditary in the sense that if $[A]\binom{n}{k}$ is endowed with a certain canonical (A,k)–equivalence relation, then the restriction to an arbitrary $f \in [A]\binom{n}{m}$ yields the same canonical (A,k)–equivalence relation.

The following theorem can be viewed as a completeness theorem of canonical (A,k)–equivalence relations:

4.8 Theorem *(*Prömel, Voigt (1983), cf. also (1986)*)*
Let A, k and m be given. Then there exists $n = PV(|A|, k, m)$ such that for every mapping $\Delta : [A]\binom{n}{k} \to \mathbb{N}$ there exist a canonical (A,k)–pair Π and an $f \in [A]\binom{n}{m}$ such that for all $g, h \in [A]\binom{m}{k}$ we have $\Delta(f \cdot g) = \Delta(f \cdot h)$ if and only if $f \cdot g \approx_\Pi f \cdot h$. □

Note that the case of canonical $(A,0)$–pairs corresponds to the canonical Hales–Jewett theorem (cf. 3.8).

Finally, we consider two particular instances of theorem 4.7: Let $n = PV(0,k,m)$ and $\Delta : \binom{n}{k} \to \mathbb{N}$ be a mapping. Define $\Delta' : [A]\binom{n}{k} \to \mathbb{N}$ by $\Delta'(g) = \Delta(\Phi g)$. Then there exist a canonical $(0,k)$–pair $\Pi = (\zeta, (\approx_i)_{i \leq |\zeta|})$ and an $f' \in [A]\binom{n}{m}$ according to theorem 4.7. Let $f \in \binom{n}{m}$ be given by

$f(i) = \min f'^{-1}(\lambda_i)$ for $i = 0, \ldots, m-1$. Then for all $g, h \in \binom{m}{k}$ we have

$$\Delta(f \cdot g) = \Delta(f \cdot h) \text{ if and only if } (f \cdot g)(\zeta(i)) \text{ for all } i < |\zeta| .$$

Hence we have proved the Erdös–Rado canonization theorem for finite sets:

4.8a Theorem (Erdös–Rado (1950))
Let k and m be given. Then there exists $n = ER(k,m)$ such that for every mapping $\Delta : \binom{n}{k} \to \mathbb{N}$ there exist a subset $\zeta \subseteq \{0, \ldots, k-1\}$ and an $f \in \binom{n}{m}$ such that for all $g, h \in \binom{m}{k}$ we have $\Delta(f \cdot g) = \Delta(f \cdot h)$ if and only if $g \cdot \zeta = h \cdot \zeta$. □

In 4.3 we observed that the case $A = \{0\}$ and $k = 1$ of the Graham–Rothschild theorem corresponds to the finite union theorem (to the Rado–Folkman–Sanders theorem, resp.) From theorem 4.8 we deduce canonizing versions of these results. The point is to observe that there exist precisely three $(\{0\}, 1)$–canonical equivalence relations:

4.8b Theorem (Prömel, Voigt (1983))
Let m be a positive integer and let $n = PV(1,1,m)$. Then for every mapping $\Delta : \mathcal{B}(n) \to \mathbb{N}$ there exist m mutually disjoint and nonempty subsets $A_0, \ldots, A_{m-1} \in \mathcal{B}(n)$ such that one of the following three cases is valid for all nonempty $I, J \subseteq \{0, \ldots, m-1\}$:
(1) $\Delta(\bigcup_{i \in I} A_i) = \Delta(\bigcup_{j \in J} A_j)$
(2) $\Delta(\bigcup_{i \in I} A_i) = \Delta(\bigcup_{j \in J} A_j)$ *iff* $\min I = \min J$
(3) $\Delta(\bigcup_{i \in I} A_i) = \Delta(\bigcup_{j \in J} A_j)$ *iff* $I = J$. □

For sake of completeness we also state the canonizing sum theorem explicitely:

4.8c Theorem (Prömel, Voigt (1983))
Let m be a positive integer and let $n = PV(1,1,m)$. For every mapping $\Delta : \{0, \ldots, 2^n - 1\} \to \mathbb{N}$ there exist mutually distinct integers $a_0, \ldots, a_{m-1}$ such that one of the following three cases is valid for all nonempty $I, J \subseteq \{0, \ldots, m-1\}$:
(1) $\Delta(\sum_{i \in I} a_i) = \Delta(\sum_{j \in J} a_j)$
(2) $\Delta(\sum_{i \in I} a_i) = \Delta(\sum_{j \in J} a_j)$ *iff* $\min I = \min J$
(3) $\Delta(\sum_{i \in I} a_i) = \Delta(\sum_{j \in J} a_j)$ *iff* $I = J$. □

4.9 Some Open Problems

It is an open question whether, in general, a restricted canonizing version of the Graham–Rothschild theorem holds true. In particular, coming back again to arithmetic progressions, we find the following conjecture very attractive:

Conjecture.
Let k and g be positive integers. Then there exists a set S of positive integers such that girth $(AP_{S,k}) > g$ but still for every coloring $\Delta : \delta \longrightarrow \omega$ there exists a progression $A \in AP_{S,k}$ such that $\Delta \rceil A$ either is a constant or a one-to-one coloring.

Another open problem is to determine the canonical total orders on k-parameter sets for $k > 0$, thus to prove a higher dimensional analogue to theorem 3.9. Canonical total orders on k-element sets are investigated in Leeb, Prömel (preprint 1983). Using the functor $\Phi : [A]\binom{n}{k} \to \binom{n}{k}$ we get from this some canonical orders for $[A]\binom{n}{k}$. Some other we obtain from the canonizing version of Hales–Jewett's theorem (considering the parameters as letters). But a complete characterization of the canonical total orders on k-parameter sets is not yet known. Recall that theorem 4.7 is not a canonizing version of the full Graham–Rothschild theorem but only of its special case $G = \{e\}$. It is not known how a canonizing version looks like if we do not restrict ourselves to the trivial group. There seems to be a difference in the canonizing pattern depending on whether G is an abelian group or a non–abelian one.

Every theorem is accompanied by a Ramsey–type function, e.g., the Graham–Rothschild theorem by $GR(.,.,.,)$. Almost nothing is known about the (minimal) growthrate of the functions. Taylor (1981) gives an upper bound for $FU(.,.)$. Compare also Graham, Rothschild, Spencer 1980.

5. Infinite Versions

We consider the question to what extend results from previous sections admit infinite versions, like, typically, Ramsey's theorem does. It turns out that, in general, the answer depends on the set theoretic axioms.

If the set of possible monochromatic configurations has the cardinality of the continuum and also the number of colored (partitioned) objects is that large, straightforward diagonalization methods, involving the axiom of choice, produce bad colorings without any monochromatic configuration. However, positive results can be established restricting to colorings which are defined without using the axiom of choice. This can be made precise using topological notions coming from descriptive set theory. The situation can be compared with, e.g., classical real analysis. The topic of real analysis is not to study all mappings $\mathbb{R} \to \mathbb{R}$ but rather those functions which are continuous, resp., can be derived from continuous functions in a natural way. We should mention here René Baire's thesis 'Sur le fonctions de variables réelles' (Baire (1899)) the hierarchy of Borel–measurable functions. It turns out that for our purposes the concept of Borel–measurable mappings can be extended in two ways, using the classical notions of category and measure (as, e.g., nicely described in Oxtoby's book 'Measure and Category' 1971.

In some cases, however, it turns out that the number of colored (partitioned) objects is countable, i.e., small compared to the number of possible monochromatic objects. Ramsey's theorem itself is an example for this. In those cases positive results can be established without any set theoretic complication.

Before we discuss specific examples we straightforwardly extend the definitions of section 2 and introduce $[A,G]\binom{\omega}{\omega}$, resp. $[A,G]\binom{\omega}{k}$ for nonnegative integers $k < \omega$.

Definition. For an ordinal $k \le \omega$ we denote by $[A]\binom{\omega}{k}$ the set of all words (mappings) $f : \omega \to A \cup \{\lambda_i \mid i < k\}$ satisfying

for every $j < k$ there exists $i < \omega$ with $f(i) = \lambda_j$ and $\min f^{-1}(\lambda_i) < \min f^{-1}(\lambda_j)$ for all $i < j < \omega$.

Composition is defined as before. Then examples 2.1 up to 2.6 carry over to the infinite situation. $[A,G]\binom{\omega}{\omega}$ and $[A,G]\binom{\omega}{k}$ are defined analogously.

5.1 An Infinite ∗-Version

Compare 3.3 and 4.5. We define, for $k < \omega$,

$$[A]^*\binom{\omega}{k} = \{g\rceil \min\ g^{-1}(k) \mid g \in [A]\binom{\omega}{k+1}\} ,$$

where $g\rceil \min\ g^{-1}(k)$ denotes the restriction of g to $\{0,\ldots,\ \min\ g^{-1}(k) -1\}$; alternatively and in accordance with our earlier definitions we could set $(g\rceil \min\ g^{-1}(k))\ (i) = g(i)$ for $i < \min\ g^{-1}(k)$ and $(g\rceil \min\ g^{-1}(k))\ (i) = *$ otherwise. For $F \in [A]\binom{\omega}{\omega}$ and $g \in [A]^*\binom{\omega}{k}$ we put

$$F \cdot g = (F \cdot \tilde{g})\rceil \min\ (F \cdot \tilde{g})^{-1}(k) ,$$

where $\tilde{g} \in [A]\binom{\omega}{k+1}$ in such that $\tilde{g}\rceil \min \tilde{g}^{-1}(k) = g$. Recall that this also agrees with the definitions of sections 3.3, resp., 4.5.

5.1 Theorem *(Carlson–Simpson lemma,* Carlson, Simpson (1984), Voigt to appear*)*
For every mapping $\Delta : [A]^*\binom{\omega}{k} \to \{0,\ldots,r-1\}$, where r and k are nonnegative integers, there exists $F \in [A]\binom{\omega}{\omega}$ such that $\Delta(F \cdot g) = \Delta(F \cdot h)$ for all $g, h \in [A]^*\binom{\omega}{k}$. □

The case $k = 0$ is due independently to Carlson, Simpson (1984) and Voigt (to appear), $k > 0$ is from Voigt (to appear). Like the finite ∗–version of Hales–Jewett's theorem, viz., theorem 3.3, the particular case $k = 0$ of theorem 5.1, also known as the Carlson–Simpson lemma, proves to be a convenient tool for establishing further partition theorems. Though slightly weaker, this particular case is closely related to the famous Halpern–Läuchli theorem (Halpern, Läuchli (1966)). Several applications and extensions of the Halpern–Läuchli theorem actually need the Carlson–Simpson lemma only, e.g., (Blass (1981)). Note that the proof of the Carlson–Simpson lemma is somewhat more transparent than those known for the full Halpern–Läuchli theorem.

5.2 Infinite A-Sequences

Next we consider 0–parameter sets in infinite dimensional cubes A^ω , resp., in our present notation $[A]\binom{\omega}{0}$. Note that $|[A]\binom{\omega}{0}| = |[A]\binom{\omega}{\omega}| = |2^\omega|$. Hence, as mentioned above, straightforward diagonalization techniques show that there exist mappings $\Delta : [A]\binom{\omega}{0} \to \{0,1\}$ such that for all $F \in [A]\binom{\omega}{\omega}$ there exist

$g, h \in [A]\binom{\omega}{0}$ with $\Delta(F \cdot g) \neq \Delta(F \cdot h)$. Of course, A should have at least two elements, otherwise everything would be trivial. However, these constructions rely on the axiom of choice. This can be made precise restricting to constructively given mappings. Basically, there exist two approaches, those of category and those of measure. Using a completely different language (and not being aware of Hales–Jewett's theorem, resp., the Graham–Rothschild generalization there of) these two approaches were suggested by Moran and Strauss (1980).

5.2.1 The Category Approach to A^ω

Let us view A^ω as a topological space, viz., the Tychonoff product with A being discrete. Thus A^ω is a metric space with $d((f_0, f_1, \ldots), (g_0, g_1, \ldots)) = \frac{1}{n+1}$ if the sequences f and g differ for the first time at position n. Recall that *meager* sets are countable unions of nowhere dense sets and that a set $\mathcal{B}$ is a *Baire set* iff there exists a meager set $\mathcal{M}$ such that the symmetric difference $(\mathcal{B}/\mathcal{M}) \cup (\mathcal{M}/\mathcal{B})$ is open (Baire sets are open modulo meager sets). The axiom of choice implies that there exist sets $\mathcal{X} \subseteq A^\omega$ which are not Baire, but as Shelah (1984) shows, it is consistent with ZF to assume that every set $\mathcal{X} \subseteq A^\omega$ is Baire.

Moran and Strauss (1980) investigated Baire mappings $\Delta : A^\omega \to \omega$.

Notation. $(A)^\omega_\omega \subseteq [A]\binom{\omega}{\omega}$ is the set of m–parameter words $F : A \to A \cup \{\lambda_i \mid i < \omega\}$ such that $|F^{-1}(\lambda_i)| = 1$ for every $i < \omega\}$, i.e., every parameter occurs exactly once.

Note that in the finite dimensional case we cannot hope to find a monochromatic parameter word where each parameter occurs exactly once (e.g, that would yield monochromatic arithmetic progressions with difference 1). However, with respect to infinite dimensional cubes the situation is somewhat different:

5.2.1 Theorem *(*Moran–Strauss (1980)*)*
Let A be a finite set. For every Baire mapping $\Delta : [A]\binom{\omega}{0} \to \omega$, i.e., $\Delta^{-1}(i)$ is a Baire *set for all $i < \omega$, there exists $F \in (A)^\omega_\omega$ such that for all $g, h \in [A]\binom{\omega}{0}$: $\Delta(F \cdot g) = \Delta(F \cdot h)$.* □

This is proved using a Baire category argument. As a matter of fact, the Moran–Strauss result is somewhat stronger than stated above.

Theorem 5.2.1 cannot be extended to infinite sets A, as there exist Borel mappings $\Delta : [\omega]\binom{\omega}{0} \to \{0, 1\}$ such that for all $F \in [\omega]\binom{\omega}{\omega}$ there exist $g, h \in [\omega]\binom{\omega}{0}$ with $\Delta(F \cdot g) \neq \Delta(F \cdot h)$ (see Moran, Strauss 1980).

5.2.2 The Measure Approach to A^ω

Let μ be the completion of the product probability measure on A^ω which is generated by equal distribution on A. We consider measurable mappings. Here Moran and Strauss (1980) prove a density result:

5.2.2 Theorem *(*Moran–Strauss (1980)*)*
Let A be a finite set and let $\mathcal{M} \subseteq A^\omega$ be a set of positive measure. Then there exists $F \in (A)^\omega_\omega$ such that $F \cdot [A]\binom{\omega}{0} \subseteq \mathcal{M}$. □

5.2.3 Canonizing Results for A^ω

We consider equivalence relations on A^ω which are topologically nice and ask about the 'typical' structures, i.e., what are the canonizing patterns? There exist two different concepts:

(a) *Baire mappings into metric spaces*

Let $\mathcal{X}$ be a metric space. A mapping $\Delta : A^\omega \to \mathcal{X}$ is a *Baire mapping* if preimages of open sets in $\mathcal{X}$ are Baire in A^ω .

5.2.3a Theorem *(*Prömel, Simpson, Voigt (1986)*)*
Let $A = 2$ be a two-element set, let $\mathcal{X}$ be a metric space and let $\Delta : 2^\omega \to \mathcal{X}$ be a Baire mapping. Then there exists $F \in (2)^\omega_\omega$ and there exists an equivalence relation $\equiv$ on 2 such that for all $g = (g_0, g_1, \ldots)$, $h = (h_0, h_1, \ldots) \in 2^\omega$ it follows that $\Delta(F \cdot g) = \Delta(F \cdot h)$ if and only if $g_i \equiv h_i$ for all $i < \omega$. □

Remark. These kind of results are related to recursion theoretic investigations of Lachlan (1971) and Thomason (1970). The particular result for continuous mappings is due to some unpublished work of Silver from about 1960. It has been shown in Lefman, Voigt (to appear) that Theorem 5.2.3a does not extend to sets A with at least three elements. For sets A with $|A| \geq 3$ one should weaken the requirements on F, allowing *ascending* parameter words.

Note that some equivalence relations on A^ω which are easily described do not occur as fibres of Baire mappings into metric spaces, e.g., consider $f \approx g$ if and only if $\{i < \omega \mid f_i \neq g_i\}$ is finite.

Moreover, this equivalence relation cannot be canonized by some $F \in (A)^\omega_\omega$. Still, this example is covered by the second concept.

(b) *Baire partitions*

An equivalence relation $\mathcal{R}$ on A^ω can be visualized as a subset of $A^\omega \times A^\omega$. We call $\mathcal{R}$ a *Baire partition* iff $\mathcal{R} \subseteq A^\omega \times A^\omega$ is a Baire set with respect to product topology. In particular, every Borel set and every analytic set is restricted Baire.

5.2.3b Theorem *(*Prömel, Simpson, Voigt (1986), Prömel, Voigt to appear*)*
Let A be a finite set and let $\mathcal{R} \subseteq A^\omega \times A^\omega$ be a Baire partition. Then there exists $F \in [A]\binom{\omega}{\omega}$ and there exists an equivalence relation $\equiv$ on A such that for all $g = (g_0, g_1, \ldots)$, $h = (h_0, h_1, \ldots) \in A^{\leq}$ it follows that $(F \cdot g, F \cdot h) \in \mathcal{R}$ if and only if $g_i \equiv h_i$, for all $i < \omega$. □

Note that the assertion is weaker than in the previous theorem, as we can only claim that $F \in [A]\binom{\omega}{\omega}$.

Nothing is known with respect to measurable mappings $\Delta : [A]\binom{\omega}{0} \longrightarrow \mathcal{X}$, resp., with respect to measurable partitions in the sense of (b).

5.2.4 Canonical Orders for A^ω

In principle one should expect the same orders as in theorem 3.9. But there is still a gap in our knowledge. With respect to the notions of category only the case $|A| = 2$ is completely settled. We state this particular result using the interpretation 2.3:

5.2.4 Theorem (Prömel, Simpson, Voigt (1986))
Let $\leq$ be a Baire order on 2^ω (the power set lattice of ω), i.e., viewed as a subset of $2^\omega \times 2^\omega$ the order has the property of Baire. Then there exists a 2^ω–sublattice $\mathcal{L} \subseteq 2^\omega$ such that $\leq\rceil\mathcal{L}$ is a lexicographic order, either coming from $0 < 1$ or coming from $1 < 0$. □

With respect to $|A| > 2$ the same canonical orders as described in theorem 3.9 occur, but unfortunately we can prove this only for restricted Baire orders, although the result should be true for Baire orders in general. Also, the case of measurable orders on A^ω is still untouched.

5.3 Partitions of $[A]\binom{\omega}{k}$ for $0 < k < \omega$.

Again, except for $|A| = 0$ and $k = 1$, we have $\left|[A]\binom{\omega}{k}\right| = \left|[A]\binom{\omega}{\omega}\right| = |\mathbf{R}|$ and so we have to restrict to the concepts of category and measure. As $[A]\binom{\omega}{k} \subseteq (A \cup \{\lambda_0, \dots, \lambda_{k-1}\})^\omega$ and, moreover, this is an open non–null set, we take the induced topology, resp. measure.

Compare Prömel, Voigt (to appear) for results concerning the concept of measure. With respect to the concept of category the following is known:

5.3.1 Theorem (Prömel, Voigt (1985))
For every Baire mapping $\Delta : [A]\binom{\omega}{k} \to \{0, \dots, r-1\}$ there exists $F \in [A]\binom{\omega}{\omega}$ such that $\Delta(F \cdot g) = \Delta(F \cdot h)$ for all $g, h \in [A]\binom{\omega}{k}$. □

Remark. For Borel–mappings this has been established by Carlson and Simpson (1984). Let us mention the case $|A| = k = 1$ explicitly, using the interpretation 2.2 (cf. also 4.3).

5.3.2 Corollary (Prömel, Voigt (1985))
Let 2^ω be the powerset lattice of ω , topologically this is Cantor's discontinuum. For every Baire–mapping $\Delta : 2^\omega \to \{0, \dots, r-1\}$, where r is a nonnegative integer, there exist mutually disjoint and nonempty subsets $A_i \in 2^\omega$, $i < \omega$, such that $\Delta(\bigcup_{i \in I} A_i) = \Delta(A_0)$ for all nonempty subsets $I \subseteq \omega$. □

This is closely related to a question of Erdös (1975) who asked whether there exists a cardinal κ such that for every partition of the nonempty subsets of κ into finitely many classes there exist countably many mutually disjoint and nonempty subsets A_i, $i < \omega$, such that all their unions belong to the same class. The expected answer is negative and presently the best result is due to Galvin, Prikry and Wolfsdorf (1984) who showed that the answer is no for all $\kappa \leq \aleph_\omega$. In view of Shelah's result 'Consistency of ZF' implies 'Consistency of ZF plus every set $X \subseteq 2^\omega$ has the Baire property' (Shelah (1984)), we see that it is consistent with ZF , but contradictory to ZFC , to assume that already $\kappa = \omega$ admits an affirmative answer.

A crucial observation for proving theorem 5.3.1 is that for every meager set $\mathcal{M} \subseteq [A]\binom{\omega}{k}$ there exists $F \in [A]\binom{\omega}{\omega}$ such that $F \cdot g \notin \mathcal{M}$ for every $g \in [A]\binom{\omega}{k}$, in other words, meager sets are Ramsey null.

As every Baire mapping $\Delta : [A]\binom{\omega}{k} \to \mathcal{X}$, where $\mathcal{X}$ is a metric space, is continuous apart from a meager set (this follows, e.g., from (Emeryk, Frankiewicz, Kulpa (1979)),every Baire mapping $\Delta : [A]\binom{\omega}{k} \to \mathcal{X}$ is continuous on some $F \in [A]\binom{\omega}{\omega}$, i.e.

Lemma *(*Prömel, Voigt (1985)*)*.
For every Baire mapping $\Delta : [A]\binom{\omega}{k} \to \mathcal{X}$, where $\mathcal{X}$ is a metric space, there exists $F \in [A]\binom{\omega}{\omega}$ such that the restriction $\Delta \rceil F \cdot [A]\binom{\omega}{k}$ is continuous. □

But even more can be said, as the structure of a 'typical' continuous mapping can be specified further. Basically, it is determined by some (A,k)–canonical equivalence relation (cf. section 4.7). This yields the canonizing theorem for $[A]\binom{\omega}{k}$:

5.3.3 Theorem *(*Prömel, Simpson, Voigt (1986)*)*
For every Baire mapping $\Delta : [A]\binom{\omega}{k} \to \mathcal{X}$, where $\mathcal{X}$ is a metric space, there exists a (A,k)–canonical pair Π and there exists $F \in [A]\binom{\omega}{\omega}$ such that for all $g, h \in [A]\binom{\omega}{k}$ it follows that

$$\Delta(F \cdot g) = \Delta(F \cdot h) \text{ if and only if } g \approx_\Pi h \, . \qquad \square$$

5.4 Partitions of $[A]\binom{\omega}{\omega}$

The situation is slightly more delicate than before. It turns out that still the concepts of category and measure are too general. For example,

5.4.1 Fact

There exists a meager set $\mathcal{M} \subseteq [A]\binom{\omega}{\omega}$ (with respect to the Tychonoff product topology on $[A]\binom{\omega}{\omega}$) such that $F \cdot [A]\binom{\omega}{\omega} \cap \mathcal{M} \neq \emptyset$ for every $F \in [A]\binom{\omega}{\omega}$.

Proof. One easily observes that for $v \in \{0,1\}$ the set $\mathcal{M}_v = \{F \in [A]\binom{\omega}{\omega} \mid \min F^{-1}(\lambda_i) \equiv v \pmod 2$ for every $i < \omega\}$ is meager, but $F \cdot [A]\binom{\omega}{\omega} \cap (\mathcal{M}_0 \cup \mathcal{M}_1) \neq \emptyset$ for every $F \in [A]\binom{\omega}{\omega}$. As every subset of a meager set is meager, using the axiom of choice, an $\mathcal{M} \subseteq \mathcal{M}_0 \cup \mathcal{M}_1$ can be easily selected. □

In some sense, the defect with the Tychonoff product topology is that although cosets $F \cdot [A]\binom{\omega}{\omega}$ are homeomorphic to $[A]\binom{\omega}{\omega}$, the natural subspace embedding $\overline{F} : [A]\binom{\omega}{\omega} \to [A]\binom{\omega}{\omega}$ with $\overline{F}(G) = F \cdot G$ generally is not a homeomorphism. $\overline{F}$ is always continuous, but, in general, it is not open.

This leads to refine the Tychonoff product topology such that all subspace embeddings $\overline{F}$ become open mappings.

In connection with the structure $\binom{\omega}{\omega}$ of infinite subsets of ω this approach originally is due to Ellentuck (1974). For this reason we call the resulting topology the *Ellentuck topology*. (Note that on $\binom{\omega}{\omega}$ the Ellentuck topology agrees

with the so called exponential or Vietoris topology. In our present context this seems to be a casual incidence, but compare Prikry's suggestion (–Prikry (1984)).
Equivalently, the Ellentuck topology can be understood in the following way: The Tychonoff cones $\mathcal{T}(H,n) := \{G \in [A]\binom{\omega}{\omega} \mid G(i) = H(i) \text{ for all } i < n\}$, where $H \in [A]\binom{\omega}{\omega}$ and $n < \omega$, form a system of basic open neighborhoods for the Tychonoff product topology. A basic system for the Ellentuck topology then is given by the sets $F \cdot \mathcal{T}(H,n)$, where $F, H \in [A]\binom{\omega}{\omega}$, and $n < \omega$.
Somewhat easier to visualize is our final description of the Ellentuck topology: for $F \in [A]\binom{\omega}{\omega}$ and $q < \omega$ let $\mathcal{E}(F,q) = \{F \cdot G \mid G \in [A]\binom{\omega}{\omega} \text{ and } G(i) = \lambda_i \text{ for all } i < q\}$ be the *Ellentuck neighborhood* determined by F and q. Then a basic system for the Ellentuck topology is given by the sets of all Ellentuck neighborhoods $\mathcal{E}(F,q)$, where $F \in [A]\binom{\omega}{\omega}$ and $q < \omega$ (with respect to the Tychonoff product topology on $[A]\binom{\omega}{\omega}$).

The Ellentuck topology is closely related to the partition problem:

5.4.2 Theorem *(Carlson, Simpson (1984))*
(1) A set $\mathcal{M} \subseteq [A]\binom{\omega}{\omega}$ is meager with respect to the Ellentuck topology iff every Ellentuck neighborhood $\mathcal{E}(F,q)$ contains some $G \in \mathcal{E}(F,q)$ with $\mathcal{M} \cap \mathcal{E}(G,q) = \emptyset$.
(2) A set $\mathcal{B} \subseteq [A]\binom{\omega}{\omega}$ has the property of Baire with respect to the Ellentuck topology iff every Ellentuck neighborhood $\mathcal{E}(F,q)$ contains some $G \in \mathcal{E}(F,q)$ with $\mathcal{E}(G,q) \subseteq \mathcal{B}$ or with $\mathcal{E}(G,q) \cap \mathcal{B} = \emptyset$. □

Such an Ellentuck type theorem implies results concerning the Tychonoff product topology. Recall that analytic sets are obtained from closed sets via Souslin's $\mathcal{A}$–operation. The property of Baire is preserved under this $\mathcal{A}$–operation. Hence, in particular, assertion (2) of theorem 5.4.2 is valid for sets which are analytic with respect to the Tychonoff product topology.

Incidentally, theorem 5.4.2 can be generalized somewhat considering the structure $[A,G]\binom{\omega}{\omega}$. The notion of the Ellentuck topology is defined as before.

5.4.3 Theorem *(Voigt to appear)*
(1) A set $\mathcal{M} \subseteq [A,G]\binom{\omega}{\omega}$ is meager with respect to the Ellentuck topology iff every Ellentuck neighborhood $\mathcal{E}(F,q)$ contains some G with $\mathcal{E}(G,q) \cap \mathcal{M} = \emptyset$.
(2) A set $\mathcal{B} \subseteq [A,G]\binom{\omega}{\omega}$ has the property of Baire with respect to the Ellentuck topology iff every Ellentuck neighborhood $\mathcal{E}(F,q)$ contains some G with $\mathcal{E}(G,q) \subseteq \mathcal{B}$ or with $\mathcal{E}(G,q) \cap \mathcal{B} = \emptyset$. □

Carlson considered the following variation:

Definition. Let $[A]^{<}\binom{\omega}{\omega} \subseteq [A]\binom{\omega}{\omega}$ consist of all mappings $F : \omega \to A \cup \{\lambda_i \mid i < \omega\}$ which act surjectively onto $\{\lambda_i \mid i < \omega\}$ and satisfy additionally that $\max F^{-1}(i) < \min F^{-1}(i+1)$ for all $i < \omega$.

Such parameter words $F \in [A]^{<}\binom{\omega}{\omega}$ are called *ascending ω-parameter words* of length ω over alphabet A. Here the parameters occur ascendingly in blocks and are not intertwined. Clearly, $[A]^{<}\binom{\omega}{\omega} \cdot [A]^{<}\binom{\omega}{\omega} = [A]^{<}\binom{\omega}{\omega}$, i.e., $[A]^{<}\binom{\omega}{\omega}$ is a subsemigroup of $[A]\binom{\omega}{\omega}$. Carlson observed that also this submonoid admits an Ellentuck type theorem:

5.4.4 Theorem *(*Carlson in preparation, cf. also Prikry (1982) and Voigt (to appear)*)*

(1) A set $\mathcal{M} \subseteq [A]^{<}\binom{\omega}{\omega}$ is meager with respect to the Ellentuck topology on $[A]^{<}\binom{\omega}{\omega}$ iff every Ellentuck neighborhood $\mathcal{E}(F,q)$, where $F \in [A]^{<}\binom{\omega}{\omega}$, contains some $G \in [A]^{<}\binom{\omega}{\omega}$ with $\mathcal{E}(G,q) \cap \mathcal{M} = \emptyset$.

(2) A set $\mathcal{B} \subseteq [A]^{<}\binom{\omega}{\omega}$ has the property of Baire with respect to the Ellentuck topology on $[A]^{<}\binom{\omega}{\omega}$ iff every Ellentuck neighborhood $\mathcal{E}(F,q)$, where $F \in [A]^{<}\binom{\omega}{\omega}$, contains some $G \in [A]^{<}\binom{\omega}{\omega}$ with $\mathcal{E}(G,q) \cap \mathcal{B} = \emptyset$ or $\mathcal{E}(G,q) \cap [A]^{<}\binom{\omega}{\omega} \subseteq \mathcal{B}$.

For $A = \emptyset$ this theorem reduces to the original Ellentuck theorem. For $|A| = 1$ this is a result of Milliken (1975). It turns out that the cases $|A| \geq 2$ are particularly more difficult to prove.

Let us say that a subsemigroup $\mathbf{S} \subseteq [A]\binom{\omega}{\omega}$ is *Ramsey* if it satisfies an Ellentuck type theorem. So, Carlson's theorem says that $[A]^{<}\binom{\omega}{\omega}$ is Ramsey.

Problem. Characterize those subsemigroups $\mathbf{S} \subseteq [A]\binom{\omega}{\omega}$ which are Ramsey.

6. Other Structures

In this final section we very briefly mention some results which are closely related to the concept of Graham–Rothschild parameter words.

6.1 Hindman's Theorem

Graham and Rothschild (1971) conjectured an infinite generalization of the Rado–Folkman–Sanders theorem (cf. 4.3).

Notation. A family $(A_i)_{i<\omega}$ of finite and nonempty subsets of ω is an *ascending family* if $\max A_i < \min A_{i+1}$ for all $i < \omega$. By $FU((A_i)_{i<\omega})$ we denote the set of all finite unions of the family $(A_i)_{i<\omega}$, i.e., $FU((A_i)_{i<\omega}) = \{\bigcup_{j\in S} A_j \mid S$ a finite and nonempty subset of $\omega\}$. E.g., if $A_i = \{i\}$ then $FU((\{i\})_{i<\omega})$ is just the set of finite and nonempty subsets of ω .

6.1.1 Theorem *(Hindman's finite union theorem ,* Hindman (1974)*)*
For every mapping $\Delta : FU((A_i)_{i<\omega}) \to \{0,\ldots,r-1\}$, where $(A_i)_{i<\omega}$ is an ascending family and r a positive integer, there exists an ascending family $(B_i)_{i<\omega}$ with each $B_i \in FU((A_i)_{i<\omega})$ such that $\Delta\rceil FU((B_i)_{i<\omega})$ is a constant mapping, i.e., all finite unions of the $(B_i)_{i<\omega}$ get the same color. □

Again, using binary expansions of integers, this can be formulated in terms of finite sums:

6.1.2 Theorem *(Hindman's finite sum theorem,* Hindman (1974)*)*
For every mapping $\Delta : \mathbb{N} \to \{0, \dots, r-1\}$, where r is a positive integer, there exist infinitely many numbers $a_0, a_1, a_2, \dots$ such that all finite sums $\sum_{i \in S} a_i$ get the same color. □

Several proofs have been given for this theorem, e.g., by Baumgartner (1974) using some kind of combinatorial forcing, by Glazer (see Hindman (1979)) using idempotent ultrafilters in $\beta\mathbb{N}$ and by Fürstenberg and Weiss (1978) using topological dynamics, compare also Fürstenberg's book (1981). Taylor (1976) considered a canonizing version of Hindman's theorem. For simplicity we only state the sum version:

6.1.3 Theorem *(*Taylor (1976)*)*
For every mapping $\Delta : \mathbb{N} \to \mathbb{N}$ there exist infinitely many numbers $a_0, a_1, a_2 \dots$ such that one of the following five cases holds for all finite and nonempty subsets $S, T \subseteq \mathbb{N}$:
(1) $\Delta(\sum_{i \in S} a_i) = \Delta(\sum_{i \in T} a_i)$
(2) $\Delta(\sum_{i \in S} a_i) = \Delta(\sum_{i \in T} a_i)$ *if and only if* $\min S = \min T$
(3) $\Delta(\sum_{i \in S} a_i) = \Delta(\sum_{i \in T} a_i)$ *if and only if* $\max S = \max T$
(4) $\Delta(\sum_{i \in S} a_i) = \Delta(\sum_{i \in T} a_i)$ *if and only if* $\min S = \min T$ *and* $\max S = \max T$
(5) $\Delta(\sum_{i \in S} a_i) = \Delta(\sum_{i \in T} a_i)$ *if and only if* $S = T$. □

Compare this with the canonizing version of the Rado–Folkman–Sanders theorem (viz., 4.7).

6.2 Ascending Parameter Words

We already introduced the notion of ascending parameter words at the end of section 5.4. Let us recall the definition

Definition.

$$[A]^<\binom{\omega}{k} := \{f \in [A]\binom{\omega}{k} \mid \max f^{-1}(\lambda_i) < \min f^{-1}(\lambda_{i+1})$$
$$\text{for all } i < k < -1 \text{ and } f^{-1}(\lambda_{k-1}) \text{ is finite}\}$$
$$[A]^<\binom{\omega}{\omega} := \{F \in [A]\binom{\omega}{\omega} \mid (F^{-1}(\lambda_i))_{i<\omega} \text{ is an ascending family}\} .$$

Composition is defined as before.

Hindman's theorem can be formulated using one–element alphabets and partitioning ascending one–parameter words (compare 4.3). This has been generalized by Milliken (1975) and, independently, by Taylor (1976) who also considered partitions of ascending k–parameter words:

6.2.1 Theorem *(Milliken 1975)*
Let $A = \{0\}$. For every mapping $\Delta : [\{0\}]^{<}\binom{\omega}{k} \to \{0,\ldots,r-1\}$, where k and r are positive integers, there exists a monochromatic $F \in [\{0\}]^{<}\binom{\omega}{\omega}$, i.e., $\Delta(F \cdot g) = \Delta(F \cdot h)$ for all $g, h \in [\{0\}]^{<}\binom{\omega}{k}$. □

Lefman (1985) considered a canonizing version of Milliken's result, thus extending Taylor's canonizing theorem 6.1.3. It turns out that all canonizing patterns on $[\{0\}]\binom{\omega}{k}$ can be obtained as follows: first take some $h \in [\{0\}]\binom{k}{j}$, where $j \leq k$; for a given $g \in [\{0\}]\binom{\omega}{k}$ this leaves us with $g \cdot h \in [\{0\}]\binom{\omega}{j}$; additionally, the patterns from Taylor's theorem can be applied to each $(g \cdot h)^{-1}(\lambda_i)$, $i < j$. Clearly, such patterns are hereditary and thus cannot be avoided. Lefman's result states that this is best possible.

If $[A] \geq 2$, again the axiom of choice prevents Ramsey type partition theorems. Using topological tools, as explained in section 5.3, positive results can be established:

6.2.2 Theorem *(Prömel, Voigt (1985))*
For every Baire–mapping $\Delta : [A]^{<}\binom{\omega}{k} \to \{0,\ldots,r-1\}$, where k and r are positive integers, there exists a monochromatic $F \in [A]^{<}\binom{\omega}{\omega}$. □

Nothing is known for measurable partitions. Also, no canonizing version of this theorem is known.

6.3 Vector Spaces

Let $\mathcal{F}$ be a finite field and let $\mathcal{F}\binom{n}{k}$ be the set of k–dimensional vector spaces in $\mathcal{F}^n$, the n–dimensional vector space over $\mathcal{F}$. At the end of the 60's much effort has been put on proving a Ramsey type partition theorem for finite vector spaces. Eventually a success has been achieved by Graham, Leeb and Rothschild (1972). We mentioned already some connections between parameter sets and vector spaces (e.g. 2.5). The close relationship will be transparent if we consider subspaces $M \in \mathcal{F}\binom{n}{m}$ represented by matrices. There is a general feeling that analogous methods and results as for parameter sets also apply for vector spaces. We do not go into any details, but refer the reader to the literature, e.g. Leeb (1973), Spencer (1979), Deuber, Voigt (1982), Voigt (1984, 1985), Carlson (1987), Prömel (1986).

6.4 Finite Abelian groups

Here the same remarks as before apply. However, concerning Ramsey type partition theorems, finite abelian groups have not yet been studied systematically. What is known is a partition theorem (Deuber, Rothschild (1975), Voigt (1980)) and an induced partition theorem (Prömel (1982)). But no canonizing version and no infinite version is known.

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Shelah's Proof of the Hales-Jewett Theorem

Alon Nilli

This communication contains a description of Shelah's recent proof for the Hales–Jewett Theorem, in a condensed (and yet self contained) form. For simplicity we include here only the proof of the one dimensional case of the theorem, which solves a problem of Graham by showing that the Hales–Jewett function is primitive recursive. The general cases will appear in the full paper of Shelah.

Definitions. $h_1(n) = 2n$; for $i > 1$, $h_i(n) = h_{i-1}(h_{i-1}(\dots h_{i-1}(1)))$, where h_{i-1} is taken n times. (For example, $h_2(n) = 2^n$.) $\Lambda = \{0, 1, \dots, m-1\}$. A *root* is a word in the symbols $\Lambda \cup \{x\}$, with at least one α, denoted $\tau(x)$. For $i \in \Lambda$, $\tau(i)$ is the word obtained from $\tau(x)$ by replacing each x by i. A *combinatorial line* is a set of m words $\{\tau(i) : i \in \Lambda\}$, where $\tau(x)$ is a root. $H(c,m)$ is the minimum n so that in each coloring of Λ^n in c colors there is a monochromatic combinatorial line.

Theorem *(Shelah's version of the Hales-Jewett Theorem). For every $c, m \geq 1$, $H(c,m) \leq \frac{1}{cm} h_4(c+m+2)$.*

Proof. For each fixed c, we apply an induction on m. For $m = 1$ the theorem is trivial. Assuming it holds for $m-1$ (and c) we prove it for m. Put $\ell = H(c, m-1)$. Define a sequence $n_i, i = 1, \dots, \ell$, by $n_1 = c^{m^\ell}, n_i = c^{m^{\ell + \sum_{j=1}^{i-1} n_j}}$. Put $n = \sum_{i=1}^{\ell} n_i$. We now show that $H(c,m) \leq n$. Let C be a coloring by c colors of Λ^n. For ℓ roots $\tau_1(x), \dots, \tau_\ell(x)$, where the length of $\tau_i(x)$ is n_i, and for a word $\nu = \nu_1 \nu_2 \dots \nu_1 \in \Lambda^\ell$ we denote $\tau(\nu) = \tau_1(\nu_1)\tau_2(\nu_2)\dots\tau_\ell(\nu_\ell)$. Note that $\tau(\nu) \in \Lambda^n$. We claim that there are ℓ roots $\tau_1(x), \dots, \tau_\ell(x)$ as above, such that:

(*) If $\nu^0, \nu^1 \in \Lambda^\ell$ are two words which differ only in one coordinate, say, the i–th in which $\nu^0_i = 0$ and $\nu^1_i = 1$, then $C(\tau(\nu^0)) = C(\tau(\nu^1))$.

(Notice that this clearly implies that the last equality holds for any $\nu^0, \nu^1 \in \Lambda^\ell$, whenever ν^0 is obtained from ν^1 by changing some of the coordinates of

ν^1 from 1 to 0.) We prove the existence of the required roots τ_i by descending induction on i. Suppose we have already defined τ_i for $i > i_0$. For $0 \leq j \leq n_{i_0}$, let σ_j be the word consisting of j zeros followed by $n_{i_0} - j$ ones. Denote $\ell_i = \sum_{j=1}^{i-1} n_j$. For each j as above define a coloring $f_j : \Lambda^{\ell_{i_0}+(\ell-i_0)} \to \{0,1,\ldots,c-1\}$ by $f_j(\nu_1,\nu_2,\ldots,\nu_{\ell_{i_0}},\nu'_{i_0+1},\ldots,\nu'_{\ldots}) = C(\nu_1,\nu_2,\ldots,\nu_{\ell_{i_0}},\sigma_j,\tau_{i_0+1}(\nu'_{i_0+1})\cdots\tau_\ell(\nu'_\ell))$.

Clearly, the total number of possibilities for f_j is at most n_{i_0}. Thus, there are $j_2 > j_1$ such that $f_{j_1} = f_{j_2}$. Define, now $\tau_{i_0}(x)$ to be the root consisting of j_1 0's, followed by $j_2 - j_1$ x's, followed by $n_{i_0} - j_2$ 1's.

One can easily check that the roots $\tau_1,\ldots,\tau_\ell$ defined by this procedure satisfy $(*)$. We can now complete the proof using the roots τ_i. Define a coloring C^* of $\{\Lambda\backslash\{0\}\}^\ell$ by $C^*(\nu) = C(\tau(\nu))$. Since $\ell = H(c,m-1)$ there is a monochromatic (with respect to C^*) combinatorial line $\{\nu(i) : i \in \Lambda\backslash\{0\}\}$, where $\nu(x) = \nu_1\ldots\nu_\ell$ is a root of length ℓ. Note that at least one ν_j is an x. Clearly $\tau(\nu)$ is a root of length n and $\{\tau(i) : i \in \Lambda\}$ is a monochromatic combinatorial line with respect to C.

It remains to estimate the upper bound obtained by this for $H(c,m)$. Clearly, $\ell_i = \ell_{i-1}+c^{m^{\ell-\ell_{i-1}}} \leq c^{m^{2(\ell+\ell_{i-1})}}$ and this gives $n = \ell_{\ell+1} \leq \frac{1}{cm}h_3(cm\ell)$. By the induction hypothesis $\ell = H(c,m-1) \leq \frac{1}{cm}h_4(c+m+1)$ and hence $H(c,m) \leq n \leq \frac{1}{cm}h_3(h_4(c+m+1)) = \frac{1}{cm}h_4(c+m+2)$. This completes the proof. □

Remark. The idea in the proof is that if n is large enough then in any coloring C of the cube Λ^n there is an *ℓ–subcube* in which the colors are indifferent to the replacement of 0's by 1's. Here an ℓ–subcube is a set of m^ℓ vertices of the large cube in which all but ℓ pairwise disjoint sets of coordinates are fixed and the coordinates in these sets vary so that all those in each set attain the sam evalue.

Acknowledgement

I would like to thank N. Alon and G. Dugon for explaining me Shelah's proof in details. I am also grateful to N. Alon for his help in writing this manuscript.

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Part IV

Noncombinatorial Methods

Partitioning Topological Spaces

William Weiss

1. Introduction

The study of partitions of topological spaces is a relatively new addition to Ramsey theory, but one which promises interesting things in the future. We partition topological spaces and hope to obtain a homogeneous set which is topologically relevant – for instance, a set homeomorphic to a well known topological space. We thus add something new to the ordinary partition calculus of set theory. We do, however, borrow the arrow notation. We write

$$X \to (Y)^n_\lambda$$

to mean the following statement.

> 'X and Y are topological spaces, λ is a cardinal and n is a positive integer. For each function $f : [X]^n \to \lambda$ there is $H \subseteq X$ such that H is homeomorphic to Y and f is constant on $[H]^n$.'

Also, as in the ordinary partition calculus

$$X \to [Y]^n_\lambda$$

denotes the modification in which we only require that $f''[H]^n$ is not all of λ.

When certain structures such as the reals $\mathbb{R}$, the rationals $\mathbb{Q}$ or cardinals κ are involved, we use the letters *top* to denote the topological, rather than order-theoretic or cardinality properties. For example, we have

$$\mathbb{R} \to (top\,\mathbb{Q})^2_2.$$

By the way, this last statement is false. I hope it is the only false one in the article. The negation of a partiton relation is written in the usual way, for example

$$\mathbb{R} \not\to (top\,\omega_1)^1_2.$$

Note that, unlike the ordinary partition calculus, a superscript 1 does not always signal an easy problem!

I have tried to make this article as self-contained as possible so that the benignly critical reader who has read this far should be able to enjoy it. Nevertheless, a good place to look for undefined terms is the textbook 'Set Theory' by K. Kunen, or any of its competitors.

There is one more comment about terminology. In this article, as in set-theoretic topology, space means *regular* space (i.e. singleton points are closed sets and any point not in a closed set has a neighbourhood whose closure misses the closed set). In fact our intuition will not go astray if you assume that each space has a basis of sets which are clopen (that is, both open and closed). Just to see what can happen when this topological restriction is relaxed, we present this one misleading but clever example of J. Nešetřil, J. Pelant and V. Rödl.

Let Y be the set of points on the usual Cartesian plane, with the topology generated by sets of the form

$$\{(x,y) : a < x < p\} \cup \{(x,y) : x = p,\ c < y < d\} \cup \{(x,y) : p < x < b\},$$

where a, b, c, d and p are real numbers. It is not difficult to check that

$$Y \rightarrow (top\,\mathbb{R})^1_2.$$

However, "in real life" homogeneous homeormorphic copies of $\mathbb{R}$ are not, as we shall see, so easily obtained.

Several results about partitioning topological spaces have never been published and have been lying around in manuscript form. I have included those proofs which are not likely to be otherwise found in the literature. The bibliography of the subject is small enough that I have been able to comment upon each entry.

I would like to thank Alan Dow, Jim Baumgartner, Peter Komjáth, John Merrill, Saharon Shelah, Boban Veličković and especially Stevo Todorčević for helpful and interesting discussions about the material presented here.

2. Partitioning Singletons

We begin by partitioning topological spaces to study the relation $X \rightarrow (Y)^1_\lambda$. The simplest non-discrete spaces we can use for Y are just the countable ordinals with the usual order topology; the simplest of these is just the convergent sequence $\omega + 1$. To get a feeling for the subject, the reader might like to verify the following relations:

$$\omega^2 \not\longrightarrow (top\ \ \omega + 1)^1_2 \text{ and } w^2 + 1 \longrightarrow (top\ \ \omega + 1)^1_2.$$

By partitioning richer topological spaces we can find any countable ordinal.

2.1 Theorem. *For each countable ordinal α we have:*
(i) $\mathbb{Q} \to (top\,\alpha)^1_n$ *for each natural number* $n \geq 1$,
(ii) $\mathbb{R} \to (top\,\alpha)^1_{\aleph_0}$, *and* $X \to (top\,\alpha)^1_{\aleph_0}$ *for any* $X \in [\mathbb{R}]^{\aleph_1}$
(iii) $\omega_1 \to (top\,\alpha)^1_{\aleph_0}$.

Proof. For parts (i) and (ii) just recall the topological theorem that any denumerable metric space without isolated points is homeomorphic to $\mathbb{Q}$, and note that $\mathbb{Q}$ contains a homeomorphic copy of each countable ordinal. Part (iii) is also easy; it will follow directly from Lemma 1.3. □

2.2 Definitions. A *closed unbounded* subset C of an ordinal α is one which is closed (in the order topology) and unbounded (in the order). We write that C is a c.u.b. subset of α and when $\alpha = \omega_1$ we usually omit mention of it. A *stationary* subset S of α is one which has non-empty intersection with each c.u.b. subset of α. We think of c.u.b. sets as large and stationary sets as medium-sized. If a stationary subset of ω_1 is partitioned into countably many pieces, one of the pieces must be stationary.

2.3 Lemma. *If α is a countable ordinal and S is a stationary subset of ω_1, then S contains a homeomorphic copy of α.*

Proof. We prove this by induction on $\alpha < \omega_1$. The cases when α is a limit ordinal or α is a successor of a successor are easy. For the case when α is the successor of a limit ordinal β, we notice that for any stationary set S, the set of all $\gamma \in \omega$, such that there is some $B \subseteq S$, homeomorphic to β with the property that for all $\delta < \gamma$ the subspace $B \cap (\delta + 1)$ is compact, is c.u.b. Intersecting this set with S gives a point which can be used to build the homeomorphic copy of $\beta + 1$. □

While Theorem 2.1 is hardly surprising, since $\mathbb{Q}, \mathbb{R}$ and ω_1 are all relatively rich topological spaces compared to countable ordinals, it is interesting to investigate the relation $\alpha \to (top\,\beta)^1_2$ when α and β are *both* countable infinite ordinals. Topological properties of ordinals in some instances differ dramatically from their order type properties. For any ordinal α we have $\alpha + 1 \simeq \alpha + 2$. We have $\omega^2 + 1$, a subspace of $\omega^2 + \omega$, but homeomorphic to $\omega^2 + \omega + 1$.
Thus

$$\omega^2 + \omega \to (top\ \ \omega^2 + \omega + 1)^1_1$$

Is it possible that for some α, $\alpha \to (top\ \ \alpha + 1)^1_2$? No.

2.4 Lemma. *For any countable ordinal α*
(i) $\alpha \not\to (top\ \ \alpha + 1)^1_2$
(ii) $\alpha^k \not\to (top\ \ \alpha + 1)^1_k$ *for all $k \geq 1$ if α satisfies the additional condition that for each $\delta < \alpha$, $\alpha \backslash \delta \to (top\,\alpha)^1_1$.*

Proof. We first prove (i) by assuming that α is the least ordinal for which (i) holds and drawing out a contradiction. Fix any $\beta < \alpha$. By the minimality of α, $\beta \not\to (top\,\alpha)^1_1$. Hence the partition

$$f(\gamma) = \begin{cases} 0 & \text{if } \gamma < \beta \\ 1 & \text{if } \gamma \geq \beta \end{cases}$$

immediately shows that $\alpha \backslash \beta \rightarrow (top \quad \alpha + 1)^1_1$.

In particular for $\beta = 0$, we have a continuous one-to-one $h : \alpha + 1 \rightarrow \alpha$. By continuity there is some $\delta < \alpha$ such that $h(\alpha)$, i.e. the set of ordinals less than $h(\alpha)$, satisfies $h(\alpha) \rightarrow (top\, \alpha \backslash \delta)^1_1$. Hence $h(\alpha) \rightarrow (top\, \alpha)^1_1$ contradicting the minimality of α.

Before proving (ii), notice that the ordinal space α^k is homeomorphic to the topological space which is the lexicographic order topology on $\alpha^k = \alpha \times \alpha \times \ldots \times \alpha$. We deal with this product; a typical element is $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_k)$. We say that $\boldsymbol{\beta}$ is a point of type j if

$$\beta_j \neq 0 \text{ but } \beta_i = 0 \text{ for all } i > j.$$

Note that for each j, the set of points of α^k of type j is a union of a collection of open sets, each homeomorphic to α, such that the pairwise intersection of closures of sets in the collection is empty.

We now suppose that α is the least ordinal satisfying $\alpha^k \rightarrow (top\, \alpha + 1)^1_k$ and $\alpha \backslash \delta \rightarrow (top \quad \alpha)^1_1$ for all $\delta < \alpha$. A natural partition of α^k is given by requiring that

$$f(\boldsymbol{\beta}) = j \text{ if } \boldsymbol{\beta} \text{ is of type } j.$$

Since for some j, the set of points of type j must contain a homeomorphic copy of $\alpha + 1$, we see that for some $\delta < \alpha$

$$\alpha \rightarrow (top(\alpha + 1) \backslash \delta)^1_1$$

and so by hypothesis $\alpha \rightarrow (top \quad \alpha + 1)^1_1$. An argument used in part (i) now shows that the minimality of α is contradicted. □

There are also some positive relations.

2.5 Lemma.

(i) $\omega \rightarrow (top\, \omega)^1_2$

(ii) If α is a countable ordinal such that $\alpha \rightarrow (top\, \alpha)^1_2$ then for all $n < \omega$ and all $m \geq 2$ we have

$$\alpha^{mn+1} \rightarrow (top\, \alpha^{n+1})^1_m$$

(iii) If $\langle \alpha_n \rangle$ and $\langle \beta_n \rangle$ are sequences of ordinals such that for each n, $\alpha_n \rightarrow (top \quad \beta_n + 1)^1_2$, then $\alpha \rightarrow (top\, \beta)^1_2$ where $\alpha = \sum \alpha_n$, the sum of the α_n, and $\beta = sup\{\beta_n\}$.

Proof. (i) is maximally trivial. The case $n = 0$ is immediate for (ii) and we shall use it repeatedly in the proof for the general case. To this end fix m and n, and let $k = mn + 1$. Let $f : \alpha^k \rightarrow m$ be a partition. We wish to find an appropriate copy of α^{n+1}. For each fixed $(\beta_1, \beta_2, \ldots, \beta_{k-1})$ we define a partition

$$f' : \alpha \backslash \{0\} \rightarrow m$$

by $f'(\gamma) = f(\beta_1,\beta_2,\ldots,\beta_{k-1},\gamma)$. Using $\alpha \to (top\,\alpha)^1_m$ we obtain, for each $(\beta_1,\beta_2,\ldots,\beta_{k-1})$ a homeomorphic copy $H(\beta_1,\beta_2,\ldots,\beta_{k-1})$ of α with f' constant on $H(\beta_1,\beta_2,\ldots,\beta_{k-1})$. Replacing α^k by its homeormorphic subspace

$$\{(\beta_1,\beta_2,\ldots,\beta_{k-1},\gamma) : \gamma \in H(\beta_1,\beta_2,\ldots,\beta_{k-1}); \beta_1,\beta_2,\ldots,\beta_{k-1} \text{ in } \alpha\}.$$

We have therefore reduced the problem to the case when the partition function f has the property that if β is any point of type k (as defined in the proof of part (ii) of Lemma 2.4) then $f(\beta)$ is determined by the first $k-1$ coordinates of β. From now on we assume f has this property and continue this process.

For each $(\beta_1,\beta_2,\ldots,\beta_{k-2})$ apply $\alpha \to (top\,\alpha)^1_{m^2}$ to find a homeomorphic copy $H(\beta_1,\beta_2,\ldots,\beta_{k-2})$ of α such that f'' is constant on $H(\beta_1,\beta_2,\ldots,\beta_{k-2})$ where $f'' : \alpha\backslash\{0\} \to m^2$ is a partition defined by

$$f''(\gamma) = (f(\beta_1,\beta_2,\ldots,\beta_{k-2},\gamma,0), f(\beta_1,\beta_2,\ldots,\beta_{k-2},\gamma,1)).$$

Note that 1 could be replaced by any $1 \leq \delta < \alpha$. We now replace α^k by its homeomorphic subspace

$$\{(\beta_1,\beta_2,\ldots,\beta_{k-2},\gamma,\delta) : \gamma \in H(\beta_1,\beta_2,\ldots,\beta_{k-2}); \beta_1,\beta_2,\ldots,\beta_{k-2},\delta \text{ in } \alpha\}.$$

We have now reduced the problem to the case when the partition function f has the following property: if β is any point of either type k or type $k-1$, then $f(\beta)$ is determined only by the first $k-2$ coordinates of β and the type of β. From now on we assume the f has this property.

We continue this process, using $\alpha \to (top\,\alpha)^1_{m^j}$ at stage j. After k stages we have reduced the problem to the case when f has the property that $f(\beta)$ is determined solely by the type of β. Since $k = mn+1$ there must be $n+1$ types $t_1,t_2,\ldots,t_{n+1}$ such that f is constant on the set

$$\{\beta \in \alpha^k : \beta \text{ has one of the types } t_1,t_2,\ldots,t_{n+1}\}.$$

And this set contains a homeormorphic copy of α^{n+1}.
Since the proof of part (iii) is easy, we are finished with the proof of this lemma. $\square$

It is easy to demonstrate that in the case of order types of countable ordinals $\alpha \to (\alpha)^1_2$ if and only if $\alpha = \omega^\beta$ for some ordinal β. The following theorem is interesting.

2.6 Theorem. *For any countable ordinal α, $\alpha \to (top\,\alpha)^1_2$ if and only if*

$$\alpha = \omega^{\omega^\beta}$$

for some countable ordinal β.

Proof. We first demonstrate the "if" part by induction on β, using Lemma 2.5. Part (i) is the case $\beta = 0$. For the successor case put

$$\alpha = \omega^{\omega^\beta}$$

and let $\alpha_n = \alpha^{2n+1}$. From part (ii) we see that $\alpha_n \to (top\ \ \alpha^n + 1)^1_2$, and hence by part (iii) we have $\alpha^\omega \to (top\,\alpha^\omega)^1_2$. But

$$\alpha^\omega = (\omega^{\omega^\beta})^\omega = \omega^{\omega^{\beta+1}}$$

and the successor case follows. The limit case is a direct application of part (iii).

In order to prove the "only if " part we use Lemma 2.4. For any countable infinite ordinal γ, there are some ordinals β, and α such that $\alpha = \omega^{\omega^\beta}$, and $\alpha \leq \gamma < \alpha^n$ for some positive integer $n \geq 2$. If $\gamma \to (top\,\gamma)^1_2$ then $\alpha^n \to (top\ \alpha + 1)^1_n$ contradicting part (ii) of the lemma. This completes the proof of the lemma.

□

We now assume $\delta = 0$, and so $\gamma = \alpha^n$ and therefore $\alpha^n \to (top\ \ \alpha^n)^1_n$ and so $\alpha^n \to (top\ \alpha + 1)^1_n$, contradicting part (ii) if $n > 1$. This completes the proof.

We now consider the relation $X \to (top\,\beta)^1_{\aleph_0}$ for countable ordinals β. Certainly X will not be countable here. The weakest relation of interest is with $\beta = \omega + 1$. In some cases, and not others, this weakest relation implies the strongest. Part (i) of the following result of P. Komjáth and I greatly generalizes parts (ii) and (iii) of Theorem 2.1.

2.7 Theorem. *Remember we consider regular topological spaces.*
(i) If X is first countable and $X \to (top\ \ \omega + 1)^1_{\aleph_0}$, then for each countable ordinal β, $X \to (top\,\beta)^1_{\aleph_0}$.
(ii) Assume $MA(\aleph_1)$. If $\chi(X) = \aleph_1$ and $X \to (top\ \ \omega + 1)^1_{\aleph_0}$, then for each countable ordinal β, $X \to (top\,\beta)^1_{\aleph_0}$.
(iii) Assume $\Diamond^+$. There is a space X such that $\chi(X) = \aleph_1$ and $X \to (top\ \ \omega + 1)^1_{\aleph_0}$ but X doesn't even contain a homeomorphic copy of $\omega^2 + 1$.

$MA(\aleph_1)$ and $\Diamond^+$ are (separately) consistent with the usual axioms of set theory; $\Diamond^+$ is a consequence of Gödel's axiom of constructibility. $\chi(X) = \aleph_1$ means that each point of X has a local base of size at most $\aleph_1$. □

We now turn to finding homeomorphic copies of ω_1. Here things are quite different from the case of countable ordinals. Immediately we have the following.

2.8 Theorem. *For each ordinal α of cardinality $\aleph_1$ we have*
(i) $\alpha \not\longrightarrow (top\,\omega_1)^1_2$, and even
(ii) $\alpha \not\longrightarrow [top\,\omega_1]^1_{\aleph_1}$.

Proof. Note that (ii) is stronger than (i). Since a homeomorphic copy of ω_1 embedded in ω_1 must be a c.u.b. of ω_1, the relation $\omega_1 \not\longrightarrow [top\,\omega_1]^1_{\aleph_1}$ expresses the well-known fact that ω_1 can be partitioned into $\aleph_1$ pairwise disjoint stationary sets. The proof of (ii) now proceeds by induction on α. At successor stages or stages of countable cofinality there is no difficulty. If α has cofinality $\aleph_1$ we find a continuous increasing sequence $\langle c_\beta : \beta < \omega_1\rangle$ cofinal in α. We partition each interval $(c_\beta, c_{\beta+1})$ independently using the inductive hypothesis and partition the sequence $\langle c_\beta : \beta < \omega_1\rangle$ using $\omega_1 \not\longrightarrow [top\,\omega_1]^1_{\aleph_1}$. □

This theorem can be extended in two ways, considering either larger ordinals or more complex spaces of size $\aleph_1$. In both cases we have only consistency results. $\diamondsuit$ is another consequence of Gödel's axiom of constructibility, as are all the $\Box$ principles, which we formally define. All these combinatorial principles are due to R. Jensen.

2.9 Definition. *By $\Box_\kappa$ we mean the following principle:*
There is a sequence $\langle C_\lambda : \lambda < \kappa^+$ and λ a limit ordinal$\rangle$ such that
(i) C_λ is closed and unbounded in λ,
(ii) $cf(\lambda) < \kappa$ implies $|C_\lambda| < \kappa$, and
(iii) if γ is a limit point of C_λ, then $C_\gamma = \gamma \cap C_\lambda$.

2.10 Theorem.
(i) Assume $\Box_{\omega_1}$. Then $\omega_2 \not\longrightarrow (top\,\omega_1)^1_2$.
(ii) Assume $\diamondsuit$. If X is any space of size $\aleph_1$, then $X \not\longrightarrow (top\,\omega_1)^1_2$.

Proof. Part (i) was noticed by J. Silver. Let S_0 and S_1 be disjoint stationary subsets of ω_1 with $\omega_1 = S_0 \cup S_1$. The reader may like to verify that the partition $f : \omega_2 \to 2$ where:

$$f(\alpha) = \begin{cases} 0 & \text{if } cf(\alpha) \neq \aleph_0 \\ 0 & \text{if o.t. } C_\alpha \in S_0 \\ 1 & \text{if o.t. } C_\alpha \in S_1 \end{cases}$$

and the C_α come from $\Box_{\omega_1}$, actually works. $\Box$

Part (ii) was noticed by P. Komjáth and I; the proof is omitted here. John Merrill, however, has a better result, including the following.

2.11 Theorem. *Assume both $\diamondsuit$ and $\Box_{\omega_1}$. If X is any space of cardinality $\aleph_2$, then $X \not\longrightarrow (top\,w_1)^1_2$.*

It seems easier to partition ordinals than general topological spaces. Prikry and Solovay proved that, under $V = L$, $\kappa \not\longrightarrow (top\,\omega_1)^1_2$ for each ordinal κ. This can be compared with the following result of S. Shelah. SPFA, the semi-proper forcing axiom, is a strengthening of $MA(\aleph_1)$ which says that in any partial order which preserves stationary subsets of ω_1, one can find a generic set for any given collection of $\aleph_1$ dense subsets.

2.12 Theorem. *Assume SPFA.*
(i) $\omega_2 \to (top\,\omega_1)^1_2$
(ii) $\{0,1\}^{\omega_1} \to (top\,\omega_1)^1_2$
(iii) If the countable subsets of ω_2 are partitioned into two pieces, one piece contains a family $\{S_\alpha : \alpha \in \omega_1\}$ such that if $\alpha < \beta$ then $S_\alpha \subsetneq S_\beta$ and if β is a limit ordinal $S_\beta = \bigcup\{S_\alpha : \alpha < \beta\}$.

Proof. We will not prove (i) and (iii) here but B. Veličković has noticed that (iii) implies (ii). Indeed, let $\{A_2 : \alpha < \omega_2\}$ be a subfamily of $[\omega_1]^{\aleph_1}$ such that if $\alpha \neq \beta$ then $|A_\alpha \cap A_\beta| \leq \aleph_0$. We define $\phi : [\omega_2]^{\aleph_0} \to \{0,1\}^{\omega_1}$ by:

$$\phi(X) \text{ is the characteristic function of } \bigcup\{A_\alpha : \alpha \in X\}.$$

It is easy to show that ϕ is one-to-one; hence a partition of $\{0,1\}^{\omega_1}$ induces a partition of $[\omega_2]^{\aleph_0}$. The image of the family $\{S_\alpha : \alpha \in \omega_1\}$ given by (iii) is a homeomorphic copy of ω_1. $\square$

In connection with part (iii) of Theorem 2.12, we immediately get that if X is the $\sum$-product of $\{0,1\}^{\omega_2}$, then SPFA implies that $X \to (top\,\omega_1)^1_2$. In the negative direction, Shelah has proven the following.

2.13 Theorem. *For any positive integers m and $n > 2m$, there is a partition $f : [\omega_2]^{\aleph_0} \to n$ such that if $\{S_\alpha : \alpha \in \omega_1\}$ is a family as described in part (iii) of Theorem 2.12, then $|\{f(S_\alpha) : \alpha \in \omega_1\}| > m$. Consequently, the result of part (iii) of Theorem 2.12 does not hold if the partition is into 3 pieces.*

In order to emphasize the difficulty in obtaining homeomorphic copies of ω_1, we present the following result attributed to J. Silver.

2.14 Theorem. *Let M be a model of set theory and $M[G]$ be the model obtained by forcing with the partial order $F_n(\omega,\omega_1)$ which collapses ω_1. In $M[G]$ we have that for any ordinal κ, there is a partition $\kappa = A \cup B$ such that neither A nor B contain a homeomorphic copy of ω_1.*

Proof. Our forcing terminology is taken from K. Kunen's textbook. We work inside the model M. Let κ be given; the partition of κ will actually be in M. Let $A = \{\alpha \in \kappa : cf(\alpha) = \aleph_0\}$, and $B = \kappa \backslash A$. Suppose there is $p \in F_n(\omega,\omega_1)$ such that

$$p \; \dot{g} \text{ is a 1 to 1 continuous function; dom } \dot{g} = \dot{\omega}_1; \text{ range } \dot{g} \subseteq \check{A}.$$

Note that we can replace $\dot{\omega}_1$ with $\check{\omega}_2$. Since $F_n(\omega,\omega_1)$ has cardinality $\aleph_1$ there is a $q \le p$ and $S \in [\omega_2]^{\aleph_2}$ such that for each $\alpha \in S$ there is $a_\alpha \in A$ and $q \; \dot{g}(\check{\alpha}) = \check{a}_\alpha$. We pick an increasing sequence $\langle \alpha_\zeta : \zeta < \omega_1 \rangle$ from S and let γ be the supremum. By continuity

$$q \; \dot{g}(\check{\gamma}) = sup\{\check{a}_{\alpha_\zeta} : \zeta < \omega_1\}$$

but clearly $sup\{a_{\alpha_\zeta} : \zeta < \omega_1\}$ cannot be in A.

A similar proof shows that B contains no homeomorphic copy of ω_1; this time choosing a countable increasing sequence. $\square$

If it is difficult to find homeomorphic copies of ω_1, it is even more difficult to find homeomorphic copies of the Cantor set, which we identify with the $\aleph_0$ power of the two point discrete space, 2^ω. More than eighty years ago F. Bernstein proved that the real line can be partitioned into two disjoint sets, neither of which contains a homeomorphic copy of the Cantor set. Somewhat later, I was able to extend this result.

2.15 Theorem. *Suppose X is a space such that for all $2^{\aleph_0} < \lambda < |X|$ with $cf(\lambda) = \aleph_0$ we have $\lambda^{\aleph_0} \le \lambda^+$ and $\square_\lambda$. Then $X \not\to (top\, 2^\omega)^1_2$.*

The result is even true for spaces which are not regular but only Hausdorff. We get two immediate corollaries.

2.16 Corollary. *If X is a space with $|X| \leq \aleph_\omega$, then $X \not\longrightarrow (top\, 2^\omega)^1_2$.*

2.17 Corollary. *If any one of the following hypotheses hold, then for all spaces X, $X \not\longrightarrow (2^\omega)^1_2$.*
(i) $\lambda^{\aleph_0} = \lambda^+$ and $\Box_\lambda$ holds for each cardinal λ of cofinality $\aleph_0$ with $\lambda > 2^{\aleph_0}$.
(ii) Gödel's axiom of constructibility.
(iii) There are no measurable cardinals.

Proof. (i) is immediate and well-known work of Jensen and Dodd and Jensen shows that (ii) or (iii) imply (i). □

Since a (correct) proof of Theorem 2.15 has not appeared in print, we take this opportunity to present one. The proof will be immediately attainable from Lemmas 2.20, 2.21, 2.22 and 2.23 which constitute the inductive steps for a proof by induction on $|X|$. First, however, we need to make two temporary definitions.

2.18 Definition. *Z is said to be an F^ω_σ subset of the space X iff there is a sequence $\{Z_n : n \in \omega\}$ of subsets of X such that $Z = \bigcup\{Z_n : n \in \omega\}$ and each Z_n contains the closure of each of its countable subsets.*

2.19 Definition. *We say that $f : X \to 2$ is a Cantor set partition iff f is not constant on any homeomorphic copy of the Cantor set in X.*

Let $IH(\kappa)$ denote the following statement.

> "If X is a space and $|X| \leq \kappa$ and Z is an F^ω_σ subset of X and if there exists $g : Z \to 2$ which is a Cantor set partition of Z, then there exists a Cantor set partition $f : X \to 2$ extending g."

The first lemma is a simple extension of Bernstein's result.

2.20 Lemma. *$IH(c)$ holds, where $c = 2^{\aleph_0}$.*

Proof. Let X be a space of cardinality c. There are at most c distinct homeomorphic copies of the Cantor set in X. Enumerate these as $\{H_\alpha : 1 \leq \alpha < c\}$. Let $g : Z \to 2$ be given as in $IH(c)$.
Recursively construct $\{g_\alpha : \alpha < c\}$ and $\{Z_\alpha : \alpha < c\}$ such that $g_0 = g$, $Z_0 = Z$ and for all $\alpha \leq \beta$:
(1) $Z_\alpha \subseteq Z_\beta$ and $|Z_\beta \backslash Z| < c$
(2) $g_\beta : Z_\beta \to 2$ extending g_α
(3) $|H_\beta \cap Z_\beta| \geq 2$ and $g_\beta \mid (H_\beta \cap Z_\beta)$ is not constant.

A stage β simply consider H_β.

Case 1. $H_\beta \subseteq \bigcup\{Z_\alpha : \alpha < \beta\}$ For some $K \subseteq H_\beta$ which is homeomorphic to the Cantor set, $K \subseteq Z$. Now just let $Z_\beta = \bigcup\{Z_\alpha : \alpha < \beta\}$ and $g_\beta = \bigcup\{g_\alpha : \alpha < \beta\}$.

Case 2. $H_\beta \not\subseteq \bigcup\{Z_\alpha : \alpha < \beta\}$. Pick $p \in H_\beta \backslash \bigcup\{Z_\alpha : \alpha < \beta\}$ and $q \in H_\beta$. If $g_\alpha(q)$ is defined for some $\alpha < \beta$, let $g_\beta(q) = g_\alpha(q)$, otherwise let $g_\beta(q) = 0$. Now, let $Z_\beta = \bigcup\{Z_\alpha : \alpha < \beta\} \cup \{p, q\}$ and $g_\beta = \bigcup\{g_\alpha : \alpha < \beta\} \cup \{< q, g_\beta(q) >, < p, 1 - g_\beta(q) >\}$.

We finished let $Z_c = \bigcup\{Z_\alpha : \alpha < c\}$ and $g_c = \bigcup\{g_\alpha : \alpha < c\}$. Extend g_c to f arbitrarily over $X \backslash Z_c$. □

2.21 Lemma. *If $\kappa^{\aleph_0} = \kappa$ and $IH(\kappa)$ holds then $IH(\kappa^+)$ holds.*

Proof. Let X, Z and g be as in $IH(\kappa^+)$. Enumerate X as $\{x_\alpha : \alpha < \kappa^+\}$. We will construct $\{h_\alpha : \alpha < \kappa^+\}$ and $\{Y_\alpha : \alpha < \kappa^+\}$ such that for all $\alpha < \beta$:
(1) $Y_\alpha \subseteq Y_\beta$,
(2) $|Y_\beta| \leq \kappa$,
(3) $x_\alpha \in Y_{\alpha+1}$,
(4) $Y_\beta = \{x \in X : x \in \overline{C}$ for some countable $C \subseteq Y_\beta\}$,
(5) $h_\alpha \subseteq h_\beta$,
(6) h_β is a Cantor set partition of Y_β,
(7) $h_\beta | Y_\beta \cap Z = g | Y_\beta \cap Z$.

We begin the inductive construction by letting $Y_0 = \emptyset = h_0$. At stage $\beta + 1$, let $Y_{\beta+1} = Y_\beta \cup \{x_\beta\}$. Let

$$h_{\beta+1} = \begin{cases} h_\beta & \text{if } x_\beta \in Y_\beta \\ h_\beta \cup \langle x_\beta, g(x_\beta)\rangle & \text{if } x_\beta \in Z \\ h_\beta \cup \langle x_\beta, 0\rangle & \text{otherwise.} \end{cases}$$

At stage λ with $cf(\lambda) = \omega$ obtain $\lambda_n \nearrow \lambda$. Let $Y_\lambda = \{x \in X : x \in \overline{C}$ for some countable $C \subseteq \bigcup\{Y_{\lambda_n} : n \in \omega\}\}$. Let

$$h^* = g \mid (Y_\lambda \cap Z) \cup \bigcup\{h_{\lambda_n} : n \in \omega\}.$$

It is straightforward to check that we can use $IH(\kappa)$ on Y_λ, $dom\, h^*$ and h^* to obtain h_λ.

At stage γ for $cf(\gamma) > \omega$, simply let $Y_\gamma = \bigcup\{Y_\beta : \beta < \gamma\}$ and $h_\gamma = U\{h_\beta : \beta < \gamma\}$. At each stage it is straightforward to show that the proper inductive hypothesis is satisfied. We finish by letting $f = h_{\kappa^+}$. □

The following lemma has a proof similar to the preceding one.

2.22 Lemma. *If $cf(\kappa) \neq \omega$ and for all $\lambda < \kappa, \lambda^{\aleph_0} < \kappa$ and $IH(\lambda)$, then $IH(\kappa^+)$.*

2.23 Lemma. *If $\square_\kappa$, $cf(\kappa) = \omega$ and for all $\lambda < \kappa, \lambda^{\aleph_0} < \kappa$ and $IH(\lambda)$, then $IH(\kappa^+)$.*

Proof. Let X, Z and g be as in $IH(\kappa^+)$. Let $\{x_\alpha : \alpha < \kappa^+\}$ enumerate X. Obtain $\kappa_n \nearrow \kappa$ such that $\kappa_n^{\aleph_0} = \kappa_n$. Let $\langle C_\lambda : \lambda \in \kappa^+$ and λ a limit $\rangle$ witness $\square_\kappa$.

We will construct $\{Y_\alpha^n : \alpha < \kappa^+, n < \omega\}$ and $\{h_\alpha^n : \alpha < \kappa^+, n < \omega\}$ such that for all $n < \omega$ and $\alpha < \beta$:

(1) $Y_\beta^n \subseteq Y_\beta^{n+1}$,

(2) $\left|Y_\beta^n\right| \leq \kappa_n$,

(3) $x_\alpha \in Y^n_{\alpha+1}$,
(4) $Y^n_\beta = \{x \in X : x$ is in the closure of a countable subset of $Y^n_\beta\}$,
(5) $h^n_\beta \subseteq h^{n+1}_\beta$,
(6) $\bigcup\{h^n_\alpha : n \in \omega\} \subseteq \bigcup\{h^n_\beta : n \in \omega\}$,
(7) h^n_β is a Cantor set partition of Y^n_β,
(8) $h^n_\beta | Y^n_\beta \cap Z = g | Y^n_\beta \cap Z$,
(9) $\bigcup\{Y^n_\eta : \eta \in C_\beta\} \subseteq Y^n_\beta$, if β is a limit ordinal and $|C_\beta| \leq \kappa_n$.

We begin the inductive construction by letting $Y^n_0 = \emptyset = h^n_0$ for each $n < \omega$. At stage $\beta + 1$, let $Y^n_{\beta+1} = Y^n_\beta \cup \{x_\beta\}$. Let

$$h^n_{\beta+1} = \begin{cases} h^n_\beta & \text{if } x_\beta \in Y^n_\beta \\ h^n_\beta \cup \langle x_\beta, g(x_\beta)\rangle & \text{if } x_\beta \in Z \\ h^n_\beta \cup \langle x_\beta, h^k_\beta(x_\beta)\rangle & \text{if } x_\beta \in Y^k_\beta \text{ for some } k > n \\ h^n_\beta \cup \langle x_\beta, 0\rangle & \text{otherwise.} \end{cases}$$

At stage λ for $cf(\lambda) = \omega$, obtain $\lambda_k \nearrow \lambda$ such that $\{\lambda_k : k \in \omega\} \subseteq C_\lambda$ and for each n such that $|C_\lambda| \leq \kappa_n$, $\bigcup\{Y^n_\eta : \eta \in C_\lambda\} = \bigcup\{Y^n_{\lambda_k} : k \in \omega\}$. To do this we may need to use (9). Now for these n, let

$$Y^n_\lambda = \{x \in X : x \text{ is in the closure of some countable subset of } \bigcup\{Y^n_\eta : \eta \in C_\lambda$$

Define

$$h^* = g|(\bigcup\{Y^n_\lambda : n < \omega\} \cup \bigcup\{h^n_{\lambda_k} : k < \omega,\ n < \omega\}.$$

We wish to define h^n_λ by induction on n. For each $\kappa_n < |C_\lambda|$, let $h^n_\lambda = \emptyset = Y^n_\lambda$. Suppose h^{n-1}_λ is defined; consider $h^{-1}_\lambda = \emptyset = Y^1_\lambda$. Apply $IH(\kappa_n)$ to Y^n_λ,

$$\bigcup\{Y^n_{\lambda_k} : k < \omega\} \cup (Y^{n-1}_\lambda \cap Y^n_\lambda) \cup (Z \cap Y^n_\lambda)$$

and

$$(h^* \bigcup h^{n-1}_\lambda)| \bigcup\{Y^n_{\lambda_k} : k < \omega\} \bigcup (Y^{n-1}_\lambda \cap Y^n_\lambda) \bigcup (Z \cap Y^n_\lambda)$$

and obtain $h^n_\lambda : Y^n_\lambda \to 2$.

At stage γ for $cf(\gamma) > \omega$, let $Y^*_\gamma = \bigcup\{Y^n_\alpha : \alpha \in C_\gamma$ and $n \in \omega\}$ and let $h^*_\gamma = \bigcup\{h^n_\alpha : \alpha \in C_\gamma$ and $n \in \omega\}$. Also, let m be the least integer such that $\kappa_m \geq |C_\gamma|$. For $n < m$, let $Y^n_\gamma = h^n_\gamma = \emptyset$. For $n \geq m$, let

$$Y^n_\gamma = \{x \in X : x \text{ is in the closure of a countable subset of } \bigcup\{Y^n_\alpha : \alpha \in C_\gamma\}\}.$$

Then $\bigcup\{Y^n_\gamma : n \in \omega\} = Y^*_\gamma$. For each $n \geq m$, let $h^n_\gamma = h^*_\gamma | Y^n_\gamma$. It is straightforward to check that at each stage the proper inductive hypotheses are fulfilled. To finish the construction, let

$$f = \bigcup\{h^n_\beta : n < \omega \text{ and } \beta < \kappa^+\}.$$

□

This completes the sequence of lemmas needed to prove Theorem 2.15. It is interesting to compare Corollary 2.17, part (i) with the following theorem of Bregman, Šapirovski and Šostak. The conclusions are the same, but neither hypothesis implies the other.

2.24 Theorem. *Assume that $2^{\aleph_0} > \aleph_1$ and that for each cardinal λ of cofinality $\aleph_0$, the following set has size less than $2^{\aleph_0}$*

$$\{\kappa : \kappa \text{ is a cardinal and } \quad \lambda < \kappa < \lambda^{\aleph_0}\}$$

Then for any space X, $X \not\longrightarrow (2^\omega)^1_2$.

We can demonstrate a connection between these studies and other problems in combinatorics by showing how Theorem 2.15 and 2.23 give a consistent verification of a conjecture of P. Erdös.

2.25 Theorem. *Assume that $2^\kappa \not\longrightarrow (top\, 2^\omega)^1_2$. We can then partition $P(\kappa)$, the power set of κ, into two pieces such that neither piece contains a family of disjoint sets $\{A_\kappa : \kappa < \omega\}$ such that all unions belong to the same piece.*

Proof. By identifying a set with its characteristic function we can identify $P(\kappa)$ with 2^κ. Such a family $\{A_k : k < \omega\}$ would give rise to a homeomorphic copy of the Cantor set in 2^κ. □

We end this section with an unpublished positive relation due to S. Todorčević; we include his proof.

2.26 Theorem. *Let A_κ denote the one point compactification of a discrete space of size κ. For any infinite cardinal κ*

$$\{0,1\}^\kappa \rightarrow (top\, A_\kappa)^1_{cf(\kappa)}$$

Proof. A Δ-system of subsets of κ is a collection of distinct subsets of κ such that there is some $R \subseteq \kappa$ which is the intersection of any two members of the collection. R is called the root of the Δ-system. By identifying subsets of κ with their characteristic functions we see that the theorem is a consequence of the following statement which we temporarily call $(*)$

> "Whenever the subsets of κ are partitioned into $cf(\kappa)$ many pieces, there is one piece which contains a Δ-system of size κ and its root."

We will first prove $(*)$ for the case $\kappa = \aleph_1$ and only then later indicate how to prove the general case.

Let $P(\omega_1) = \bigcup\{P_\gamma : \gamma < \omega_1\}$ be a partiton of the power set of ω_1. Let $\omega_1 = \bigcup\{S_\alpha : \alpha < \omega_1\}$ be any partition of ω_1 into $\aleph_1$ pairwise disjoint uncountable sets. From this we define for each $\zeta < \omega_1$ a proper σ-ideal I_ζ on $P(\omega_1)$ by letting I_ζ be all those $X \subseteq \omega_1$ such that
(a) for all $\alpha < \omega_1$, $|X \cap S_\alpha| < \aleph_1$, and
(b) for all $\alpha < \zeta$, $X \cap S_\alpha = \emptyset$.

Note that if $\eta < \zeta$, then $I_\eta \supseteq I_\zeta$.

We now begin the recursive construction of a sequence of triples $\langle X_\zeta, Y_\zeta, i(\zeta)$ where X_ζ and Y_ζ are subsets of ω_1 and $i(\zeta) \in \omega_1$. We will show that if at any stage ζ we cannot continue this construction, then we will be able to build the required Δ-system and its root. On the other hand, we will show that if this construction can be continued for all $\zeta < \omega_1$, then some $X \subseteq \omega_1$ was not put into any piece of the partition. For the construction, we insist that the following inductive hypotheses, which we call $INDHYP(\zeta)$, are satisfied at stage ζ.

(i) $X_\zeta \cap Y_\zeta = \emptyset$
(ii) $\bigcup\{X_\eta : \eta < \zeta\} \subsetneqq X_\zeta$, and $X_\zeta \setminus \bigcup\{X_\eta : \eta < \zeta\} \in I_\zeta$
(iii) $Y_\zeta \in I_0$ and $\bigcup\{Y_\eta : \eta < \zeta\} \subseteq Y_\zeta$
(iv) $X_\zeta \in P_{i(\zeta)}$ and for all $\gamma < i(\zeta)$ there is no set $X \in P_\gamma$ such that (i), (ii) and (iii) hold for some Y_γ and X replacing X_ζ.
(v) if $\emptyset \neq Z \in I_\zeta$ and $Z \cap (Y_\zeta \cup X_\zeta) = \emptyset$, then $(Z \cup X_\zeta) \notin P_{i(\zeta)}$.

First, suppose we have constructed $\langle X_\eta, Y_\eta, i(\eta)\rangle$ for all $\eta < \zeta$, but we cannot find an appropriate $\langle X_\zeta, Y_\zeta, i(\zeta)\rangle$. Note that each $X_\eta \in I_0$ and hence there is some x with

$$x \in X_\zeta \setminus \bigcup\{X_\eta \cup Y_\eta : \eta < \zeta\}$$

We let $X_\zeta^1 = \{x\} \cup \bigcup\{X_\eta : \eta < \zeta\}$ and $Y_\zeta^1 = \bigcup\{Y_\eta : \eta < \zeta\}$, and note that X_ζ^1 and Y_ζ^1 satisfy (i), (ii) and (iii) of $INDHYP(\zeta)$. Hence there must be some triple $\langle X_\zeta, Y_\zeta, i(\zeta)\rangle$ satisfying (i), (ii), (iii) and (iv) of $INDHYP(\zeta)$ and that means that (v) must fail for this triple. Therefore, there is some non-empty $Z_0 \in I_\zeta$ such that $Z_0 \cap (X_\zeta \cup Y_\zeta) = \emptyset$ and $Z_0 \cup X_\zeta \in P_{i(\zeta)}$. We can now recursively build $\{Z_\beta : \beta \in \omega_1\}$ by, at stage β, considering the triple $\langle X_\zeta, Y_\zeta^\beta, i(\zeta)\rangle$ where $Y_\zeta^\beta = Y_\zeta \cup \bigcup\{Z_\gamma : \gamma < \beta\}$. $\{Z_\beta : \beta \in \omega_1\}$ forms a Δ-system with root X_ζ.

Second, suppose $\langle X_\zeta, Y_\zeta, i(\zeta)\rangle$ have been constructed for all $\zeta < \omega_1$. Define

$$X_{\omega_1} = U\{X_\zeta : \zeta < \omega_1\}$$

and set $X_{\omega_1} \in P_\gamma$ in order to derive a contradiction. Note that for each $\zeta < \omega_1$,

$$X_\zeta \setminus \bigcup\{X_{eta} : \eta < \zeta\} \in I_\zeta$$

and hence

$$X_{\omega_1} \setminus \bigcup\{X_\eta : \eta < \zeta\} \in I_\zeta.$$

Therefore X_{ω_1} and Y_ζ satisfy (i), (ii) and (iii) of $INDHYP(\zeta)$, and so $\gamma \geq i(\zeta)$. Similarly each pair $X_{\zeta+1}, Y_\zeta$ satisfies (i), (ii) and (iii) of $INDHYP(\zeta)$, and so $i(\zeta+1) \geq i(\zeta)$. Also, by considering $Z = X_{\zeta+1} \setminus X_\zeta$ in (v), we see that $i(\zeta) \neq i(\zeta+1)$. This shows that $\gamma \notin \omega_1$ and completes the proof for the case $\kappa = \aleph_1$.

In general, when κ is regular the proof is similar to the case $\kappa = \aleph_1$. When κ is singular, we find an increasing cofinal sequence $\langle \kappa_\alpha : \alpha < cf(\kappa)\rangle$ of cardinals greater $cf(\kappa)$. The following changes need to be made. We partition $\kappa = \bigcup\{S_\alpha : \alpha < cf(\kappa)\}$ with each $|S_\alpha| = \kappa$.

We replace (a) by
(a^1) for all $\alpha < cf(\kappa)$, $|X \cap S_\alpha| \leq \kappa_\alpha$.

We can no longer deal with just triples. We deal with quadruples $\langle X_\zeta, Y_\zeta, i(\zeta), \delta(\zeta)\rangle$ where the new coordinate $\delta(\zeta)$ is an ordinal less than $cf(\kappa)$. We change I_ζ to $I_{\delta(\zeta)}$ in (ii) and (v) of $INDHYP(\zeta)$ and we add
(vi) for each $\eta < \zeta$ $\delta(\eta) < \delta(\zeta)$.

The proof now proceeds similarly to the case $\kappa = \aleph_1$. □

2.27 Corollary. *Suppose X is a dyadic space with weight having uncountable cofinality. Then*

$$X \to (top\, A_\kappa)^1_{cf(\kappa)}$$

where A_κ is the one point compactification of a discrete space of size κ.

Proof. This uses Theorem 2.26 and the topological fact that if X is a dyadic space and the cofinality of its weight κ is uncountable, then X contains a copy of $\{0,1\}^\kappa$. □

Earlier, G. Elekes, P. Erdös, and A. Hajnal had used a result of J. Baumgartner to give an elegant proof of (*) for the special case when κ is regular. In fact, they were able to prove more, but we do not include the proof here.

2.28 Theorem. *Suppose $\lambda < \kappa$ are cardinals. Whenever the subsets of κ are partitioned into κ many pieces there is one piece which contains a Δ-system of size λ and its root. Hence*

$$\{0,1\}^\kappa \longrightarrow (top\, A_\lambda)^1_\kappa.$$

3. Partitioning Pairs

We now consider the relation $X \longrightarrow (Y)^2_\lambda$. Far less is known here than for the case of singletons. There are few positive results; but an interesting one is the following, essentially due to Erdös and Rado.

3.1 Theorem. $\mathbb{R} \longrightarrow (top \;\; \omega + 1)^2_2$

Proof. Let $f : [\mathbb{R}]^2 \longrightarrow 2$ be a partition of the pairs from $\mathbb{R}$. Let $\{x_\alpha : \alpha < \omega_1\}$ enumerate an uncountable subset of $\mathbb{R}$.

For each $\alpha < \omega_1$ we try to construct a sequence σ_α of ordinals as follows. Let $\alpha_0 < \alpha$ such that $f(\{\alpha_0, \alpha\}) = 0$. If $\alpha_0, \ldots, \alpha_n$ have been constructed, let α_{n+1} be such that
(i) $\alpha_n < \alpha_{n+1} < \alpha$,
(ii) $|x_{\alpha_{n+1}} - x_\alpha| < \frac{1}{n+1}$, and
(iii) $f''[\{x_{\alpha_0}, \ldots, x_{\alpha_n}, x_{\alpha_{n+1}}, x_\alpha\}] = \{0\}$

If $\langle \alpha_n : n < \omega\rangle$ can be constructed for some α, then we have a homogeneous homeomorphic copy of $\omega + 1$ with x_α as the limit point. Therefore we can assume that for each $\alpha < \omega_1$ there is a finite sequence σ_α which cannot be extended. We define a function g on ω_1 by

$$g(\alpha) = \begin{cases} 0 & \text{if } \sigma_\alpha = \emptyset \\ \text{the last element of } \sigma_\alpha, & \text{otherwise.} \end{cases}$$

By the pressing down lemma, there is an uncountable $S \subseteq \omega_1$ and $\gamma \in \omega_1$ such that $g(\alpha) = \gamma$ for all $\alpha \in S$. We can then find an uncountable $T \subseteq S$ and a finite sequence σ such that
(iv) $\sigma_\alpha = \sigma$ for all $\alpha \in T$, and
(v) the diameter of T is less than $\frac{1}{|\sigma|+1}$, where $|\sigma|$ is the length of σ

Now pick $\beta < \alpha$ in T; if $f(\{x_\beta, x_\alpha\}) = 0$, then the sequence σ_α can be extended with β. Hence $f''[T]^2 = \{1\}$ and we can use part (ii) of Theorem 2.1 to extract a homeomorphic copy of $\omega + 1$ (or even more). □

The method of proof of this theorem now belongs to the folklore of the subject. This same method proves the following

3.2 Theorem. $\omega_1 \longrightarrow (top\ \ \omega + 1)^2_2$

A subset $A \subseteq \mathbb{R}$ is said to be second category if it is not the union of countably many nowhere dense sets. A classical construction of N. Luzin uses the continuum hypothesis, CH, to construct a subset $A \subseteq \mathbb{R}$ for which every uncountable $B \subseteq A$ is second category. Starting with such a set A and using the method of proof of Theorem 3.1, one can show the following.

3.3 Theorem. *Assume CH.* $\mathbb{R} \longrightarrow (top\ \ \omega + 1,\ \text{second category})^2$

Partitioning in $\aleph_0$ pieces gives mainly negative relations.

3.4 Theorem.
(i) $\mathbb{R} \not\longrightarrow (top\ \ \omega + 1)^2_{\aleph_0}$
(ii) For any ordinal κ, $\kappa \not\longrightarrow (top\ \ \omega + 1)^2_{\aleph_0}$

Proof. Part (i) follows from cardinality considerations and the ordinary partition calculus. Part (ii) needs a short proof. For each $\alpha < \kappa$ of cofinality $\aleph_0$, fix an increasing sequence $\langle \alpha_n : \kappa < \omega \rangle$ cofinal in α. For $\beta < \alpha < \kappa$, let

$$f(\{\beta, \alpha\}) = \begin{cases} n & \text{if } \alpha_n \leq \beta < \alpha_{n+1} \\ 0 & \text{otherwise.} \end{cases}$$

Verifying that f witnesses the negative partition relation completes the proof. □

We now state some results of J. Baumgartner, who has studied the square bracket relations on countable spaces.

3.5 Theorem. *For any positive integer n* $\mathbb{Q} \not\longrightarrow [top\ \ \omega^n + 1]^2_{2n}$.

3.6 Theorem. $\mathbb{Q} \not\longrightarrow [top\,\mathbb{Q}]^2_{\aleph_0}$ *and in fact even* $\mathbb{Q} \not\longrightarrow [top\,\omega^\omega]^2_{\aleph_0}$.

However, he also has this positive result, which shows that Theorem 3.5 is best possible.

3.7 Theorem. $Q \longrightarrow [top\ \ \omega^n + 1]^2_{k,2n}$ *for all* $k, n \geq 1$.

This last theorem says that if the pairs of rationals are partitioned into k pieces then there is a homeomorphic copy of $\omega^n + 1$ such that the pairs from it are distributed among at most $2n$ pieces. In fact, relating to Theorem 3.7, Baumgartner is able to determine for a given n which ordinals γ have the property $\gamma \longrightarrow [top\ \ \omega^n + 1]^2_{k,2n}$ for all k.

4. Open Problems

In conformity with the two previous sections, I would have liked to title this one 'Partitioning Triples', but then I wouldn't have had anything to write. The study of partitioning topological spaces is so new that triples have not yet been investigated. Furthermore, we have enough questions about partitioning singletons and pairs to keep us quite busy. I will explicitly mention four problems which intrigue me and have captured the interest of others as well.
The question of Nešetřil and Rödl has still not been completely answered.

4.1 Question. *Is it consitent that there is a space* X *such that* $X \longrightarrow (top\, 2^\omega)^1_2$?

A negative answer in the case of metric spaces X would still be interesting! The following problem is attributed to F. Galvin.

4.2 Question. *Does* $\mathbb{R} \longrightarrow [top\ Q]^2_3$?

This one is attributed to R. Laver.

4.3 Question. *Does* $\omega_1 \longrightarrow (top\, \alpha)^2_2$ *for all* $\alpha < \omega_1$?

To which I add the following.

4.4 Question. *Does* $\mathbb{R} \longrightarrow (top\, \alpha)^2_2$ *for all* $\alpha < \omega_1$?

There is some evidence for positive answers to the latter two questions.

4.5 Notation. We write $X \overset{*}{\longrightarrow} (Y)^m_n$ if whenever f is an n-to-one function with domain $[X]^m$, there is, in X, a homeomorphic copy H of Y such that f restricted to $[H]^m$ is one-to-one.

4.6 Lemma. $X \longrightarrow (Y)^m_n$ *implies* $X \overset{*}{\longrightarrow} (Y)^m_n$.

Proof. Given an n-to-one function f with domain $[X]^m$, define a function $g : [X]^m \longrightarrow n$ such that f is one-to-one on each $g^{-1}(j)$. Now a homogeneous Y for the first partition relation immediately gives one for the second. □

S. Todorčević has proven the following.

4.7 Theorem. *Let* α *be any countable ordinal and* n *any positive integer. Then*
(i) $\omega_1 \overset{*}{\longrightarrow} (top\, \alpha)^2_n$, *and*
(ii) $\mathbb{R} \overset{*}{\longrightarrow} (top\, \alpha)^2_n$.

There are many more questions to be answered before we can build a theory of partitions of topological spaces as rich as the theory of partitions of order types. For example, it is not known if there is an absolute example of a non-discrete space X of size $\aleph_1$ such that

$$X \longrightarrow (top\, X)^1_{\aleph_0}.$$

There is of course related to the Toronto seminar problem of whether there is an uncountable non-discrete space which is homeomorphic to each of its uncountable subspaces. There are rules for working on this latter problem. The problem can be worked upon only in groups of three or more mathematicians, and it is required that alcohol, preferably beer, be present during this time. Contact anyone in the Toronto Set Theory Seminar for the current status of the problem. It may never be solved.

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Topological Ramsey Theory

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We survey the interplay between topology and Ramsey Theory which began with Ellentuck's Theorem (Ellentuck 1974) (and was anticipated by work of Nash-Williams (1965), Galvin and Prikry (1973) and Silver (1970) by giving a fairly abstract treatment of what have become known as Ellentuck type theorems.

Section 1 is introductory. Section 2 contains an abstract version of Ellentuck's Theorem which reduces the verification of an Ellentuck type theorem to its "combinatorial part". Section 2 also contains the definition of the term "Ramsey space" – a topological structure which satisfies an Ellentuck type theorem. Finitary consequences of Ellentuck type theorems are discussed in section 3. The necessity of using the axiom of choice in constructing sets without the property of Baire in any of a large class of Ramsey spaces is discussed in section 4. In section 5, "finite dimensional" versions of Ellentuck type theorems are discussed. Section 6 contains a treatment of canonical partitions in connection with Ramsey spaces.

1. Introduction

The main purpose of this paper is to present a unified treatment of Ellentuck type theorems and closely related matters. We have concentrated on general principles and examples which illustrate them. No detailed proofs are given, and no attempt to comment on all, or even most, of the relevant results has been made.

Following the custom among logicians, a natural number n is identified with the set of its predecessors: $n = \{0, 1, \ldots, n-1\}$. ω is the set of natural

1 partially supported by NSF grant MCS 19-07774.
2 partially supported by NSF grant MCS 8107867.

numbers. $\langle a_0, a_1, \ldots, a_{n-1}\rangle$ denotes the sequence of length n whose i^{th} term is a_i. For sets X and Y , ${}^XY$ denotes the collection of functions from X into Y. So ${}^nY$ is the collection of all sequences of length n with values in Y. Functions with domain ω are called *infinite sequences.* $X^{<\omega}$ is the collection of finite sequences with values in X.

E is a *partition of a set I* if E is a collection of pairwise disjoint nonempty subsets of I and the union of the members of E is I. The elements of a partition E are called *blocks* of E. Note that the empty set is a partition of itself.

$[\omega]^\omega$ is the collection of infinite subsets of ω.

2. Ramsey Spaces and Ellentuck's Theorem

We begin by recalling Ellentuck's theorem. For A an infinite set of natural numbers and a a finite subset of A let $E(a, A)$ be the collection of all infinite sets of natural numbers which begin with a and are subsets of A. If n is a natural number $E(n, A)$ is defined to be $E(a, A)$ where a consists of the first n elements of A. The collection of sets $E(n, A)$ is a basis for the *Ellentuck topology* on $[\omega]^\omega$. We will write $\mathcal{E}$ for $[\omega]^\omega$ endowed with the Ellentuck topology. A subset X of $\mathcal{E}$ is *Ramsey* if for each basic set $E(n, A)$ there is B in $E(n, A)$ such that $E(n, B)$ is either contained in or disjoint from X. X is *Ramsey null* if for each basic set $E(n, A)$ there is B in $E(n, A)$ such that $E(n, B)$ is disjoint from X.

Ellentuck's Theorem. (Ellentuck 1974). *Suppose X is a subset of $\mathcal{E}$. X is Ramsey iff X has the property of Baire (with respect to the Ellentuck topology).*

A simple argument shows that $\mathcal{E}$ is a Baire space. Hence, the theorem implies that a subset of $\mathcal{E}$ is Ramsey null iff it is meager. Ellentuck's theorem is a very strong partition theorem for $[\omega]^\omega$ which immediately implies the Galvin-Prikry theorem.

Galvin-Prikry Theorem. (Galvin, Prikry 1973). *If $[\omega]^\omega$ is partitioned into finitely many Borel sets there exists $A \in [\omega]^\omega$ such that all infinite subsets of A are in the same block.*

Borel refers to the usual topology on $[\omega]^\omega$ which is induced from the product toplogy on ${}^\omega 2$ under the identification of a set with its characteristic function.

Definition 1. Suppose $\mathcal{I}$ is any set, p is a function with domain $\omega \times \mathcal{I}$ and $\leq$ is a partial order on $\mathcal{I}$. $\langle \mathcal{I}, p, \leq\rangle$ is called a *partial order with approximations* provided assumptions A1–A3 below hold.

A1. $p(0, A) = p(0, B)$ for all A and B in $\mathcal{I}$.
A2. If A and B are distinct elements of I then $p(n, A) \neq p(n, B)$ for some n.
A3. If $p(n, A) = p(m, B)$ then $n = m$ and $p(i, A) = p(i, B)$ for all $i < n$.

$p(n, A)$ is the n^{th} *approximation of A*. The *depth* of $p(n, A)$, $\#(p(n, A))$, is defined to be n.

We will use $p(A)$ to denote the sequence of approximations of A of positive depth. So $p(A)$ is a function with domain ω whose value at n is $p(n+1, A)$. Note that $\langle \mathcal{I}, p, \leq \rangle$ is a partial order with approximations iff it is isomorphic (in the obvious sense) to some $\langle \mathcal{J}, r, \preceq \rangle$ where $\mathcal{J}$ is a set of sequences of length ω, $\preceq$ is a partial ordering of $\mathcal{J}$ and r is the restriction function given by $r(n, S) = S\lceil n$.

Let P be the range of p. The elements of P will often be referred to simply as *approximations*. Give P the discrete topology and ${}^{\omega}P$ the product topology. The map which takes A to $p(A)$ induces a topology on $\mathcal{I}$ which we will somewhat inappropriately refer to as the *product topology* on $\mathcal{I}$.

We now describe a basis for a second topology on $\mathcal{I}$. If A and B are elements of $\mathcal{I}$ then we will say B is a *reduction* of A if $B \leq A$. If A is an element of $\mathcal{I}$ and b is an approximation then we will write $R(b, A)$ for the set of all reductions B of A such that b is an approximation of B. Let $R(n, A)$ denote $R(p(n, A), A)$. The collection of $R(n, A)$ is a base for a toplogy on $\mathcal{I}$ which we will call the *natural topology* on $\mathcal{I}$. Note that the natural toplogy is stronger than the product topology.

$\langle [\omega]^{\omega}, p, \leq \rangle$ is a partial ordering with approximations where $p(n, A)$ is the set consisting of the first n elements of A and $A \leq B$ iff A is a subset of B. In this case, $R(a, A) = E(a, A)$ and $R(n, A) = E(n, A)$. This leads to the following definition.

Definition 2. Suppose $\langle \mathcal{I}, p, \leq \rangle$ is a partial ordering with approximations and X is a subset of $\mathcal{I}$. X is *Ramsey* if for all $R(n, A)$ there exists $B \in R(n, A)$ such that $R(n, B)$ is either contained in or disjoint from X. X is *Ramsey null* if for all $R(n, A)$ there exists $B \in R(n, A)$ such that $R(n, B)$ is disjoint from X.

We next define a Ramsey space to be a partial ordering with approximations which satisfies an Ellentuck type theorem. Note that if $\langle \mathcal{I}, p, \leq \rangle$ is a partial ordering with approximations then every Ramsey set has the property of Baire and every Ramsey null set is nowhere dense (with respect to the natural topology).

Definition 3. Suppose $\langle \mathcal{I}, p, \leq \rangle$ is a partial order with approximations. $\langle \mathcal{I}, p, \leq \rangle$ is a *Ramsey space* if every set with the property of Baire is Ramsey and every meager set is Ramsey null.

The Ramsey spaces that have been studied in the literature have a great deal of structure in common which we attempt to capture by some additional assumptions on partial orders with approximations.

Assume $\langle \mathcal{I}, p, \leq \rangle$ is a partial order with approximations and $\trianglelefteq$ is a partial ordering of P, the set of approximations. If $a \trianglelefteq b$ we will say that a is a *reduction* of b. We will be concerned with the following assumptions on $\trianglelefteq$.

A4. If A and B are in $\mathcal{I}$ then A is a reduction of B iff every approximation of A is a reduction of an approximation of B.

A5. If a is an approximation then a has only finitely many reductions.

A6. The collection of $p(A)$ is a closed subset of ${}^{\omega}P$ i.e. if S is an infinite sequence of approximations and every finite initial segment of S is an initial segment of some $p(A)$ then S equals $p(A)$ for some A.

A7. If $B \in R(b, A)$ where b is a reduction of the n^{th} approximation of A then there is A' in $R(n, A)$ such that $R(b, A')$ is a subset of $R(b, B)$.

Theorem 1. (Carlson). *Suppose $\langle \mathcal{I}, p, \leq \rangle$ is a partial order with approximations which satisfies assumptions A4–A7. $\langle \mathcal{I}, p, \leq \rangle$ is a Ramsey space iff*

A8. If X is a set of approximations of depth $n+1$ and A is in $\mathcal{I}$ there exists $B \in R(n, A)$ such that either all the $(n+1)^{\text{th}}$ approximations of elements of $R(n, B)$ are in X or none of the $(n+1)^{\text{th}}$ approximations of elements of $R(n, B)$ are in X.

Generally, when proving a particular partial ordering with approximations is a Ramsey space assumptions A1–A7 are immediate. Theorem 1 is an abstract version of Ellentuck's theorem which is proved in essentially the same way. To derive Ellentuck's theorem from Theorem 1, define an approximation a to be a reduction of an approximation b if either both are empty or a is a nonempty subset of b with the same maximal element as b (this last condition is needed for A7). All assumptions A1–A8 are obvious.

Example 1 (Carlson). Suppose L is a finite alphabet. Fix a symbol v not in L which we will call a variable. A *variable word* (over L) is a finite sequence of symbols which are either in L or are v in which v occurs at least once. If t is a variable word and x is a symbol then $t(x)$ is the result of replacing all occurrences of v in t by x. In particular, $t(v)$ is just t. Let $\mathcal{I}$ be the collection of all infinite sequences of variable words and let p be the usual restriction function given by $p(n, S) = S \mid n$. So the approximations are simply finite sequences of variable words. We first define reductions between the approximations and then extend to the elements of $\mathcal{I}$ so that assumption A4 holds. Given an approximation $s = t_1, t_2, \ldots, t_n$ and a variable word $u = a_1, a_2, \ldots, a_n$ define $s(u)$ to be the variable word $t_1(a_1) * t_2(a_2) * \ldots * t_n(a_n)$ i.e. substitute the i^{th} symbol of n into the i^{th} term of s for each i and then concatenate the results. Similarly, if $u_1, u_2, \ldots, u_k$ are variable words such that the sum of the lengths of the u_i is the length of s then define $s(u_1, u_2, \ldots, u_k)$ to be a sequence of k variable words (the first term of $s(u_1, u_2, \ldots, u_k)$ is $s_1(u_1)$ where s_1 is the initial segment of s whose length is the same as the length of u_1 etc.). Extend the notion of reduction to I so that A4 holds. Assumptions A1–A7 are easily verified while A8 requires a delicate argument. If L is empty we obtain Ellentuck's theorem (for which A8 is trivial) and if L has exactly one symbol we obtain a theorem of Milliken (1975) (for which A8 is equivalent to Hindman's theorem (Hindman 1974)).

Note that any basic neighborhood in a Ramsey space has a natural Ramsey space structure. Suppose $\langle \mathcal{I}, p, \leq \rangle$ is a Ramsey space. If $n \in \omega$ and $A \in \mathcal{I}$ then $\langle R(n, A), q, \leq \rangle$ is a Ramsey space where $q(m, B) = p(n+m, B)$. This allows one to derive "local" versions of the results in the following sections. Also notice that a dense subset of a Ramsey space is a Ramsey space with the induced structure.

Axioms A4–A7 where chosen primarily in order to allow the usual arguments concerning Ellentuck type theorems to be carried out and then for their simplicity. Some of the axioms can be replaced by weaker, but more obscure, formulations. For example, in Theorem 1 A6 can be replaced by the assumption that the collection of $p(A)$ is sufficiently dense in its closure: If $n \in \omega$ and S is in the closure of the collection of $p(A)$ then there is $B \in \mathcal{I}$ such that $p(B) \mid n = S \mid n$ and every approximation of B is a reduction of a coordinate of S or of the approximation of depth 0.

3. Finitary Consequences of Ellentuck Type Theorems

Suppose $\langle \mathcal{I}, p, \leq \rangle$ is a Ramsey space and $\trianglelefteq$ is a partial ordering of the approximations. Under very general conditions we can derive a finitary version of the fact that $\langle \mathcal{I}, p, \leq \rangle$ is a Ramsey space.

Definition 4. Suppose n, h, e, c are natural numbers. The statement

$$n \to (h)^e_c \bmod \langle \mathcal{I}, p, \leq \rangle$$

menas that whenever the reductions of an approximation a of depth n are colored with c colors there exists a reduction b of a of depth h such that all reductions of b of depth e have the same color.

Theorem 2. *Assume that the notion of reduction is extended to the approximations so that*

A9. *If a is a reduction of an approximation of A then $R(a, A)$ is nonempty.*

A10. *Every approximation is a reduction of an approximation which is maximal (that is, which is a reduction only of itself) and no two maximal approximations have a common reduction.*

A11. *If a and b are approximations of the same depth then there is a correspondence between the reductions of a and the reductions of b which preserves order and depth.*

For all natural numbers h, e, c there is a natural number n such that $n \to (h)^e_c \bmod \langle \mathcal{I}, p, \leq \rangle$.

The proof of the theorem is a straightforward proof by contradiction. Assumptions A9–A11 are usually easy to verify. The result of applying this theorem to Ellentuck's theorem is essentially the finite version of Ramsey's theorem.

Example 2 (Carlson). Fix a finite field F. ${}^{\omega}F$ is a vector space under coordinatewise operations. Give F the discrete topology and give ${}^{\omega}F$ the corresponding product topology. Let $\mathcal{I}$ be the collection of infinite dimensional subspaces of ${}^{\omega}F$ which are closed in the product topology on ${}^{\omega}F$. There are natural projection maps of ${}^{\omega}F$ onto ${}^{n}F$ for each natural number n obtained by restriction to the first n values. Let $p(n+1, A)$ be the projection of A to ${}^{m}F$ where m is maximal such that the projection has dimension n. Choose the approximation of depth 0 arbitrarily so long as it is distinct from the other approximations. An approximation a is defined to be a reduction of an approximation b if it is a subspace of b or if both have depth zero. If A and B are in $\mathcal{I}$ define A to be a reduction of B if A is a subspace of B. $\langle \mathcal{I}, p, \leq \rangle$ is shown to be a Ramsey space in (Carlson). The conclusion of Theorem 2 in this case is the Graham-Leeb-Rothschild theorem for linear spaces (Graham, Leeb, Rothschild 1972) (the Graham-Leeb-Rothschild theorem for affine spaces can be derived from Theorem 2 by restricting this example to a basic neighborhood).

4. The Axiom of Choice and the Construction of Non-Ramsey Sets

Using the axiom of choice non-Ramsey sets can be constructed for almost any partial ordering with approximations. In fact, if $\leq$ is a partial ordering of a set $\mathcal{I}$ such that $\mathcal{I}$ has no minimal elements (that is, each element of $\mathcal{I}$ has an element of $\mathcal{I}$ which is strictly below it) then from a well-ordering of I one can construct a subset x of $\mathcal{I}$ such that any element of $\mathcal{I}$ has an element below it in x and an element below it which is not in x (to prove this it suffices to consider two cases: P is inversely well founded, i.e. there is no strictly increasing sequence, or each element of P has a strictly increasing sequence below it).

The use of the axiom of choice has been shown to be necessary for constructing non-Ramsey sets for certain Ramsey spaces (Milliken 1975, Carlson, Simpson 1984). We will extend this to a large class of Ramsey spaces in this section.

Suppose T is a collection of finite sequences and $\trianglelefteq$ is a partial ordering of T. Construct a partial order with approximations $\langle [T], r, \leq \rangle$ as follows (recall that [T] is the collection of all infinite sequences A such that every finite initial segment of A is in T). Let r be the usual restriction function with $r(n, A) = A \mid n$, and define $\leq$ on $[T]$ so that A4 holds i.e. so that $A \leq B$ iff every finite initial segment of A is less than or equal to a finite initial segment of B.

Note that if $\langle \mathcal{I}, p, \preceq \rangle$ is a partial ordering with approximations such that A4 and A6 hold for some partial ordering of the approximations then $\langle \mathcal{I}, p, \preceq \rangle$ is isomorphic to $\langle [T], r, \leq \rangle$ for some T and $\trianglelefteq$. This is accomplished by identifying each A in $\mathcal{I}$ with the infinite sequence $p(A) = \langle p(1, A), p(2, A), \ldots, p(n, A), \ldots \rangle$. T is the collection of all finite sequences $\langle p(1, A), \ldots, p(n, A) \rangle$ with the case $n = 0$ interpreted as the empty sequence. Since r is the restriction function, T will be the set of approximations for $\langle [T], r, \leq \rangle$. The ordering $\trianglelefteq$ on T is simply the ordering induced from the approximations for $\langle \mathcal{I}, p, \preceq \rangle$ under the identification of the unique approximation of depth 0 with the empty sequence and $p(n+1, A)$ with $\langle p(1, A), \ldots, p(n+1, A) \rangle$. Moreover, if the set of approximations of $\langle \mathcal{I}, p, \preceq \rangle$ is countable we may assume that T is a subset of $\omega^{<\omega}$ by identifying each approximation with a natural number. This leads us to the following definition.

Definition 5. Suppose T is a subset of $\omega^{<\omega}$ which contains the empty sequence and all initial segments of any of its members, each element of T has a proper extension in T and $\trianglelefteq$ is a partial ordering of T. $\langle T, \trianglelefteq \rangle$ is a *standard Ramsey space* if $\langle [T], r, \leq \rangle$ as constructed above satisfies A1–A8 with the ordering $\trianglelefteq$ of T.

Note that if T and $\trianglelefteq$ are as in the assumption of Definition 5 then $\langle [T], r, \leq \rangle$ and $\trianglelefteq$ already satisfy A1–A4 and A6.

Every Ramsey space which satisfies A4–A7 with some partial ordering of its approximations is locally isomorphic to a standard Ramsey space. Speicifically, if $\langle \mathcal{I}, p, \preceq \rangle$ is a Ramsey space which satisfies A4–A7 and $A \in \mathcal{I}$ then the collection of approximations of reductions of A is countable so by the remark above Definition 5 $\langle R(0, A), p, \preceq \rangle$ is isomorphic to a standard Ramsey space. Of course, "$\langle T, \trianglelefteq \rangle$ is isomorphic to $\langle \mathcal{I}, p, \preceq \rangle$" is intended to mean "$\langle [T], r, \leq \rangle$ is isomorphic to $\langle \mathcal{I}, p, \preceq \rangle$" here.

Theorem 3. *If ZFC plus "there exists an inaccessible cardinal" is consistent so is ZFC plus "if $\langle T, \trianglelefteq \rangle$ is a standard Ramsey space then every subset of $[T]$ which is definable from a sequence of ordinals is Ramsey".*

Recall that ZF is Zermelo-Fraenkel set theory and ZFC is Zermelo-Fraenkel set theory plus the axiom of choice. For the definition of "definable from a sequence of ordinals" we refer the reader to section 15 of Jech (1978). Suffice it to say for now that this is an extremely broad notion of definability and that ZFC plus "every set is definable from a sequence of ordinals" is consistent (though not simultaneously with the line in quotes at the end of Theorem 3 of course).

The proof of Theorem 3 proceeds by collapsing an inaccessible cardinal to become ω_1 and follows the pattern laid down in Solovay (1970) and elaborated in Mathias (1977) and Carlson, Simpson (1984). One introduces a notion of forcing for each standard Ramsey space $\langle T, \trianglelefteq \rangle$ which adds a generic element of $[T]$ in such a way that any reduction of a generic element is also generic.

Following Solovay (1970) Corollary 1 below can be derived.

Corollary 1. *If ZFC plus "there is an inaccessible cardinal" is consistent then so is ZF plus DC plus "if $\langle \mathcal{I}, p, \leq \rangle$ is a Ramsey space such that there exists a partial ordering of the approximations satisfying A4–A7 then every subset of $\mathcal{I}$ is Ramsey".*

DC is the axiom of dependent choices which says: if R is a subset of $A \times A$ for some set A and for each x in A there is a y in A such that (x, y) is in R then there exists x_n for $n \in \omega$ such that (x_n, x_{n+1}) is in R for all n. DC is a sufficient replacement for the axiom of choice in a large part of standard mathematics.

5. Finite Dimensional Analogues of Ellentuck Type Theorems

For many Ramsey spaces $\langle \mathcal{I}, p, \leq \rangle$ the elements of $\mathcal{I}$ can be viewed as infinite dimensional objects of some sort (in the present discussion one might keep in mind Example 2 from section 3). Moreover, there is usually a natural collection $\mathcal{F}$ of "finite dimensional" reductions of the elements in $\mathcal{I}$. Suppose that the notion of reduction is extended to $\mathcal{F}$ i.e. assume that $\leq$ is a partial ordering not just of $\mathcal{I}$ but of $\mathcal{F} \cup \mathcal{I}$. A finite dimensional analogue of the fact that $\langle \mathcal{I}, p, \leq \rangle$ is a Ramsey space might be something like: If $\mathcal{F}$ is partitioned into finitely many "nice" sets then there exists an element of $\mathcal{I}$ all of whose reductions in $\mathcal{F}$ are in the same block of the partition. This usually isn't the right approach since often $\mathcal{F}$ can be partitioned into sets $\mathcal{F}_k (k \in K)$ in a natural way such that every element of $\mathcal{I}$ contains reductions in each $\mathcal{F}_k$. In Example 2, each $\mathcal{F}_k$ consists of all finite dimensional subspaces of ${}^{\omega}F$ of some fixed dimension. The solution is to replace $\mathcal{F}$ by $\mathcal{F}_k$ for any k :

(∗) If $\mathcal{F}_k$ is partitioned into finitely many "nice" sets then there exists an element of $\mathcal{I}$ all of whose reductions are in the same block of the partition.

The problem is to determine which choices of "nice" make (∗) true.

A form of (∗) can often be deduced by chosing an appropriate projection of $\mathcal{I}$ to $\mathcal{F}_k$. Suppose that $f : \mathcal{I} \to \mathcal{F}_k$ has the property that

(∗∗) each A in $\mathcal{I}$ has a reduction B in $\mathcal{I}$ such that every reduction of B in $\mathcal{F}_k$ is of the form $f(C)$ for some reduction C of A.

We can then take "nice" to mean that the inverse image under f of each block of the partition has the property of Baire (and is therefore Ramsey). For we can choose A in $\mathcal{I}$ such that $f(C)$ is in the same block of the partition for all reductions C of A, and B can then be chosen so that all reductions of B in $\mathcal{F}_k$ have the form $f(C)$ for some reduction C of A.

In the case of Ellentuck's theorem $\mathcal{F}$ is the collection of all finite subsets of ω, and let $\mathcal{F}_k$ be $[\omega]^k$, the collection of all subsets of ω of size k, if $k \in \omega$. Define $f(A)$ to be the set consisting of the first k elements of A. (∗∗) holds with $B = A$. (∗) is the infinite version of Ramsey's theorem in this case.

Example 3 (Carlson, Simpson 1984). $(\omega)^\omega$ denotes the collection of all partitions of ω into infinitely many blocks. If A is an element of $(\omega)^\omega$ then a natural number n is called a *leader* of A if n is the least element of some block of A. $p(n+1, A)$ is the restriction of A to $\{0, 1, \ldots, m-1\}$ where m is the n^{th} leader of A. Define B to be a reduction of A if B is a coarser partition than A. In Carlson, Simpson (1984), $\langle (\omega)^\omega, p, \leq \rangle$ is shown to be a Ramsey space (in fact, if an approximation a is defined to be a reduction of an approximation b iff a is a coarsening of the partition b then A1–A8 hold).

For each positive natural number k let $\mathcal{F}_k$ be $(\omega)^k$, the collection of all partitons of ω into k blocks, and let $\mathcal{F}$ be the union of the $\mathcal{F}_k$ $(k \geq 1)$. If A is an element of $\mathcal{F} \cup \mathcal{I}$ then the blocks of A are assumed to be listed according to their minimal elements. So the 0^{th} block of A is the block of A which contains the 0^{th} leader of A (namely, 0) etc. Extend the notion of reduction to $\mathcal{F} \cup \mathcal{I}$ in the natural way by letting B be a reduction of A iff B is coarser than A.

Fix a positive natural number k. Define $f(A)$ for A in $\mathcal{F}$ to be the reduction of A in $\mathcal{F}_k$ whose i^{th} block is the same as the i^{th} block of A whenever $0 < i < k$. In other words, all of the blocks of A form the k^{th} block on are combined with the 0^{th} block to get $f(A)$. One easily ckecks that (**) holds.

There is a natural topology on $\mathcal{F}_k$ under which f is continuous (even with respect to the product topology on $\mathcal{I}$). Each element of $\mathcal{F}_k$ can be identified with an equivalence relation on ω which in turn can be identified with its characteristic function which is a member of ${}^{\omega\times\omega}2$. Under this identification, the product topology on ${}^{\omega\times\omega}2$ induces a property of Baire in $\mathcal{I}$ form a σ-algebra, the inverse image under f of any Borel subset of $\mathcal{F}_k$ has the property of Baire. Hence, (*) holds when we interpret "nice" as Borel.

Prömel and Voigt have shown that not only is (*) true with "nice" interpreted as Borel but more generally if "nice" is interpreted as the property of Baire. This can be derived directly as in (Prömel and Voigt (1985a)), or one can derive this from the instance of (*) with "Borel" by showing that meager subsets of $\mathcal{F}_k$ can be neglected. More specifically, if x is a meager subset of $\mathcal{F}_k$ then the collection of A in $\mathcal{I}$ which have a reduction in x is meager with respect to the product topology on $\mathcal{I}$ (see Carlson).

6. Canonical Partitions

If E is a partition of I we will identify E with an equivalence relation on I. This allows us to view E as a subset of $I \times I$.

Fix a Ramsey space $\langle \mathcal{I}, p, \leq \rangle$. If A is in $\mathcal{I}$ let *Red*(A) be the collection of reductions of A (this is the same as $R(0, A)$ but we will modify this definition later). If E is a partition of $\mathcal{I}$ (or of *Red*(A) for some A) and B is in $\mathcal{I}$ (is a reduction of A) we call the restriction of E to *Red*(B) a *reduction* of E.

If E is a partition of $\mathcal{I}$ into finitely many blocks each of which has the property of Baire then, since $\langle \mathcal{I}, p, \leq \rangle$ is a Ramsey space, there exists A such that *Red*(A) is contained in one block of E. Generally, there is no hope of finding such A if E has infinitely many blocks. Nevertheless, perhaps if E is nice enough we can find an A in $\mathcal{I}$ such that the partition E restricted to *Red*(A) is as simple as possible in some sense. When should we say a restriction to *Red*(A) is "as simple as possible"? We need a way of comparing a partition of *Red*(A) with a partition of *Red*(B).

Usually the sets *Red*(A) are naturally order isomorphic with each other. We assume there are maps $h_{A,B}$ for $A, B \in \mathcal{I}$ such that for all A, B, C in $\mathcal{I}$

(a) $h_{A,B}$ is an order preserving bijection from *Red*(A) to *Red*(B).
(b) $h_{A,C} = h_{B,C} \circ h_{A,B}$.
(c) $h_{B,A}$ is the inverse of $h_{A,B}$.

Let $\mathcal{H}$ be the system of maps $h_{A,B}$. If E is a partition of *Red*(A) define $h_{A,B}(E)$ to be the partiton of *Red*(B) induced by E via $h_{A,B}$ i.e. two reductions of B are in the same block of $h_{A,B}(E)$ iff their preimages under $h_{A,B}$ are in the same block of E. A partition E of *Red*(A) is *canonical* (with respect to $\mathcal{H}$) if for all reductions B of A the restriction of E to *Red*(B) equals $h_{A,B}(E)$ i.e. two reductions of A are in the same block of E iff their images under $h_{A,B}$ are.

Ideally, all "nice" partitions of $\mathcal{I}$ would reduce to canonical partitions. Unfortunately, there are usually very simple partitions which don't have canonical reductions. In the case of Ellentuck's space, say that two infinite subsets of ω are equivalent iff they have the same minimal element, call it n, and they have the same first n elements. This partition is an open subset of $[\omega]^\omega \times [\omega]^\omega$ (even using the product topology on $[\omega]^\omega$) but has no canonical reductions with respect to the natural system of maps induced by the order preserving bijections between elements of $[\omega]^\omega$.

Even though we cannot prove such strong results about canonical partitions there are two profitable alternatives.

The first alternative is to show that every nice partition reduces to a partition which is very simple in some sense (see Prömel and Voigt (1985)). In this situation we say that the "simple" partitions form a *basis* for the "nice" partitions. Note that the collection of reductions of the partition given above does not have a basis which is minimal under inclusion. This implies that no collection of partitions which includes all reductions of the partition above can have a minimal basis.

The second alternative is to carry out the scenario above for partitions of "finite dimensional" reductions of the elements of $\mathcal{I}$. So suppose that $\leq$ is extended to $\mathcal{F} \cup \mathcal{I}$ for some set $\mathcal{F}$ which is disjoint form $\mathcal{I}$. We will say that elements of $\mathcal{F}$ are *finite dimensional* and elements of $\mathcal{I}$ are *infinite dimensional.* We also assume that $\mathcal{F}$ is partitioned into sets $\mathcal{F}_k (k \in K)$ (see the discussion in section 5). The elements of $\mathcal{F}_k$ are said to have *dimension* k. We will modify the definitions made earlier in this section. Let *Red*(A) be defined for all elements of $\mathcal{F} \cup \mathcal{I}$ to be the collection of reductions of A i.e. the collection of B in $\mathcal{F} \cup \mathcal{I}$ with $B \leq A$. If $k \in K$ let *Red*(k, A) be the collection of reductions of A of dimension k. We assume that whenever A and B have the same dimension (infinite dimensional objects are considered to have the same dimension) there is a map $h_{A,B}$ such that for all A, B, C, D with A, B, C of the same dimension (a)–(c) above hold along with

(d) $h_{A,B}$ preserves dimension.
(e) If D is a reduction of A then the restriction of $h_{A,B}$ to *Red*(D) is $h_{D,D'}$ where $D' = h_{A,B}(D)$.

Redefine $\mathcal{H}$ to be this system of maps $h_{A,B}$. If A and B are infinite dimensional and E is a partition of *Red*(k, A) then define $h_{A,B}(E)$ to be the partition of *Red*(k, B) which is induced by E via $h_{A,B}$, and if B is a reduction of A the restriction of E to *Red*(k, B) will be called a *reduction* of E. E is *canonical* (with respect to $\mathcal{H}$) if $h_{A,B}(E)$ equals the restriction of E to $Red(k, B)$ for all infinite dimensional reductions B of A.

Fix $k \in K$. We want to prove

$(***)$ Every "nice" partition of $\mathcal{F}_k$ reduces to a canonical partition.

for some broad interpretation of "nice". This can often be done either by arguments similar to those outlined in section 5 for establishing $(*)$ or by first proving a version of $(*)$ and then deriving $(***)$ as a corollary. A derivation of $(***)$ from $(*)$ is usually based on two facts

(i) Each finite dimensional object has only finitely many reductions.
(ii) There's $d \in K$ such that whenever $A, B \in \mathcal{F}_k$ and $A, B \leq C \in \mathcal{I}$ then there is $D \in \mathcal{F}_d$ such that $A, B \leq D \leq C$.

Given a parition E of $\mathcal{F}_k$ one can then partition $\mathcal{F}_d$ so that A and B in $\mathcal{F}_d$ are equivalent iff the restriction of E to $R(k, A)$ and $R(k, B)$ are of the same type, meaning that $h_{A,B}$ (the restriction of E to $Red(k, A)$) equals the restriction of E to $Red(k, B)$. This partition of $\mathcal{F}_d$ has only finitely many blocks so $(*)$ can be applied if E is nice enough. If $Red(d, A)$ is in one block of the partition on $\mathcal{F}_d$ then the restriction of E to $Red(k, A)$ will be canonical.

Establishing $(***)$ is the easy part. One then wants to investigate the structure of canonical partitions and perhaps show they all admit very simple descriptions (see Erdös, Rado (1950) and Prömel, Simpson, Voigt (1984)).

Let's consider the version of $(***)$ we get from Ellentuck's theorem. For $k \in \omega$ let $\mathcal{F}_k$ be $[\omega]^k$, the collection of subsets of ω of cardinality k. Let $A \leq B$ iff A is a subset of B. The maps $h_{A,B}$ are induced from the order preserving bijection between A and B when they have the same dimension (which simply means they have the same cardinality). (i) and (ii) clearly hold. This allows us to conclude from $(*)$, which is Ramsey's theorem in this case, that $(***)$ holds for arbitrary partitions of $[\omega]^k$. Erdös and Rado (1950) showed that the canonical partitions of $[\omega]^k$ are very simple: A partition E of $[\omega]^k$ is canonical iff there exists a set $s \subseteq \{0, 1, \ldots, k-1\}$ such that A and B are equivalent just in case the i^{th} element of A equals the i^{th} element of B for all i in s.

The discussion in this section should be compared with remarks on canonical partition theorems in Voigt (1984) and Prömel, Voigt (1983).

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Ergodic Theory and Configurations in Sets of Positive Density

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1. Introduction

We shall present here two examples from "geometric Ramsey theory" which illustrate how ergodic theoretic techniques can be used to prove that subsets of Euclidean space of positive density necessarily contain certain configurations. Specifically we will deal with subsets of the plane, and our results will be valid for subsets of "positive upper density". For any measurable subset $E \subset \mathbb{R}^2$ we let S range over all squares in the plane and we set

$$\overline{D}(E) = \limsup_{\ell(s)\to\infty} m(S \cap E)/m(S)$$

where $\ell(S)$ denotes the length of a side of S. $\overline{D}(E)$ is the *upper density* of E and we shall be concerned with sets E having $\overline{D}(E) > 0$. Our first result is

Theorem A. *If $E \subset \mathbb{R}^2$ has positive upper density then $\exists \ell_0$ so that for any $\ell > \ell_0$ one can find a pair of points $x, y \in E$ with $\|x - y\| = \ell$.*

We could say that the configuration $\{x, y\}$ is *congruent* to the configuration $\{0, \ell\} \subset \mathbb{R}$.
The next result deals with triangles.

Theorem B. *Let $E \subset \mathbb{R}^2$ have positive upper density and let E_δ denote the points of distance $< \delta$ from E. Let $u, v, \in \mathbb{R}^2$, then $\exists \ell_0$ so that for $\ell > \ell_0$ and any $\delta > 0$ there exists a triple $\{x, y, z\} \subset E_\delta$ forming a triangle congruent to $\{0, \ell u, \ell v\}$.*

Theorem A answers a question posed by L. Szekely (1983). Since we announced this result it has been proved by other methods by J. Bourgain who has also given several refinements (Bourgain 1986) and also by Falconer and Marstrand (1986). Bourgain also has shown by an example that the result of

Theorem B cannot be improved to finding triangles in E itself (instead of the "thickened" set of E_δ). We shall reproduce this in Section 7. The underlying idea in our proof of both Theorems A and B is the possibility of attaching to a measurable subset of the plane a measure preserving action of $\mathbb{R}^2$ on a particular measure space. If $E \subset \mathbb{R}^2$ is the subset in question we shall obtain an "$\mathbb{R}^2$ measure preserving system" $(X, \mathcal{B}, \mu, T_u)$, where $(X, \mathcal{B}, \mu)$ denotes the measure space, $T_u, u \in \mathbb{R}^2$ denotes the measure preserving action, and a subset $\tilde{E} \subset X$ so that every "recurrence" of the set $\tilde{E}$:

(*) $\mu(\tilde{E} \cap T_{u_1}^{-1}\tilde{E} \cap \ldots \cap T_{u_k}^{-1}\tilde{E}) > 0$
implies a "reccurence" of the thickened set E_δ :
(**) $E_\delta \cap (E_\delta - u_1) \cap \ldots \cap (E_\delta - u_k) \neq 0$.

Ergodic theory will enable us to establish results of the form $(*)$ and the foregoing correspondence then guarantees the existence of points $x \in E_\delta$ with

$$x + y_1, x + u_2, \ldots, x + u_k \in E_\delta.$$

In the case of Theorem A we shall be able to pass from the existence of configurations in E_δ to the existence of (simpler) configurations in E. In the case of Theorem B we shall have to be satisfied with results regarding E_δ.

2. Correspondence Between Subsets of $\mathbb{R}^2$ and $\mathbb{R}^2$-Actions

Let $E \subset \mathbb{R}^2$ be an arbitrary subset and set

$$\varphi'(u) = \varphi'_E(u) = \text{dist}(u, E) = \inf\{\|u - v\| \mid v \in E\}.$$

Where $\|u - v\|$ denotes the euclidean metric in $\mathbb{R}^2$. If

$$\varphi(u) = \min\,\{\varphi'(u), 1\}$$

then $\varphi(u)$ is a bounded uniformly continuous function on $\mathbb{R}^2$ with

$$|\varphi(u_1) - \varphi(u_2)| \leq \|u_1 - u_2\|. \tag{2.1}$$

The functions $\Psi_v(u) = \varphi(u + v)$ form an equicontinuous family and have compact closure in the topology of uniform convergence over bounded sets in $\mathbb{R}^2$. Denote this closure by X; thus $\Psi \in X$ if there is a sequence $\{v_n\} \subset \mathbb{R}^2$ with

$$\varphi(u + v_n) \to \Psi(u)$$

uniformly for $d(u, 0) < R$, for each $R < \infty$.

$\mathbb{R}^2$ acts on X with $T_v\Psi(u) = \Psi(u+v)$ for $\psi \in X, u, v \in \mathbb{R}^2$. The function φ belongs to X and its orbit $\{T_v\varphi\}_{v \in \mathbb{R}^2}$ is dense in X. X is a compact metrizable space and we can identify borel measures on X with functionals on $C(X)$.

Suppose now that $\overline{D}(E) > 0$ so that there exists a sequence of squares $S_n \subset \mathbb{R}^2$ with $\ell(S_n) \to \infty$ and

$$\frac{m(S_n \cap E)}{m(S_n)} \to \overline{D}(E), \tag{2.2}$$

with $\overline{D}(E) > 0$. Using the sequence of squares $\{S_n\}$ we shall define a probability measure on X. Namely refine the sequence $\{S_n\}$ so that

$$\lim_{k\to\infty} \frac{1}{m(S_{n_k})} \int_{S_{n_k}} f(T_v\varphi)dm(v)$$

exists for every $f \in C(X)$. Such a subsequence can be found since it can be found simultaneously for a countable dense set of functions $f \in C(X)$. We now define the measure μ on X by

$$\int_X f d\mu = \lim_{k\to\infty} \frac{1}{m(S_{n_k})} \int_{S_{n_k}} f(T_v\varphi)dm(v). \tag{2.3}$$

Set $f_0(\Psi) = \Psi(0)$. By definition of the topology on X, f_0 is a continuous function. We define $\tilde{E} \subset X$ by

$$\Psi \in \tilde{E} \Leftrightarrow f_0(\Psi) = 0 \Leftrightarrow \Psi(0) = 0.$$

$\tilde{E}$ is a closed subset of X and we have

$$\mu(\tilde{E}) = \lim_{\ell\to\infty} \int_X (1 - f_0(\Psi))^\ell d\mu(\Psi). \tag{2.4}$$

Lemma 2.1. $\mu(\tilde{E}) \geq \overline{D}(E)$.

Proof. By (2.4), it suffices to show that for any ℓ,

$$\int_X (1 - f_0(\Psi))^\ell d\mu(\Psi) \geq \overline{D}(E).$$

By (2.3),

$$\begin{aligned}\int_X (1 - f_0(\Psi))^\ell d\mu(\Psi)) &= \lim_{k\to\infty} \frac{1}{m(S_{n_k})} \int_{S_{n_k}} (1 - f_0(T_v\varphi))^\ell dm(v) \\ &= \lim_{k\to\infty} \frac{1}{m(S_{n_k})} \int_{S_{n_k}} (1 - \varphi(v))^\ell dm(v).\end{aligned}$$

Since $\varphi(v) = 0$ for $u \in E$, the last expression is at least

$$\lim_{k\to\infty} \frac{m(E \cap S_{n_k})}{m(S_{n_k})} = \overline{D}(E). \qquad \square$$

We now establish the correspondence between E and $\tilde{E}$ described in the Introduction.

Proposition 2.2. *Let $E \subset \mathbb{R}^2$ and $\tilde{E} \subset X$ be as above. If for a k-tuple of vectors, $\{u_1, \ldots, u_k\}$ we have*

$$\mu(\tilde{E} \cap T_{u_1}^{-1}\tilde{E} \cap \ldots \cap T_{u_k}^{-1}\tilde{E}) > 0, \tag{2.5}$$

then for all $\delta > 0$,

$$E_\delta \cap (E_\delta - u_1) \cap \ldots \cap (E_\delta - u_k) \neq \emptyset.$$

Proof. Define the function $g(\Psi)$ on X by

$$g(\Psi) = \begin{cases} \delta - f_0(\Psi), & \text{if } f_0(\Psi) < \delta \\ 0, & \text{if } f_0(\Psi) \geq \delta. \end{cases}$$

$g(\Psi)$ is positive for $\Psi \in \tilde{E}$ and (2.5) implies that

$$\int g(\Psi)g(T_{u_1}\Psi)\ldots g(T_{u_k}\Psi)d\mu > 0.$$

In particular for some v, the integrand is > 0. Since $g(T_w\varphi) > 0 \leftrightarrow \varphi(w) < \delta \Leftrightarrow w \in E_\delta$ this implies for some v,

$$v \in E_\delta,\ v + u_1 \in E_\delta,\ \ldots,\ v + u_k \in E_\delta$$

and this is the assertion of the proposition. □

3. Ergodic Averages for Subsets of $\mathbb{R}^2$

The following is the variant of the ergodic theorem which we use.

Theorem 3.1. *Let G be a locally compact abelian group and let $T_g, g \in G$ be a measure preserving action on a probability space $(X, \mathcal{B}, \mu)$. Let m_u be a sequence (one parameter family, etc.) of probability measures on G such that*

$$\lim_{u \to \infty} \hat{m}_u(\gamma) = 0, \gamma \in \hat{G}, \gamma \neq 0. \tag{3.1}$$

Denote by P the orthogonal projection of $L^2(X)$ on the subspace of G-invariant elements. Then

$$T^{(u)} = \int T_g dm_u(g) \to P \tag{3.2}$$

in the strong topology.

Proof. For $f \in L^2(X), < T_g f, f >$ is positive-definite on G and is in fact the Fourier transform of $\nu_f(\gamma)$–the spectral measure of f on $\hat{G}$.

Recall that $\|f\|^2 = \int d\nu_f$, that ν_{Pf} is the part of ν_f carried by $\{0\} \in \hat{G}$, and that $T^{(u)}Pf = Pf$ for all u. Thus

$$T^{(u)}f - Pf = T^{(u)}(f - Pf)$$

and by (3.1)

$$\|T^{(u)}f - Pf\|^2 = \int |\hat{m}_u(\gamma)|^2 \, d(\nu_f - \nu_{Pf}) \to 0. \qquad \square$$

Theorem 3.2. *Let T_z be an $\mathbb{R}^2$-action on $(X, \mathcal{B}, \mu)$, and let P denote the orthogonal projection of $L^2(X)$ onto the subspace of T_z-invariant functions. Then for $0 \le a < \beta \le 2\pi$*

$$\frac{1}{\beta - a} \int_a^\beta T_{Re^{i\theta}} \, d\theta = P \tag{3.3}$$

as $R \to \infty$, in the strong operator topology for $L^2(X)$.

Proof. Apply Theorem 3.1. The parameter $u = R$ and m_u is normalized arc length in the arc $Re^{i\theta}, a \le \theta \le \beta$. To check condition (3.1) we write $\zeta = (\xi, \mu) = re^{i\varphi}$ and

$$\hat{m}_R(\zeta) = \frac{1}{\beta - a} \int_a^\beta e^{iR(\xi \cos\theta + \mu \sin\theta)} d\theta = \frac{1}{\beta - a} \int_a^\beta e^{iRr\cos(\theta - \varphi)} d\theta.$$

Apply Van der Corput's Lemma (Zygmund 1955) to obtain

$$M_R(\zeta) = O(r^{-1/2} R^{-1/2}) \text{ as } R \to \infty. \qquad \square$$

4. First Application to Subsets of Positive Density in $\mathbb{R}^2$

Theorem 4.1. *Let $E \subset \mathbb{R}^2$ be a subset of positive density, $\overline{D}(E) > 0$. Let $\epsilon > 0$ be given as well as numbers $0 \le a_1 < \beta_1 < a_2 < \beta_2 < \ldots < a_N < \beta_N \le 2\pi$. Then for all sufficiently large R there exists points $z_0, z_1, \ldots, z_N \in E$ so that writing $z_j - z_0 = r_j e^{i\theta_j}, 0 \le \theta_j < 2\pi$, we have*

(i) $|r_j - R| < \epsilon$

(ii) $a_j < \theta_j < \beta_j$ *for* $j = 1, 2, \ldots, N$.

Proof. If δ is small and we find points $z'_0, z'_1, \ldots, z'_N$ in E_δ satisfying

(i') $r_j = R$

(ii) $a'_j < \theta'_j < \beta'_j$ for $j = 1, 2, \ldots, N$ with $z'_j - z'_0 = r'_j e^{i\theta'_j}$ and with $a_j < a'_j < \beta'_j < \beta_j$, then there will be points $z_0, z_1, \ldots, z_N$ in E as required. We shall use Proposition 2.2 to show the existence of the desired configurations in E_δ by way of intersection properties of $\tilde{E}$. To obtain the relevant properties of $\tilde{E}$ we make use of Theorem 3.2.

The operator P in Theorem 3.2 is a positive self-adjoint operator, so that $< Pf, f > \ge 0$ for all $f \in L^2(X)$. Also $P1 = 1$. Setting $f = 1_A - \mu(A)$ for a subset $A \subset X$ we deduce that $< P1_A, 1_A > \ge \mu(A)^2$. This implies that for a.e. $x \in A, P1_A(x) > 0$. For if $B = \{x \in A \mid P1_A(x) = 0\}$ we will have $< P1_B, 1_B > \leq < P1_A, 1_B >= 0$, so that $\mu(B) = 0$.

Now apply Theorem 3.2 to the function $1_{\tilde{E}}$ to obtain for each $j, j = 1, 2, \ldots, N$,

$$\frac{1}{\beta'_j - a'_j} \int_{a'_j}^{\beta'_j} T_{Re^{i\theta}} 1_{\tilde{E}} d\theta \to P1_{\tilde{E}} \tag{4.1}$$

in $L^2(X, \mathcal{B}, \mu)$ as $R \to \infty$, where $(X, \mathcal{B}, \mu, T_u)$ is the system attached to $E \subset \mathbb{R}^2$. Since the function of the right in (4.1) is positive for almost every $x \in \tilde{E}$, it follows that if R is sufficiently large, the expression on the left will also be positive for all $x \in \tilde{E}$ but for, say, a subset of measure $< \frac{\mu(\tilde{E})}{2N}$. Hence for at least half of the points $x \in \tilde{E}$, all N of the averages in (4.1), $j = 1, 2, \ldots, N$ are positive and the product

$$\frac{1}{\Pi_1^N(\beta'_j - a'_j)} \int_{a'_1}^{\beta'_1} \cdots \int_{a'_N}^{\beta'_N} \Pi_1^N(T_{Re^{i\theta_j}} 1_{\tilde{E}})\, d\theta_1 \ldots d\theta_N$$

is positive. Multiplying by $1_{\tilde{E}}$ and integrating over X, we conclude that for some $\theta_1 \in (a'_1, \beta'_1), \theta_2 \in (a'_2, \beta'_2), \ldots, \theta_N \in (a'_N, \beta'_N)$,

$$\mu(\tilde{E} \cap T^{-1}_{Re^{i\theta_1}} \tilde{E} \cap \ldots \cap T^{-1}_{Re^{i\theta_N}} \tilde{E}) > 0.$$

Proposition 2.2 now gives the desired result. □

Because of the approximative nature of Theorem 4.1 it is easily seen that the result will be valid for a set E if it is true for arbitrarily small thickenings E_δ. Thus it will be true if each $E_\delta, \delta > 0$, has positive upper density. This happens, for example, if E is a subset of the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ which, relative to the lattice has positive upper density. This gives the following.

Theorem 4.2. *Let E be a subset of a lattice $\lambda = c\mathbb{Z}^2$ with positive upper density. Let $\epsilon > 0$ be given as well as numbers $0 \leq a_1 < \beta_1 < a_2 < \beta_2 < \ldots < a_N < \beta_N \leq 2\pi$. Then for all sufficiently large R there exists points $z_0, z_1, \ldots, z_N \in E$ so that writing $z_j - z_0 = r_j e^{i\theta_j}, 0 \leq \theta_j < 2\pi$, we have*
(i) $|r_j - R| < \epsilon$
(ii) $a_j < \theta_j < \beta_j$ for $j = 1, 2, \ldots, N$. □

5. Proof of Theorem A

We can now prove Theorem A. Let $E \subset \mathbb{R}^2$ be measurable and with positive upper density. Denote by $\mathbb{Q}_k$ the partition of $\mathbb{R}^2$ into squares of side 2^{-k}, determined by the lattice $2^{-k}\mathbb{Z}^2$. A number $\beta > 0$ is *eligible* for $\mathbb{Q}_k$ if the set of squares $Q \in \mathbb{Q}_k$, in which the relative measure of E exceeds β, has positive upper density. It is clear that if β is eligible for $\mathbb{Q}_k$, it is eligible for $Q_{k+\ell}$ for $\ell > 0$, since every $Q \in Q_k$ in which the relative measure of E exceeds β contains at least one $Q' \in \mathbb{Q}_{k+l}$ with the same property. Define $\beta^* = \sup\{\beta; \exists k$ such that β is eligible for $\mathbb{Q}_k\}$, the positive upper density of E insures that $\beta^* > 0$. The key observation now is that if $\beta < \beta^*$ but is very close to it and if we partition a typical square of $\mathbb{Q}_k$ in which the relative measure of E exceeds β into its $\mathbb{Q}_{k+\ell}$ subsquares, all of the $2^{2\ell}$ subsquares will have the property that the relative measure of E in them exceeds $\beta^*/2$. The proper order of quantifiers here is: for any $\ell > 0$ there exists $\beta_\ell < \beta^*$ such that the above is valid for $\beta > \beta_\ell$. The observation is that if the relative measure of E in one of the subsquares is lower than the average, other subsquares have to conpensate but none of them, typically, can exceed the average by more than $\beta^* - \beta$.

We now set $N = 30(\beta^*)^{-2}, \ell >> \log N, \beta^* > \beta > \beta_\ell, k$ large enough so that β is eligible for $\mathbb{Q}_k$ and denote by F the set of lower left hand corners of the squares $Q \in \mathbb{Q}_k$ in which the relative density of E exceeds β, and are "typical" in the sense discussed above.

$F \subset 2^{-k}\mathbb{Z}^2$ and has there positive upper density. We apply Theorem 4.2 with $\epsilon << 2^{-k}, a_j = \frac{\pi j}{3N} - \frac{1}{N^2}, \beta_j = \frac{\pi j}{3N} + \frac{1}{N^2}, j = 1, \dots, N$ and obtain $R_0 > 0$ such that given any $R > R_0$ we have a configuration $z_0, \dots, z_N \in 2^{-k}\mathbb{Z}^2$ such that $z_j - z_0 = r_j e^{iv_j}$ with $|r_j - R| < \epsilon$ and $\left|v_j - \frac{\pi j}{3N}\right| < \frac{1}{N^2}$. We denote by $Q_0, Q_1, \dots, Q_N$ the corresponding $\mathbb{Q}_k$ squares and set $E_j = E \cap Q_j, j = 0, \dots, N$; by the definition of F, $m(E_j) > \beta m(Q_j) = \beta 2^{-2k}$, and for convenience we normalize all measures by a factor 2^k so that $m(Q_j) = 1$ and $m(E_j) > \beta$. Our final step is to evaluate the measure of the set of points in Q_0 which are at distance R from some point in $\bigcup_{j=1}^N E_j$ and show that the measure exceeds $1 - \beta$. Once we do that we are done because it implies that this set must have positive intersection with E_0.

Denote by G_j the subset of Q_0 of points whose distance from some point in E_j is R. G_j is a union of circular arcs, intersection of Q_0 with circles of radius R centered at points in E_j. Divide Q_j into its $\mathbb{Q}_{k+\ell}$ subsquares and in each of the principal diagonal subsquares find a subset of E_j contained in a horizontal sugment of full length and of relative linear measure equal to β. Denote the set so obtained by E'_j and the corresponding subset of Q_0 by G'_j. We shall estimate the (planar) measures of G'_i and $G'_i \cap G'_j$. G'_j is a union of arcs from circles of radius R and to estimate the measures in question we approximate these arcs by line segments. The following is now an approximate description of G'_j. Through the lower–left and upper–right vertices of Q_0 pass two lines orthogonal to the direction $\theta = \frac{\pi j}{3N}$. Divide the strip formed by these lines into 2^ℓ equal strips. Let S_j be the union of the lines also orthogonal to $\theta = \frac{\pi j}{3N}$, such that S_j meets each of the 2^ℓ strips in a fixed proportion β_j of the strip. The β_j will be bounded from below. Then G'_j is approximately $S_j \cap Q_0$.

Let P be a parallelogram formed by intersecting one of the narrow strips corresponding to $\theta = \frac{\pi j}{3N}$ with one of the narrow strips corresponding to $\theta = \frac{\pi i}{3N}$.

We have

$$m(S_j \cap P) = \beta_j m(P), m(S_j \cap P) = \beta_j m(P)$$
$$m(S_i \cap S_j \cap P) = \beta_i \beta_j m(P).$$

By choosing ℓ very large we can assume that Q_0 differs by an arbitrarily small amount from the union of parallelograms such as P. Hence we will have

$$m(G'_j) \approx \beta_j,\ m(G'_i) \approx \beta_i,\ m(G'_i \cap G'_j) \approx \beta_i \beta_j.$$

Let $\beta' = \inf\ \beta_i$. We now show that for ℓ large

$$m(\bigcup G'_j) > 1 - \frac{1}{N\beta'^2} - \delta \tag{5.1}$$

for arbitrarily small $\delta > 0$. Set $\varphi_i = \beta_i - 1_{G'_i}$ on Q_0. We consider the situation $m(G'_i) = \beta_i$ and $m(G'_i \cap G'_j) = \beta_i \beta_j$. On $Q_0 \setminus \bigcup G'_j, \sum \varphi_j \geq N\beta'$ and so

$$m(Q_0 \setminus \bigcup G'_j) N^2 \beta'^2 \leq \| \sum \varphi_i \|^2_{Q_0}.$$

Now

$$< \varphi_i, \varphi_i >_{Q_0} = \beta_i - \beta_i^2, < \varphi_i, \varphi_j > = 0$$

so that $\| \sum \varphi_i \|^2_{Q_0} = \sum \beta_i - \sum \beta_i^2 \leq N$ and

$$m(Q_0 \setminus \bigcup G'_j) \leq \frac{1}{N\beta'^2}$$

this would give

$$m(\bigcup G'_i) \geq 1 - \frac{1}{N\beta'^2}$$

and since ℓ can be choosen very large we have (5.1).

Finally we note that from our construction

$$\beta = \frac{\beta \cos \frac{\pi j}{3N}}{\cos \frac{\pi j}{3N} + \sin \frac{\pi j}{3N}} \geq \frac{\beta}{1 + \tan \frac{\beta}{3}}.$$

Hence choosing N large after β has been prescribed we can assure that some G'_i meets E_0. This proves Theorem A. □

6. A Recurrence Property of $\mathbb{R}^2$–Actions

In this section we prove a certain recurrence result for $\mathbb{R}^2$–actions from which Theorem B will follow.

We begin with a lemma regarding mean values of vectors in a Hilbert space.

Lemma 6.1. *Let $\{u_m\}$ be a bounded sequence of vectors in a Hilbert space $\mathcal{H}$; assume that*

(6.1)
$$\gamma_m = \lim_{N\to\infty} \frac{1}{N} \sum < u_n, u_{n+m} >$$

exists, and that

(6.2)
$$\lim \frac{1}{M} \sum_1^M \gamma_m = 0.$$

Then

$$\frac{1}{N} \sum_1^N u_n \to 0$$

in the norm of $\mathcal{H}$.

Proof. We choose M large so that the average in (6.2) is small and we choose N large with respect to M. Having done so the two expressions

$$\frac{1}{N} \sum_{n=1}^N u_n, \quad \frac{1}{NM} \sum_{n=1}^M \sum_{m=1}^M u_{n+m}$$

will be close, the vectors u_n being bounded. In general one has

$$\|\frac{1}{N} \sum_1^N y_n\|^2 \le \frac{1}{N} \sum_1^N \|y_m\|^2.$$

So up to a small error

$$\|\frac{1}{N} \sum_{n=1}^N u_n\|^2$$

will be bounded by

$$\frac{1}{N} \sum_{m=1}^N \|\frac{1}{M} \sum_{m=1}^M u_{n+m}\|^2 = \frac{1}{NM^2} \sum_{n=1}^N \sum_{m_1,m_2=1}^M < u_{n+m_1}, u_{n+m_2} > .$$

Let $N \to \infty$ and it is easily seen that this expression goes to 0. □

The same proof yields the following uniform version.

Lemma 6.2. *For each* ξ *in some index set* Ξ *let* $u_n(\xi) \in \mathcal{H}$, *such that all the* $u_n(\xi)$ *are uniformly bounded. Assume for each* m *the limits*

$$\gamma_m(\xi) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N < u_n(\xi), u_{n+m}(\xi) >$$

exist uniformly, and that

$$\lim_{M\to\infty} \frac{1}{M} \sum_1^M \gamma_m(\xi) = 0$$

uniformly. Then

$$\frac{1}{N}\sum_{n=1}^{N} u_n(\xi) \to 0$$

in $\mathcal{H}$ uniformly in ξ.

We shall need the following notion.

Definition. An action of a locally compact abelian group G by measure preserving transformations T_g of a measure space $(X, \mathcal{B}, \mu)$ is a *Kronecker action* if X is a compact abelian group, μ = Haar measure on X, and we have a homomorphism $\tau, \tau : G \to X$ with $\tau(G)$ a dense subgroup of X and

$$T_g(x) = \tau(g) + x.$$

Theorem 6.3. *If $(X, \mathcal{B}, \mu, T_g)$ is an ergodic measure preserving action of a abelian group G then there is a map $\pi : X \to Z$ where Z is a compact abelian group, and a Kronecker action T_g on Z so that $T_g\pi(x) = \pi(T_g x)$ for $x \in X$. For every character χ on Z the function $\chi'(x) = \chi(\pi(x))$ satisfies*

$$\chi'(T_g x) = \chi(\tau(g) + \pi(x)) = \chi(\tau(g))\chi'(x)$$

and so is an eigenfunction of the G–action, and, moreover, every eigenfunction of the G–action comes about this way.

We refer the readers to Fürstenberg (1981) for the proof of this.

The next theorem is a consequence of the fact that the eigenvectors of the tensor product of two unitary operators are spanned by tensor products of the eigenvectors.

Theorem 6.4. *Let T be a measure preserving transformation on the space $(X, \mathcal{B}, \mu)$ and let S be a measure preserving transformation on the space $(Y, \mathcal{D}, \nu)$. If $F \in L^2(X \times Y, \mathcal{B} \times \mathcal{D}, \mu \times \nu)$ satisfies $F(Tx, Sy) = F(x, y)$ a.e., and if $f \in L^2(X, \mathcal{B}, \mu)$ is orthogonal to all eigenfunctions of the transformation $f \to Tf$ where $Tf(x) = f(Tx)$, then*

$$\int F(x, y) f(x) d\mu(x) = 0$$

a.e. *on Y.*

We now take $G = \mathbb{R}^2$ and we consider an ergodic $\mathbb{R}^2$–action on a space $(X, \mathcal{B}, \mu)$. The following proposition is also presented without proof. The proof is based on the notion of the spectral measure (class) of an $\mathbb{R}^2$–action and the manner in which this determines the spectral measure for the restriction of the action of subgroups on $\mathbb{R}^2$.

Proposition 6.5. *Let $(X, \mathcal{B}, \mu, T_u)$ denote an ergodic $\mathbb{R}^2$–action. But for a countable set of 1-dimensional subgroups of $\mathbb{R}$, each subgroup $\mathbb{R}u_0$ contains at most a countable subset of elements v for which T_v is not ergodic on $(X, \mathcal{B}, \mu)$.*

Lemma 6.6. *Let $(X, \mathcal{B}, \mu, T_u)$ denote an ergodic action, and suppose T_v is ergodic on $(X, \mathcal{B}, \mu)$. Then every eigenfunction of T_v is an eigenfunction for the $\mathbb{R}^2$–action.*

Proof. $\varphi(T_u x) = \lambda\varphi(x)$. Then for each u,

$$\varphi(T_u T_v x) = (T_v T_u x) = \lambda\varphi(T_u x).$$

Since $|\lambda| = 1, \varphi(T_u x)\overline{\varphi(x)}$ is seen to be invariant under $x \to T_v x$. By ergodicity of T_v this is constant. Moreover $|\varphi(x)|$ is constant by ergodicity so that

$$\overline{\varphi(x)} = \frac{c}{\varphi(x)}$$

and we conclude $\varphi(T_u x) =$ constant $\varphi(x)$. □

The foregoing results are combined in the following.

Proposition 6.7. *Let $(x, \mathcal{B}, \mu, T_v)$ be an ergodic action of $\mathbb{R}^2$ and let v_1, v_2 be such that T_{v_1} and $T_{v_2-v_1}$ act ergodically. Let f, g be bounded measurable functions on X and suppose that f is orthogonal to all eigenfunctions of the $\mathbb{R}^2$–action. Then for all $w_1, w_2 \in \mathbb{R}^2$*

$$\frac{1}{N}\sum_{n=1}^{N} T_{w_1+nv_1} f T_{w_2+nv_2} g \to 0 \tag{6.4}$$

in $L^2(X, \mathcal{B}, \mu)$, uniformly in (w_1, w_2).

Proof. This will be an application of Lemma 6.2 with $\xi = (w_1, w_2)$, $\mathcal{H} = L^2(x, \mathcal{B}, \mu)$ and

$$u_n(\xi) = T_{w_1+nv_1} f T_{w_2+nv_2} g.$$

We have

$$\begin{aligned} < u_n(\xi), u_{n+m}(\xi) > &= \int T_{w_1+nv_1} f T_{w_1+nv_1+mv_1} \overline{f} T_{w_2+nv_2} g T_{w_2+nv_2+mv_2} \overline{g} d\mu \\ &= \int (T_{w_1-w_2} f T_{w_1-w_2+mv_1} \overline{f}) T^n_{v_2-v_1}(g T_{mv_2}\overline{g}) d\mu. \end{aligned}$$

Since $T_{v_2-v_1}$ acts ergodically

$$\frac{1}{N}\sum_{n=1}^{N} T^n_{v_2-v_1}(g T_{mv_2}\overline{g}) \to \int g T_{mv_2}\overline{g} d\mu$$

and this expression is independent of (w_1, w_2). Hence

$$\frac{1}{N}\sum_{n=1}^{N} < u_n(\xi), u_{n+m}(\xi) > \to \gamma_m$$

where

$$\gamma_m = \int T_{w_1-w_2} f T_{w_1-w_2+mv_1} \bar{f} d\mu \int g T_{mv_2} \bar{g} d\mu$$
$$= \int f T_{mv_1} \bar{f} d\mu \int g T_{mv_2} \bar{g} d\mu$$

and this convergence is uniform in (w_1, w_2). Finally

$$\frac{1}{M} \sum_1^M \gamma_m \to \int \int f(x) g(y) F(x,y) d\mu(x) d\mu(y)$$

where

$$F(x,y) = \lim \frac{1}{M} \sum \bar{f}(T_{v_1}^m x) \bar{g}(T_{v_2}^m y)$$

which is well defined by the ergodic theorem. Now by Lemma 6.6 since f is orthogonal to all eigenfunctions of the $\mathbb{R}^2$–action, it is orthogonal to all eigenfunctions of T_{v_1}. But clearly

$$F(T_{v_1} x, T_{v_2} y) = F(x,y)$$

so that we may apply Theorem 6.4 to conclude that

$$\int f(x) F(x,y) d\mu(x) = 0$$

and hence that

$$\frac{1}{M} \sum_1^M \gamma_m \to 0.$$

This yields the proposition. □

Let $(X, \mathcal{B}, \mu, T_v)$ denote an ergodic $\mathbb{R}^2$–action and let (Z, T_v) represent the Kronecker factor of $(X, \mathcal{B}, \mu, T_v)$. We have a map $\pi : X \to Z$ which defines a "disintegration" of the measure μ to measures $\mu_z, z \in Z$, with μ_z supported for each z by $\pi^{-1}(z)$. The map

$$f \to \int f d\mu_z = \hat{f}(z)$$

takes $L^2(X)$ to $L^2(Z)$. If we lift functions on Z to X then the foregoing map represents the projection of $L^2(X)$ to $L^2(Z) \circ \pi \subset L^2(X)$ and by Theorem 6.3, $L^2(Z) \circ \pi$ is the subspace of $L^2(X)$ spanned by eigenfunctions of $\{T_u\}$. It follows that for each $f \in L^2(X)$, $f - f \circ \pi$ is orthogonal to the subspace of $L^2(X)$ spanned by eigenfunctions. From this it is easy to deduce from the foregoing proposition:

Theorem 6.8. *Let $(X, \mathcal{B}, \mu, T_v)$ be an ergodic $\mathbb{R}^2$–action; let (Z, T_v) denote its Kronecker factor, and let $f, g, h \in L^\infty(X, \mathcal{B}, \mu)$. If $\hat{f}, \hat{g}, \hat{h}$ denote the correspon-ding functions on Z and if T_{v_1} and $T_{v_1-v_1}$ act ergodically on $(X, \mathcal{B}, \mu)$, then*

$$\frac{1}{N}\sum_{n=1}^{N}\int f(T_{w_1+nv_1}x)g(T_{w_2+nv_2}x)h(x)d\mu$$
$$-\frac{1}{N}\sum_{n=1}^{N}\int_Z \hat{f}(T_{w_1+nv_2}\xi)\hat{g}(T_{w_2+nv_2}\xi)\hat{h}(\xi)d\xi$$

converges to 0 as $N \to \infty$, uniformly in w_1, w_2.

The main result of this section is the following theorem. In formulating this theorem we identify $\mathbb{R}^2$ with $\mathbf{C}$ and we can then multiply elements of $\mathbf{C}$.

Theorem 6.9. *Let $(X, \mathcal{B}, \mu, T_z)$ be a measure preserving action of C. Let $A \in \mathcal{B}$ with $\mu(A) > 0$ and let $\omega \in C$. There exists ℓ_0 so that for all $\ell > \ell_0$ there will exist $z \in C$ with $|z| = \ell$ for which*

$$\mu(A \cap T_z^{-1}A \cap T_{\omega z}^{-1}A) > 0.$$

Proof. One first shows by a standard argument that it suffices to treat the case of an ergodic $\mathbf{C}$–action. By Proposition 6.5 we can find z_1 so that T_{z_1} and $T_{(\omega-1)z_1}$ are both ergodic. We will later impose a further restriction on z_1 which will be consistent with the present restriction.

Set $f = 1_A$ and consider $\hat{f}(\xi)$ defined on Z. We see that $\hat{f}(\xi)$ is a non–negative function which is strictly positive on a set of positive measure of Z. There will be a neighborhood W of the identity of Z so that if $w_1, w_2 \in W$,

$$\int \hat{f}(\xi)\hat{f}(\xi + w_1)\hat{f}(\xi + w_2)d\xi > a > 0,$$

for some appropriate a.
Define a homomorphism $\sigma : \mathbf{C} \to Z \times Z$ by

$$\sigma(z) = (\tau(z), \tau(\omega z))$$

(see the definition of a Kronecker action), and let $\Omega = \overline{\sigma(\mathbf{C})}$. A non–trivial character on Ω restricts to a non–trivial character on $\mathbf{C}$ which has the form $\chi(z) = e^{i\mathcal{R}zz'}$. An element z will have $\{\sigma(nz)\}_{n\in\mathbb{Z}}$ dense in Ω unless one of these characters is trivial on z, and so we see that for z outside of a countable set of lines $\sigma(z)$ generates a dense subgroup of Ω. Let us suppose that z_1 satisfies this restriction.

From the fact that $\sigma(z_1)$ generates a dense subgroup of Ω it is easy to show that there exists a number L so that for each $(\xi_1, \xi_2) \in \Omega$, some

$$(\xi_1 + \tau(nz_1), \xi_2 + \tau(n\omega z_1)) \in W \times W$$

with $0 \le n < L$. W econclude that

$$\frac{1}{N}\sum_{n=1}^{N}\int \hat{f}(\xi)\hat{f}(\xi + \tau(w_1 + nz_1))\hat{f}(\xi + \tau(\omega w_1 + n\omega z_1))d\xi > \frac{a}{2L}$$

whenever $N \ge L$, for all $w_1 \in \mathbf{C}$.

Now apply Theorem 6.8 to deduce that there exists N_0 so that if $N \geq N_0$

$$\frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T_{w_1+nz_1}^{-1} A \cap T_{\omega w_1+n\omega z_1}^{-1} A) > \frac{a}{3L},$$

for all $w_1 \in \mathbf{C}$. In particular, for each $w_1 \in \mathbf{C}$, $\exists n \leq N_0$ with

$$\mu(A \cap T_{w_1+nz_1}^{-1} A \cap T_{\omega w_1+n\omega z_1}^{-1} A) > \frac{a}{3L}.$$

Our G–actions are required to be continuous in the sense that for any measurable set $A \subset X, \epsilon > 0, \exists$ a neighborhood of the identity $V \subset G$ so that for $g \in V, \mu(T_g A \Delta A) < \epsilon$. Take $\epsilon < \frac{a}{8L}$. Then if w' is sufficiently close to $w_1 + nz_1$ we will have

$$\mu(A \cap T_{w'}^{-1} A \cap T_{\omega w'}^{-1} A) > \frac{a}{3L} - \frac{a}{4L} = \frac{a}{12L}. \tag{6.6}$$

To prove the theorem, choose w_1 a vector length 1 which is orthogonal to z_1. Then for large ℓ

$$|\ell w_1 + nz_1| = \ell + \frac{n^2\,|z_1|^2}{|\ell w_1 + nz_1| + \ell} = \ell + 0(\frac{1}{\ell})$$

for n in restricted range. It follows that for large ℓ we can find w' with $|w'| = \ell$ so that (6.6) is true. This proves the theorem. □

7. Proof of Theorem B

Theorem B is an immediate consequence of the foregoing theorem and Proposition 2.2. For $E \subset \mathbb{R}^2$ we form the $\mathbb{R}^2$–action described in Section 2. Consider $u, v \in \mathbb{R}^2$ as complex numbers and with $v = \omega u$, and apply Theorem 6.9 to $A = \tilde{E}$ and this ω. For each z with

$$\mu(\tilde{E} \cap T_z^{-1} \tilde{E} \cap T_{\omega z}^{-1} \tilde{E}) > 0$$

and for any $\delta > 0$ there exists $a, \beta, \gamma \in E_\delta$ with $\beta - a = z,\ \gamma - a = \omega z$ and so the triangle $\{a, \beta, \gamma\}$ is congruent to $\{0, z, \omega z\}$ which is congruent to $\{0, \ell u, \ell v\}$ with $\ell = |z| \,/\, |u|$.

We conclude with the example given by J. Bourgain (1986) which shows that the configurations of Theorem B may not exist in E itself.

Let $E = \{(x, y) \in \mathbb{R}^2 \mid \exists n \in \mathbb{Z}$ with $|x^2 + y^2 - n| < \frac{1}{10}\}$, and let the "triangle" of Theroem B be $\{0, u, 2u\}$. It is easily checked that E has (uniform) density $\frac{1}{5}$. Since for vectors v', v'',

$$\|v' + v''\|^2 + \|v' - v''\|^2 = 2\|v'\|^2 + 2\|v''\|^2$$

and so

$$(7.1) \qquad \|v'\|^2 + \|v''\|^2 - 2\|\frac{v'+v''}{2}\|^2 = 2\|\frac{v'-v''}{2}\|^2,$$

then if $v', v'', \frac{v'+v''}{2} \epsilon E$, the expressions to the left of (7.1) differs from an integer by less than $\frac{2}{5}$ dnd so $2\|\frac{v'-v''}{2}\|^2$ cannot be $\frac{1}{2}$ an odd integer. This means that $\|v' - v''\|$ does not attain all possible large values in such a triple.

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Part V

Variations and Applications

Topics in Euclidean Ramsey Theory

Ronald L. Graham

1. Introduction

Many questions in Ramsey Theory can be placed in the following context. We are given a set X, a family $\mathcal{F}$ of distinguished subsets of X, and a positive integer r. We would like to decide whether or not the following statement holds: For any partition of $X = X_1 \cup \ldots \cup X_r$ into r classes, there is an $F \in \mathcal{F}$ and an index i such that $F \subseteq X_i$.

Such an F is usually called *homogeneous* (or *monochromatic*, if the partition of X is thought of as an r-coloring of X; we will use both terminologies interchangeably).

The key feature which distinguishes *Euclidean* Ramsey Theory from other branches of Ramsey Theory is the use of the Euclidean metric in determining the structure of $\mathcal{F}$. More precisely, X is usually taken to be Euclidean n-space $\mathbb{E}^n$ for some n, and $\mathcal{F} = \mathcal{F}(C)$ consists of all subsets F which are *congruent* to a given point set $C \subseteq \mathbb{E}^n$.

The requirement that the homogeneous set be congruent to C is quite stringent. For example, if C consists of three equally spaced collinear points then it turns out (as we shall see) that for any $n, \mathbb{E}^n$ can always be 4-colored with no monochromatic congruent copy of C formed, whereas monochromatic *homothetic* copies of C must always exist, as shown by van der Waerden's theorem, for example.

In this chapter I will survey some of the basic results in Euclidean Ramsey Theory as well as describing some very recent theorems and numerous open problems.

2. Preliminaries

Let us say that $R(C, n, r)$ holds if any r-coloring of $\mathbb{E}^n$ contains a monochromatic set congruent to C. Thus, for example, if C' is the set of three vertices of a unit equilateral triangle then $R(C', 4, 2)$ holds (by considering the five vertices of a unit simplex in $\mathbb{E}^4$) while $R(C', 2, 2)$ does not hold (by partitioning $\mathbb{E}^2$ into two classes of alternating strips of width $\sqrt{3/2}$, each open on the top and closed on the bottom.

Slightly more interesting is the following.

2.1 Theorem. *If S is the set of four vertices of a unit square then $R(S, 6, 2)$ holds.*

Proof. Consider the set $X \subseteq \mathbb{E}^6$ defined by $X = \{(x_1, \ldots, x_6) : x_i = 1/\sqrt{2}$ for exactly two values of i, and $x_i = 0$ for all other values of $i\}$. Any partition of $\mathbb{E}^6$ into two classes, say $\chi : \mathbb{E}^6 \to \{0, 1\}$, also partitions X into two classes. To each point $(x_1, \ldots, x_6) \in X$, we can associate a pair $\{i, j\}$ by letting i and j be the indices of the nonzero coordinates of $(x_1, \ldots, x_6)$. Thus, χ induces a 2-coloring of the edges of K_6, the complete graph on six vertices. It is a standard result in (Ramsey) graph theory that in any such 2-coloring, a monochromatic 4-cycle, say $c_1 \to c_2 \to c_3 \to c_4 \to c_1$, must be formed. It is now straightforward to check that this 4-cycle corresponds to the four vertices of a unit square in X, and the theorem is proved. □

It is no accident that in the examples we have given up to this point, proofs that $R(C, n, r)$ holds for some C were always accomplished by selecting only a suitable *finite* subset of $\mathbb{E}^n$ and coloring it (rather than all of $\mathbb{E}^n$). A standard compactness argument (see Graham, Rothschild, Spencer 1980) shows that this is always the case, although it is often far from obvious what the appropriate finite subset should be.

Before proceeding to more general considerations, we first discuss a tantalizing question which besides being among the most fundamental in the theory, illustrates quite clearly how little we [1] still know about what is going on in this area. For this example we take C to be the set C^* consisting of two points separated by distance 1.

To begin with, it is easy to see that $R(C^*, 2, 2)$ holds, simply by considering (as the suitable finite set) the set of three vertices of a unit equilateral triangle. To show that $R(C^*, 2, 3)$ holds, we need only consider the graph G (known as the Moser graph) shown in Fig. 1. Each edge $\{x, y\}$ of G denotes the fact that the distance between x and y is 1.

A simple calculation shows that the chromatic number of G is 4. Thus, any 3-coloring of $\mathbb{E}^2$ induces a 3-coloring of (infinitely many copies of) G and consequently, always produces a monochromatic pair of points at unit distance from each other, as claimed.

In the other direction, it is not difficult to 7-color the standard tiling of $\mathbb{E}^2$ by regular hexagons of side 9/10 so that no color class contains two points separated by distance 1. Thus, $R(C^*, 7, 2)$ does not hold. The least value d for which $R(C^*, d, 2)$ holds is also known as the *chromatic number* $\chi(\mathbb{E}^2)$ of $\mathbb{E}^2$, since it is the chromatic number of the (uncountable) graph formed by taking each point of $\mathbb{E}^2$ as a vertex and each pair $\{x, y\}$ with distance 1 between x

1 "we" meaning combinatorialists collectively, in this case.

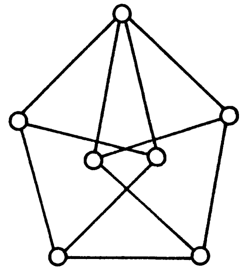

Fig. 1. The Moser graph

and y, as an edge. Thus, the best available bounds for $\chi(\mathbb{E}^2)$ are:

$$4 \leq \chi(\mathbb{E}^2) \leq 7.$$

There is some evidence that $\chi(\mathbb{E}^2) \geq 5$ from the result of Wormald (1979), who showed that $\mathbb{E}^2$ contains a (finite) graph of chromatic number 4, with all edges of length one and containing no 3–cycle and no 4–cycle.

For the chromatic number $\chi(\mathbb{E}^n)$ of $\mathbb{E}^n$, it has been recently shown by Frankl and Wilson (1981), using a powerful result on set systems with restricted intersections, that $\chi(\mathbb{E}^n)$ grows exponentially with n, verifying an earlier conjecture of Erdös. The best current bounds on $\chi(\mathbb{E}^n)$ are now:

$$(1+o(1))(6/5)^n \leq \chi(\mathbb{E}^n) \leq (3+o(1))^n.$$

3. Ramsey Sets

A basic concept in Euclidean Ramsey Theory is that of a Ramsey set.

Definition. A configuration C is said to be *Ramsey* if for all r there exists an $N = N(C,r)$ such that $R(C,N,r)$ holds.

An easy argument shows that no infinite set can be Ramsey. The following result forms the basis for constructing essentially all known Ramsey sets.

3.1 Theorem *(Erdös, Graham, Montgomery, Rothschild, Spencer, Straus 1973). If C_1 and C_2 are Ramsey then the cartesian product $C_1 \times C_2$ is Ramsey.*

Proof. Fix $C_1 \subseteq \mathbb{E}^m$, $C_2 \subseteq \mathbb{E}^n$ and let r be a positive integer. Choose u so that $R(C_1,u,r)$ holds. By the compactness theorem mentioned earlier there exists a finite set $T \subseteq \mathbb{E}^u$ such that in any r-coloring of T, a monochromatic congruent copy of C_1 is formed. Let $t = |T|$ and let $T = \{x_1, x_2, \ldots, x_t\}$. Choose v so that $R(C_2, v, r^t)$ holds (which is possible since C_2 is Ramsey). We claim that $R(C_1 \times C_2, u+v, r)$ holds. To prove the claim, suppose $\chi : \mathbb{E}^{u+v} \to \{1,2,\ldots,r\}$ is an r-coloring of $\mathbb{E}^{u+v}$. Define an induced coloring $\chi' : \mathbb{E}^v \to \{1,2,\ldots,r^t\}$ by

$$\chi'(y) = (\chi(x_1,y), \chi(x_2,y), \ldots, \chi(x_t,y)).$$

By the choice of v there is a χ'-monochromatic congruent copy of C_2, say $\overline{C}_2$, in $\mathbb{E}^v$. Now define an induced r-coloring χ'' of T by $\chi''(x_i) = \chi(x_i, y)$ for some $y \in \overline{C}_2$. This is well-defined since $\overline{C}_2$ is χ'-monochromatic. It is now straightforward to check that T contains a monochromatic (under the original coloring χ) copy of $C_1 \times C_2$. This proves the claim and consequently, the theorem follows. □

Since any two-point set is Ramsey then arbitrary cartesian products of two-point sets, i.e., the sets of vertices of rectangular parallelepipeds, are also Ramsey (and, of course, any *subset* of these sets of vertices). An interesting question which arises in this context is that of determining which simplexes (i.e., $(n+1)$-subsets of $\mathbb{E}^n$ in general position) are subsets of the vertex set of a rectangular parallelepiped. A *necessary* condition is that no angle determined by three of its vertices should exceed 90°. This condition turns out to be *sufficient* for $n = 2$ and $n = 3$. However, it is not sufficient for $n \geq 4$. Indeed, it is not difficult to construct a five-point simplex in $\mathbb{E}^4$ with all angles determined by three points being less than 89°, and which cannot be extended to the vertex set of any rectangular parallelepiped.

An ingenious construction recently discovered by Frankl and Rödl can be used to show that *any* set of three non-collinear points is Ramsey (thus partially resolving a conjecture in Graham (1980)). The idea behind their construction is the following. For arbitrary fixed k and r, and $n = n(k, r)$ chosen suitably large, consider the subset $X \subseteq \mathbb{E}^n$ formed as follows. For each subset $I \subseteq \{1, 2, \ldots, n\}$ of size $2k-1$, say $I = \{i_1, i_2, \ldots, i_{2k-1}\}$, define $x = x_I = (x_1, x_2, \ldots, x_n)$ by taking

$$x_j = \begin{cases} j & \text{if } j = i_u \text{ for } u = 1, 2, \ldots, k, \\ 2k - j & \text{if } j = i_u \text{ for } u = k+1, \ldots, 2k-1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, a typical point x looks like:

$$x = (0, \ldots, 0, 1, 0, \ldots, 0, 2, 0, \ldots, 0, k, 0, \ldots, k-1, \ldots, 0, 1, 0, 0)$$

X is taken to be $\{x_I : I \subseteq \{1, \ldots, n\} \text{ with } |I| = 2k-1\}$. Consider now an arbitrary r-coloring of X. This induces an r-coloring of the set of all $(2k-1)$-subsets of $\{1, 2, \ldots, n\}$. Hence, if n is large enough then by Ramsey's Theorem there is a $(2k+1)$-subset $Y \subseteq \{1, 2, \ldots, n\}$ having all *its* $(2k-1)$-subsets in a single color. Suppose we write Y as $\{i_1, i_2, \ldots, i_{2k+1}\}$. Consider the three points x_{I_1}, x_{I_2} and x_{I_3} where

$$\begin{aligned} I_1 &= \{i_1, i_2, \ldots, i_{2k-1}\}, \\ I_2 &= \{i_2, i_3, \ldots, i_{2k}\}, \\ I_3 &= \{i_3, i_4, \ldots, i_{2k+1}\}. \end{aligned}$$

A straightforward calculation shows that distance(x_{I_1}, x_{I_2}) =distance(x_{I_2}, x_{I_3}) $= \sqrt{2k}$, distance$(x_{I_1}, x_{I_3}) = \sqrt{8k-2}$. Thus, x_{I_1}, x_{I_2} and x_{I_3} form an (arbitrarily) obtuse monochromatic isoceles triangle. Various obtuse triangles can now be formed from the sets of vertices of prisms created by taking the product of X with two-point sets. To form arbitrary obtuse triangles a similar technique is used, but with greater "shifts" of the $(1, 2, \ldots, k, \ldots, 2, 1)$ positions. Presumably, every non-degenerate simplex is Ramsey[2]. It would also be interesting to know whether such simple sets such as the five vertices of a regular pentagon are Ramsey but at present this is unknown[3].

As mentioned earlier the collinear set $C = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ with distance $(\mathbf{x}, \mathbf{y}) =$ distance$(\mathbf{y}, \mathbf{z}) = 1$ is *not* Ramsey. (Indeed, no set with three collinear points can be Ramsey, as we will see later). The proof of this is not difficult and goes as follows. For each point $\mathbf{u} \in \mathbb{E}^n$ assign the color

$$\chi(\mathbf{u}) = [\mathbf{u} \cdot \mathbf{u}] \pmod 4$$

where for $\mathbf{u} = (u_1, \ldots, u_n), \mathbf{u} \cdot \mathbf{u}$ denotes the inner product $\sum_{i=1}^n u_i^2$ and $[x]$ denotes the greatest integer not exceeding x. Suppose the set $C = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ occurs monochromatically in this 4-coloring of $\mathbb{E}^n$, say $C \subseteq \chi^{-1}(i)$. From Fig. 2, since

$$a^2 = b^2 + 1 - 2b \cos \Theta$$
$$c^2 = b^2 + 1 + 2b \cos \Theta$$

then

$$a^2 + c^2 = 2b^2 + 2$$

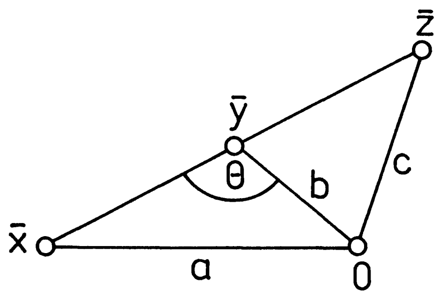

Fig. 2. The collinear set $C = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$

Since

$$\chi(\mathbf{x}) = \chi(\mathbf{y}) = \chi(\mathbf{z}) = i$$

then

2 This has now been proved by Frankl and Rödl (to appear).

3 The vertices of a regular pentagon do form a Ramsey set. This has been proved very recently by Igor Kříž (to appear).

$$a^2 = 4k_a + i + e_a, \ 0 \le e_a < 1,$$
$$b^2 = 4k_b + i + e_b, \ 0 \le e_b < 1,$$
$$c^2 = 4k_c + i + e_c, \ 0 \le e_c < 1$$

for suitable integers k_a, k_b, k_c. Thus, we have

$$4(k_a - 2k_b + k_c) - 2 = -e_a + 2e_b - e_c$$

which is easily seen to be (just barely) impossible. This proves that C is not Ramsey.

The preceding argument actually contains the kernel of an idea which when more fully developed leads to the following result.

Let us call a set $X \subseteq \mathbb{E}^n$ *spherical* if it is a subset of a sphere.

3.2 Theorem *(Erdős, Graham, Montogomery, Rothschild, Spencer, Straus 1973). If C is Ramsey then C is spherical.*

The proof, which we sketch for completeness, rests on several lemmas.

3.3 Lemma. *There exists a $(2n)$-coloring χ of $\mathbb{R}$ such that the equation*

$$\sum_{i=1}^{n}(y_i - y_i') = 1$$

has no solution with $\chi(y_i) = \chi(y_i')$, $1 \le i \le n$.

Proof. Define χ by setting $\chi(y) = j$ if $y \in [2m + j/n, 2m + (j+1)/n]$ for some integer m. Then $\chi(y_i) = \chi(y_i')$ implies

$$y_i - y_i' = 2m_i + \theta_i$$

for some θ_i with $|\theta_i| < 1/n$. Therefore

$$1 = \sum_{i=1}^{n}(y_i - y_i') = 2\sum_{i=1}^{n} m_i + \sum_{i=1}^{n} \theta_i = 2M + \theta$$

where $\theta = \sum_{i=1}^{n} \theta_i$. However, this is impossible since $0 \le |\theta| < 1$. □

3.4 Lemma *(Strauss 1975). Suppose $c_1, \ldots, c_n$ and $b \neq 0$ are arbitrary real numbers. Then there exists a $(2n)^n$-coloring χ^* of $\mathbb{R}$ such that the equation*

$$\sum_{i=1}^{n} c_i(x_i - x_i') = b \tag{1}$$

has no solution with $\chi^*(x_i) = \chi^*(x_i')$, $1 \le i \le n$.

Proof. Note that (1) holds if and only if

$$\sum_{i=1}^{n} c_i^*(x_i - x_i') = 1 \tag{2}$$

where $c_i^* = c_i b^{-1}$. Define χ^* on $\mathbb{R}$ by setting $\chi^*(\alpha) = \chi^*(\beta)$ if and only if $\chi(c_i^*\alpha) = \chi(c_i^*\beta)$ for all i, where χ is the $2n$-coloring defined in Lemma 3.3. Thus, χ^* is a $(2n)^n$-coloring. Now suppose (2) holds with $\chi^*(x_i) = \chi^*(x_i')$, $1 \le i \le n$. Then $\chi(c_j^* x_i) = \chi(c_j^* x_i')$, $1 \le i, j \le n$. In particular, $\chi(c_i^* x_i) = \chi(c_i^* x_i')$, $1 \le i \le n$. Therefore,

$$\begin{aligned}\sum_{i=1}^{n} c_i^*(x_i - x_i') &= \sum_{i=1}^{n} (c_i^* x_i - c_i^* x_i') \\ &= \sum_{i=1}^{n} (2m_i + \theta_i - 2m_i' - \theta_i') \\ &= 2M + \sum_{i=1}^{n} (\theta_i - \theta_i') \neq 1\end{aligned}$$

since $0 \le \sum_{i=1}^{n} |\theta_i - \theta_i'| < 1$. □

3.5 Lemma. *A set* $K = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k\}$ *is not spherical if and only if there exist* c_i, *not all* 0, *such that:*
(i) $\sum_{i=1}^{k} c_i(\mathbf{v}_i - \mathbf{v}_0) = 0$,
(ii) $\sum_{i=1}^{k} c_i(\mathbf{v}_i \cdot \mathbf{v}_i - \mathbf{v}_0 \cdot \mathbf{v}_0) = b \neq 0$.

Proof. Assume K is a subset of a sphere with center $\mathbf{w}$ and radius r, and suppose K satisfies (i). By the law of cosines,

$$\begin{aligned} r^2 &= (\mathbf{v}_i - \mathbf{w}) \cdot (\mathbf{v}_i - \mathbf{w}) \\ &= (\mathbf{v}_0 - \mathbf{w}) \cdot (\mathbf{v}_0 - \mathbf{w}) + (\mathbf{v}_i - \mathbf{v}_0) \cdot (\mathbf{v}_i - \mathbf{v}_0) - 2(\mathbf{v}_i - \mathbf{v}_0) \cdot (\mathbf{w} - \mathbf{v}_0)\end{aligned}$$

which implies

$$(\mathbf{v}_i - \mathbf{v}_0) \cdot (\mathbf{v}_i - \mathbf{v}_0) = 2(\mathbf{v}_i - \mathbf{v}_0) \cdot (\mathbf{w} - \mathbf{v}_0)$$

since $(\mathbf{v}_0 - \mathbf{w}) \cdot (\mathbf{v}_0 - \mathbf{w}) = r^2$. Thus,

$$\sum_{i=1}^{k} c_i(\mathbf{v}_i - \mathbf{v}_0) \cdot (\mathbf{v}_i - \mathbf{v}_0) = 2(\mathbf{w} - \mathbf{v}_0) \cdot \sum_{i=1}^{k} c_i(\mathbf{v}_i - \mathbf{v}_0) = 0$$

which contradicts (ii).

On the other hand, suppose K is not spherical. We may assume without loss of generality that K is *minimally* non-spherical, i.e., all proper subsets of K are spherical. Thus, the $k+1$ points of K cannot form a simplex since a simplex is spherical. Therefore, the vectors $\mathbf{v}_i - \mathbf{v}_0$ are linearly dependent, i.e., there exist c_i, $1 \le i \le k$, not all 0, such that

$$\sum_{i=1}^{k} c_i(\mathbf{v}_i - \mathbf{v}_0) = 0. \tag{3}$$

By the minimality assumption on K, we may assume that $c_k \neq 0$ and that $\mathbf{v}_0, \ldots, \mathbf{v}_{k-1}$ lie on some sphere, say with center $\mathbf{w}$ and radius r. Since

$$\mathbf{v}_i \cdot \mathbf{v}_i - \mathbf{v}_0 \cdot \mathbf{v}_0 = (\mathbf{v}_i - \mathbf{w}) \cdot (\mathbf{v}_i - \mathbf{w}) - (\mathbf{v}_0 - \mathbf{w}) \cdot (\mathbf{v}_0 - \mathbf{w}) + 2(\mathbf{v}_i - \mathbf{v}_0) \cdot \mathbf{w}$$

then

$$\begin{aligned}\sum_{i=1}^{k} c_i(\mathbf{v}_i \cdot \mathbf{v}_i - \mathbf{v}_0 \cdot \mathbf{v}_0) &= \sum_{i=1}^{k} c_i((\mathbf{v}_i - \mathbf{w}) \cdot (\mathbf{v}_i - \mathbf{w}) - (\mathbf{v}_0 - \mathbf{w}) \cdot (\mathbf{v}_0 - \mathbf{w})) \\ &\quad + 2\sum_{i=1}^{k} c_i(\mathbf{v}_i - \mathbf{v}_0) \cdot \mathbf{w} \\ &= c_k((\mathbf{v}_k - \mathbf{w}) \cdot (\mathbf{v}_k - \mathbf{w}) - r^2) \neq 0\end{aligned}$$

by (3) since $\mathbf{v}_k$ is *not* on the sphere of radius r centered at $\mathbf{w}$. Thus (ii) holds and the lemma is proved. □

We are now ready to complete the proof of Theorem 3.2. Assume $C = \{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ is not spherical. By Lemma 3.5, there exist $c_1, c_2, \ldots, c_n$ and $b \neq 0$ such that

$$\sum_{i=1}^{n} c_i(\mathbf{v}_i - \mathbf{v}_0) = 0, \ \sum_{i=1}^{n} c_i(\mathbf{v}_i \cdot \mathbf{v}_i - \mathbf{v}_0 \cdot \mathbf{v}_0) = b \neq 0. \tag{4}$$

Let us color each point $\mathbf{u}$ of $\mathbb{E}^N$ with χ by defining $\chi(\mathbf{u}) = \chi^*(\mathbf{u} \cdot \mathbf{u})$ where χ^* is the $(2n)^n$-coloring used in Lemma 3.4 with these values of c_i and b. Thus, if χ assigns a single color to all the $\mathbf{v}_i$ then χ^* must assign a single color to all the $\mathbf{v}_i \cdot \mathbf{v}_i$. However, this is impossible since (4) cannot hold monochromatically using the coloring χ^*. Thus, with the $(2n)^n$-coloring χ of $\mathbb{E}^N$ given above, the set C cannot occur monochromatically. Since N was arbitrary, this shows that C is not Ramsey, and the theorem is proved. □

Before concluding this section we point out that a number of analogues to the preceding results are known when instead of requiring a monochromatic set congruent to the given set C, we only require that the congruent set have at most k colors for some fixed value of k. Specifically, call a configuration *k-Ramsey* if for any r there is an $N = N(k, C, r)$ such that in any r-coloring of $\mathbb{E}^N$, some set congruent to C must occur which has at most k colors. Thus, 1-Ramsey sets are just Ramsey sets. The following analogue to Theorem 3.2 appears in Erdős et al. (1973).

3.6 Theorem. *If C is k-Ramsey then C is contained in the union of k spheres.*

4. Sphere-Ramsey Sets

Rather than take all of $\mathbb{E}^n$ as our underlying space, it is possible to consider various subsets of $\mathbb{E}^n$ instead and ask the analogous questions. A very natural choice for such subsets are unit spheres. Specifically, we denote by S^n the unit sphere in $\mathbb{E}^{n+1}$ centered at the origin, i.e.,

$$S^n = \{(x_0, \ldots, x_n) \in \mathbb{E}^{n+1} : \sum_{i=0}^{n} x_i^2 = 1\}.$$

A configuration C will then be called *sphere-Ramsey* if for any r, there is an $N = N(C,r)$ such that in any r-coloring of S^N there is always a monochromatic subset of S^N which is congruent to C. In this section we will describe several results concerning sphere-Ramsey sets which bear some similarity to those for ordinary Ramsey sets, although in general far less is known about sphere-Ramsey sets.

The strongest constraint currently known for sphere-Ramsey sets is given by the following result.

4.1 Theorem *(Graham 1983). If $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \subseteq \mathbb{E}^n$ is sphere-Ramsey then for any linear dependence $\sum_{i\in I} \alpha_i \mathbf{x}_i = \mathbf{0}$ there must exist a nonempty subset $J \subseteq I$ such that $\sum_{j\in J} \alpha_j = 0$.*

Proof. Suppose the contrary, i.e., suppose

(i) for some nonempty $I \subseteq \{1, 2, \ldots, m\}$, there exist nonzero α_i, $i \in I$, such that
$$\sum_{i\in I} \alpha_i \mathbf{x}_i = \mathbf{0};$$

(ii) for all nonempty $J \subseteq I$,
$$\sum_{j\in J} \alpha_j \neq 0.$$

We will show that there exists an $r = r(X)$ such that for any N, S^N can be r-colored with no monochromatic subset congruent to X.

To begin with, consider the homogeneous linear equation

$$\sum_{i\in I} \alpha_i z_i = 0. \tag{5}$$

By assumption (ii), Rado's results for the partition regularity of this equation over $\mathbb{R}^+$ (see Graham, Rothschild, Spencer 1980 or Rado 1933) implies that (5) is *not* regular, i.e., for some r there is an r-coloring $\chi : \mathbb{R}^+ \to \{1, 2, \ldots, r\}$ such that (5) has no monochromatic solution. Color the points of

$$S_+^N = \{(x_0, \ldots, x_N) \in S^N : x_0 > 0\}$$

with χ^* by defining

$$\chi^*(\mathbf{x}) = \chi(\mathbf{u} \cdot \mathbf{x})$$

where $\mathbf{u}$ denotes the unit vector $(1, 0, \dots, 0)$. Thus, the color of $\mathbf{x} \in S_+^N$ depends only on the distance of $\mathbf{x}$ to the "north pole" of S^N.

For each nonempty subset $J \subseteq I$, consider the equation

$$\sum_{j \in J} \alpha_j z_j = 0. \tag{6}$$

Of course, by (ii) this equation also fails to satisfy the necessary and sufficient condition of Rado for partition regularity. Therefore, there is an r_J-coloring χ_J of $\mathbb{R}^+$ so that (6) has no χ_J-monochromatic solution. As before, we can color S_+^N by giving $\mathbf{x} \in S_+^N$ the color

$$\chi_J^*(\mathbf{x}) = \chi_J(\mathbf{u} \cdot \mathbf{x}).$$

Now, we form the product coloring $\hat{\chi}$ of S_+^N by defining for $\mathbf{x} \in S_+^N$

$$\hat{\chi}(\mathbf{x}) = (\dots, \chi_J(\mathbf{x}), \dots)$$

where the index J ranges over all $2^{|I|} - 1$ nonempty subsets of I. The number of colors required by the coloring $\hat{\chi}$ is at most

$$R = \prod_{\phi \neq J \subseteq I} r_J.$$

An important property of $\hat{\chi}$ is this. Suppose we extend $\hat{\chi}$ to

$$S_0^N = \{(x_0, \dots, x_n) \in S^N : x_0 \geq 0\}$$

by assigning *all* R colors to any point in $S_0^N \backslash S_+^N$, i.e., with $x_0 = 0$. Then the only monochromatic solution to (5) in $\mathbb{R}^+ \cup \{0\}$ is $z_i = 0$ for all $i \in I$.

Next, we construct a similar coloring χ on $S_-^N = \{-\mathbf{x} : \mathbf{x} \in S_+^N\}$, but using R different colors. This assures that any set X which intersects both hemispheres S_+^N and S_-^N cannot be monochromatic.

Finally, we have left to color the equator

$$S^{N-1} = \{\mathbf{x} \in S^N : x_0 = 0\}.$$

By the construction, any monochromatic set congruent to X must be contained entirely in S^{N-1}. Hence, it suffices to color S^{N-1} avoiding monochromatic copies of X, where we may use any of $2R$ colors previously used in the coloring of $S_+^N \cup S_-^N$. By induction, this can be done provided we can so color S^1. However, if $m > 1$ then S^1 can in fact always be 3-colored without a monochromatic copy of X. This completes the proof of the theorem. □

4.2 Corollary. *If $X \subseteq S^n$ and $\mathbf{0} \in conv(X)$ then X is not sphere-Ramsey (where $conv(X)$ denotes the convex hull of X).*

Proof. If $\mathbf{0} \in conv(X)$ then there exist $\alpha_x > 0$, $\mathbf{x} \in X' \subseteq X$, such that

$$\sum_{\mathbf{x} \in X'} \alpha_{\mathbf{x}} \mathbf{x} = \mathbf{0}.$$

Since no subset of the $\alpha_{\mathbf{x}}$ can sum to 0, the result follows. □

In the other direction, it is known that the vertex set of any rectangular parallelepiped is sphere-Ramsey, provided the length of its main diagonal is at most $\sqrt{2}$. The proof has the same basic structure as the usual proofs of the Hales-Jewett theorem and can be found in Graham (1983). It seems likely that this should hold in fact for any rectangular parallelepiped with main diagonal length less than 2[4]. Here, we show this for the case of two points. Specifically, we have

4.3 Theorem. *For any λ with $0 < \lambda < 1$, the set $\{-\lambda, \lambda\}$ is sphere-Ramsey.*

Proof. It is enough to show that the graph $G(\lambda)$ with vertex set S^n and edge set $\{\{\mathbf{x}, \mathbf{y}\}: \text{ distance } (\mathbf{x}, \mathbf{y}) = \lambda\}$ has chromatic number tending to infinity with n. To prove this we use the following result of Frankl and Wilson:

4.4 Theorem *(Frankl, Wilson 1981). Let $\mathcal{F}$ be a family of k-sets of $\{1, 2, \ldots, n\}$ such that for some prime power q,*

$$|F \cap F'| \not\equiv k (mod\, q)$$

for all $F \neq F'$ in $\mathcal{F}$. Then

$$|\mathcal{F}| \leq \binom{n}{q-1}.$$

For a fixed r, choose a prime power q so that

$$\binom{2(1+e)q}{(1+e)q} > r \binom{2(1+e)q}{q-1}$$

where $\beta = \lambda/\sqrt{2q}$, and α and $e > 0$ are chosen so that

$$\alpha^2 + 2(1+e)q\beta^2 = 1$$

and $N = (1+e)q$ is an integer. Consider the set

$$S = \{(s_0, \ldots, s_{2N}) : s_0 = \alpha,\ s_i = \pm\beta, \sum_{i=1}^{2N} s_i = 0\}.$$

To each $\mathbf{s} \in S$ associate the subset

$$F(\mathbf{s}) = \{i \in \{1, \ldots, 2N\} : s_i = \beta\}.$$

Thus, the family

$$\mathcal{F} = \{F(\mathbf{s}) : \mathbf{s} \in S\}$$

4 This has now been proved by Frankl and Rödl (to appear).

consists of the $\binom{2N}{N}$ N–element subsets of $\{1,\ldots,2N\}$. If $F, F' \in \mathcal{F}$, $F \neq F'$, then

$$|F \cap F'| \equiv N \ (mod\, q)$$

if and only if

$$|F \cap F'| = N - q = eq.$$

If the elements of $\mathcal{F}$ are r-colored then some color class must contain at least

$$\frac{1}{r}\,|\mathcal{F}| = \frac{1}{r}\binom{2N}{N} > \binom{2N}{q-1}$$

elements of $\mathcal{F}$. However, by the preceding result of Frankl and Wilson, if $|F \cap F'| = eq$ never occurs, then the number of elements of $\mathcal{F}$ can be at most $\binom{2N}{q-1}$, which is a contradiction.

Therefore, $\mathcal{F}$ must contain a monochromatic pair $F(\mathbf{s}), F(\mathbf{s}')$ with

$$|F(\mathbf{s}) \cap F(\mathbf{s}')| = eq.$$

This implies that $\mathbf{s}$ and $\mathbf{s}'$ must (up to a permutation of coordinates) look like:

$$\begin{array}{lllll} & eq & q & q & eq \\ \mathbf{s} = (\alpha, & \beta,\ldots,\beta, & \beta,\ldots,\beta, & -\beta,\ldots,-\beta, & -\beta,\ldots,-\beta), \\ \mathbf{s}' = (\alpha, & \beta,\ldots,\beta, & -\beta,\ldots,-\beta, & \beta,\ldots,\beta, & -\beta,\ldots,-\beta). \end{array}$$

It now follows that

$$\text{distance}(\mathbf{s},\mathbf{s}') = \sqrt{8q\beta^2} = 2\lambda$$

and

$$\text{distance}(\mathbf{s},\mathbf{0}) = \text{distance}(\mathbf{s}',\mathbf{0}) = \alpha^2 + 2(1+e)q\beta^2 = 1.$$

Thus, $\mathbf{s}$ and $\mathbf{s}' \in S^{2N}$ and the theorem is proved. □

5. Concluding Remarks

Space limitations have prevented us from describing more than just a few of the many interesting results and problems in Euclidean Ramsey Theory. Several topics we might have discussed are the following.

Let us call a collection C of *line segments* in $\mathbb{E}^n$ *line-Ramsey* if for any r, in any partition of all the line segments in $\mathbb{E}^n$ into r classes, some class contains a set of line segments congruent to C. It is known (Erdős et al. 1973), for example, that if C is line-Ramsey then all line segments must have the same length. Another negative result is the following.

5.1 Theorem *(Graham 1983). Suppose C is a configuration of unit line segments L_i such that:*

(i) The set of endpoints of the L_i is not spherical;

(ii) The graph having the L_i as its edges is not bipartite.

Then C is not line-Ramsey.

It is not known whether four line segments forming a unit square is line-Ramsey.

Even if we restrict ourselves to $\mathbb{E}^2$, there are many unsolved problems. For example, is it true that if T is any three-point set in $\mathbb{E}^2$ which does not form an equilateral triangle, then $R(T,2,2)$ holds? The strongest conjecture would be that in *any* 2-coloring of $\mathbb{E}^2$, a congruent copy of *every* three point set must occur monochromatically, with the exception of the set of vertices of a single equilateral triangle. On the other hand, it may be true that $R(T,2,3)$ *never* holds for *any* three-point set T.

Since we have seen that $\mathbb{E}^2$ can be 7-colored so that no set congruent to a given two-point occurs monochromatically, one might wonder if there were any interesting Euclidean Ramsey properties which hold when $\mathbb{E}^2$ is partitioned into an arbitrarily large (finite) number of colors. The following result shows that there are.

5.2 Theorem *(Graham 1980). For every partition of $\mathbb{E}^2$ into finitely many classes, some class has the property that for all $\alpha > 0$, it contains three points which span a triangle of area α.*

The proof, which can be extended to the analogous result for $\mathbb{E}^n$, is surprisingly tricky.

For other Euclidean Ramsey Theory results, the reader can consult Erdős et al. (1973), Erdős, Rothschild, Straus (1983), Graham (1980), Graham (1983), Shader (1974, 1976), Straus (1975).

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On Pisier Type Problems and Results (Combinatorial Applications to Number Theory)

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Abstract

Several number theoretical results are (sometimes a bit surprisingly) consequences of purely combinatorial statements. In this paper we deal with some of these results. Particularly we solve several problems related to Pisier-type problems.

1. Introduction

1.1 In connection with his work on Sidon sets Pisier formulated (1983) the following problem which in our opinion is very interesting in itself:

A (finite or infinite) sequence of integers $A = \{a_1 < a_2 < \ldots\}$ is called *independent* if all the sums $\sum \epsilon_i a_i$, $\epsilon_i = 0$ or 1, are distinct.

Pisier posed the problem of giving a necessary and sufficient condition that a sequence should be the union of a finite number of independent sequences. He then asked:

Is it true for every $\delta > 0$ there is an absolute constant k_δ (depending only on δ) so that

(P) if for every m and every subsequence $\{a_{i_1} < a_{i_2} < \ldots < a_{i_m}\}$ there exists a further subsequence of δm terms of $a_{i_1} < \ldots < a_{i_m}$ which is independent, then A is the union of k_δ independent sequences?

The condition (P) is clearly necessary if A is a union of k independent sequences (we may put $\delta = \frac{1}{k}$). The problem is whether it is sufficient. If the answer is positive then Pisier has a characterization for Sidon sets, in other words his well known condition, see Pisier (1983), that a set should be a Sidon set is both necessary and sufficient.

We will be here concerned only with the combinatorial and number theoretic problems of Pisier type. As the reader will see many related questions can be asked in number theory or combinatorics. Essentially all our non-tivial results are of negative character.

1.2 First let us make some remarks on independent sets. One of the first conjectures of the first author states (this dates 1932 A.D. (not B.C.)):

Is it true that if $a_1 < a_2 < \ldots < a_k \leq x$ is independent then

$$k < \frac{\log x}{\log 2} + C? \tag{1}$$

In other words: Is the sequence $2^n, n = 0, 1, 2, \ldots$ in some sense the densest independent sequence?

P. Erdös offers 500 dollars for a proof or disproof of (1). The best known upper bound for k is due to P. Erdös and L. Moser and states

$$k < \frac{\log x}{\log 2} + \frac{\log\log x}{2\log 2} + C.$$

Conway and Guy found a set of 24 integers $1 \leq a_1 < \ldots < a_{24} < 2^{22}$ which is independent and it has been conjectured that (1) holds with $C = 3$. The truth or falsity of Pisier problem seems to have nothing to do with (1) and unfortunately we can make no contribution to this beautiful problem.

1.3 More than 50 years ago Sidon called a sequence $a_1 < a_2 < \ldots$ a B_r *sequence* if all the sums $\sum_{i=1}^{r} \epsilon_i a_{n_i}, \epsilon_i = 0, 1$, are distinct. Especially B_2 sequences have been studied a great deal (Ajtai, Komlós, Szemerédi 1981, Alon, Erdös 1985).

In Alon, Erdös (1985) the following Pisier-type problem is posed: Let $\delta > 0$ and suppose that A has the property that for every finite subsequence $a_{i_1} < \ldots < a_{i_n}$ there is a finite B_2 sub-subsequence having δm terms. Is it then true that there exists k_δ such that A is the union of k_δ B_2-sequences?

This conjecture can be disproved for $\delta \leq \frac{1}{4}$ by using a non-trivial Ramsey-type result of Nešetřil and Rödl (see 1979, 1981):

1.4 Theorem. *There exists a sequence A of positive integers with the following properties:*

1) *A fails to be a finite union of B_2-sequences*
2) *For every finite subsequence B of A there exists a B_2 sub-subsequence C of B such that $|C| \geq \frac{1}{4}|B|$.*

Trivially the result holds for $\delta > \frac{3}{4}$. It is open at the moment what happens for $\frac{1}{4} < \delta \leq \frac{3}{4}$. $\delta = \frac{3}{4}$ seems particularly interesting:

Is it true if every subsequence of m terms contains a B_2 sequence of $\frac{3m}{4}$ terms then A is the union of a finite number of B_2 sequences?

1.5 The methods of Nešetřil, Rödl (1985) imply that if we permit δ to be sufficiently small then all the Pisier type problems for B_r^l-sequences have negative answer, see Theorem 4.1. For the original Pisier problem this method does not apply (compare Theorem 5.3 below). So far we have no non-trivial positive result in number theory for a Pisier type problem. Just for illustration we give at least a trivial result of this type:

A sequence of integers is called *primitive* if no one divides the other Erdös, Sárközy, Szemerédi (1967). From Dilworth (1950) one gets easily the following:

Let A be a sequence. If every subsequence of A of m terms contains a primitive sub-subsequence of δm terms then A is the union of $\lceil 1/\delta \rceil$ primitive sequences.

1.6 Two more Pisier type problems will be treated in Section 4. Fix positive ϵ and let the sequence $A = (a_1 < a_2 < \ldots)$ be such that every subsequence of m terms contains a sub-subsequence of at least $(\frac{1}{2} + \epsilon)m$ terms for which

$$(*) \qquad a_u + a_v \neq a_w$$

for every choice of distinct indices. Is it then true that A is a finite union of sets with the property $(*)$?

This being trivially true for $\epsilon > 1/6$, for $0 < \epsilon \leq 1/6$ the problem appears to be much harder. The same question can be asked if $a_u + a_v \neq a_w$ is replaced by the condition that no a is the sum of distinct other entries: a sequence of integers is called *free* if no entry is the sum of other entries. Here we have a more satisfactory situation as we can prove:

1.7 Theorem. *For every positive ϵ there exists a sequence $A = (a_1 < a_2 < \ldots)$ of integers with the following properties:*

1) *For every finite paritito $A = A_1 \cup \ldots \cup A_k$ one of the classes A_i fails to be free;*
2) *For every finite subsequence $B \subseteq A$ there exists a free sub-subsequence $C \subseteq B$ with $|C| \geq (1/2 - \epsilon)\,|B|$.*

1.8 Now some problems in graph theory:
Let $G = G(n; e)$ be a graph with n vertices and e edges, H is any other fixed graph. We try to find a necessary and sufficient condition that our $G(n, e)$ is the union of a bounded number of graphs which do not contain H as a subgraph. The Pisier type condition could be: Assume that every subgraph $G(n, e_1)$ contains another subgraph $G(n, e_2)$ with $e_2 > \delta e_1$ which does not contain H as a subgraph. Is there then a C_δ for which $G(n, e)$ is the union of C_δ graphs none of which contain H as a subgraph?

The necessity of this condition is obvious the sufficiency is the difficult problem. Again we do not have any non-trivial positive result.

1.9 Let us discuss few special cases which seemed particularly interesting to us. First, let H be C_4. It follows from Ramsey-theoretic results of Nešetřil and Rödl that if $\delta \leq \frac{1}{3}$ then our Pisier type conjecture is false. For $\delta > \frac{3}{4}$ the answer is trivially positive, but for $\frac{1}{3} < \delta \leq \frac{3}{4}$ the problem is open and perhaps is difficult. Similar situation prevails for $K_{r,r}$ and probably for every bipartite graph which is not a tree.

1.10 One further question of a slightly different type seems to be of an interest:

Let $G(n,e)$ be a graph of n vertices and e edges. We would like to find a condition which would imply that there is a subgraph of δe edges without a C_4 and in fact a condition which would imply that every subgraph of e_1 edges contains another subgraph of $\geq \delta e_1$ edges without a C_4.

Here is a condition which is obviously necessary and could be sufficient: There is an absolute constant c so that every bipartite subgraph of G of a white and b black vertices, $a \leq b$, contains at most $c.a.b^{1/2}$ edges.

The necessity follows from the fact that a bipartite graph of a white and b black vertices ($a \leq b$) which has no C_4 contains at most $a.b^{1/2}$ edges.

1.11 Let us now consider the case when H is a K_r. First consider $r = 3$.

Clearly every $G(n,e)$ contains a bipartite subgraph of more than $\frac{e}{2}$ edges, thus we can hope for a positive Pisier type result only for $\delta > \frac{1}{2}$:

Is it true that if $G(n,e)$ is such that every subgraph of e_1 edges contains a subgraph of $(\frac{1}{2} + \epsilon)e_1, \epsilon > 0$, edges without a triangle then our $G(n,e)$ is the union of C_ϵ graphs without a triangle? We do not know the answer even for $\epsilon = \frac{1}{6}$. The case $\epsilon = \frac{1}{6}$ seems to be particularly interesting. Similar questions can be asked for $K_r, r > 3$.

1.12 Related question has been asked by Erdös and Nešetřil:

Is there a $G(n,e)$ which contains no K_4 but every subgraph of $(\frac{1}{2} + \epsilon)e$ edges of it contains a triangle? Frankl and Rödl proved (1986) by the probability method that such a graph exists. If we replace triangle by K_r, $r > 3$, then the problem is still open.

All our examples of solved Pisier type problems are based on a convenient combinatorial representation. Therfore we decide to include some other examples of the use of combinatorial methods in number theory. This is done in Section 2 where we survey recent results related to theorems of Ramsey and Van der Waerden.

In Section 3 we define graphical sequences of integers (introduced in Nešetřil, Rödl (1979)) and we list several applications in number theory.

In Section 4 we deal with negative solutions of Pisier type problems. Particularly we solve Alon-Erdös problem on B_r sequences.

In Section 5 we list some positive examples of Pisier type problems. In Section 6 we sketch a proof of a Ramsey-type result which is the key point in proof of Section 4.

2. Multiplicative Bases and Szemerédi-Ruzsa Theorem

Let X be a set of positive integers. We say that X is a *multiplicative base* if for every positive integer n there are $x, y \in X$ such that $n = xy$. The following was proved by P. Erdös (see Erdös 1964).

2.1 Theorem. *Let X be a multiplicative base. Then for every positive integer p there exists a positive integer n such that n can be expressed as the product of two elements of X in at least p different ways.*

Nešetřil and Rödl found a very easy proof using the well-known theorem of Ramsey: Let p be a given integer and let $[A]^p = C_1 \cup C_2$ be a partition of the set of all p-tuples of elements of an infinite set A into two parts. Then there exists i and an infinite set $B \subseteq A$ such that $[B]^p \subseteq C_i$. The set B is called *homogeneous* with respect to the partition (C_1, C_2).

Proof of Theorem 2.1. In the following we shall consider the integers which are products of distinct primes only. Such integers can be identified in a natural way with finite subsets of the set of all primes. Thus it suffices to prove the following.

2.2 Theorem. *Let A be an infinite set. Denote by $[A]^{<\omega}$ the set of all finite subsets of A. Let $\mathcal{A} \subseteq [A]^{<\omega}$ be a set of finite subsets of A such that the following holds:*
$(*)$ For every $P \in [A]^{<\omega}$ there are $Q, Q' \in \mathcal{A}$ such that $Q \cup Q' = P$ and $Q \cap Q' = \emptyset$.

Then for every integer p there is a set P which can be expressed in at least p different ways as a union of two disjoint elements of $\mathcal{A}$.

Proof of Theorem 2.2 is a straightforward consequence of Ramsey's theorem: Let p be a given positive integer, $p \geq 2$. For every $i = 1, \ldots, p-1$ consider a partition $[A]^i = C_1^i \cup C_2^i$ defined by $Q \in C_1^i$ iff $Q \in \mathcal{A}$. Let B be an infinite set which is homogeneous with respect to all partitions $C_1^i, C_2^i, i = 1, \ldots, p-1$ (such a set clearly exists by iterating the Ramsey theorem). From $(*)$ we get that there is an $i, 1 \leq i \leq p-1$, such that

$$[B]^i \subseteq C_1^i \text{ and } [B]^{p-i} \subseteq C_1^{p-i},$$

and hence every $P \in [B]^p$ can be represented as a union of at least $\binom{p}{i} \geq p$ elements of $\mathcal{A}$. □

2.3 Remark Note that the above proof gives nothing concerning the additive version of Theorem 2.1. This is an old problem of P. Erdös and Turán:

Problem. *Let X be a set of positive integers with the property that for every positive integer n there are $x, y \in X$ such that $n = x + y$. Is it true that for every positive integer p there exists a positive integer n such that n can be expressed as the sum of two elements of X in at least p different ways?*

2.4 Perhaps the most celebrated application of combinatorial methods to number theory is the well known Szemerédi theorem. Set
$\nu_k(n) = \max\{|X|; X \subseteq \{1, \ldots, n\}, X$ contains no arithmetic progression of length $k\}$.
Answering a longstanding conjecture of Erdös and Turán Szemerédi proved his famous theorem:

Theorem *(Szemerédi 1975). $\nu_k(n) = o(n)$ for every k.*

We include a proof of a special case $k = 3$ as it fits to our survey being a very beautiful application of graph theory to number theory. The proof given here is based on idea of Ruzsa and Szemerédi (1978) and is taken from Erdős, Frankl, Rödl (1986).

2.5 For a graph (V, E) let $A, B \subseteq V$ be a pair of disjoint subsets of V. The *density* of a pair (A, B) is the fraction

$$d(A,B) = \frac{e(A,B)}{|A| \cdot |B|}$$

where $e(A, B)$ is the number of edges with one end point in A and second in B. The pair (A, B) is called *ϵ-uniform* if for every $A' \subset A, B' \subset B, |A'| > \epsilon |A|, |B'| > \epsilon |B|$, $|d(A', B') - d(A, B)| < \epsilon$ holds.

Finally, the partition $V = C_0 \cup \ldots \cup C_k$ is called ϵ-uniform if
i) $|C_0| < \epsilon |V|$
ii) $|C_1| = |C_2| = \ldots = |C_k|$
iii) all but $\epsilon\binom{k}{2}$ pairs $(C_i, C_j), 1 \leq i < j \leq k$, are ϵ-uniform.

2.6 The following important result was proved by E. Szemerédi:

Theorem *(Uniformity Lemma – Szemerédi 1976). For every $\epsilon > 0$ and positive integer l, there exist positive integers $n_0(\epsilon, l)$ and $m_0(\epsilon, l)$ such that every graph with at least $n_0(\epsilon, 1)$ vertices has an ϵ-uniform partition, into k classes, where k is an integer satisfying $l < k < m_0(\epsilon, l)$.*

2.7 First we show how one can approach arithmetical progression of length 3 in a combinatorial way:

Let $A \subseteq \{1, \ldots, n\}$ and let X, Y, Z be three disjoint copies of (integer) interval $[1, 3n]$. Consider all triples $\{x, y, z\}, x \in X, y \in Y, z \in Z$, such that

$$(*) \qquad y - x = z - y = \frac{z - x}{2} \in A$$

and let G be a graph consisting of all pairs contained in these edges. This graph has $9n$ vertices, $|E(G)| \geq 3|A| \cdot n$ edges and can be decomposed into $\frac{1}{3}|E(G)|$ edge disjoint triangles.

Claim *(Ruzsa, Szemerédi 1978). If G contains a next triangle i.e. different from those of the form $(*)$ then A contains an arithmetic progression of length three.*

Proof. Let x, y, z be such traingle. Then $z - y \neq y - x$. Set $b = z - y, a = y - x$. We have $\frac{z-x}{2} = \frac{a+b}{2} \in A, a \in A, b \in B$, which gives an arithmetic progression of length 3 in A. □

2.8 Consider now a set $A \subseteq \{1, \ldots, n\}, |A| = \alpha n$ where α is a positive constant independent on n. We will prove that A contains an arithmetic progression of length three provided n is large enough.

Set $m = 9n, |E(G)| = \beta\binom{m}{2} > 3\alpha n^2$. Then β is a positive constant independent on n. Set further $\epsilon = \frac{\beta}{15}$ and $l = \lceil \frac{1}{\epsilon} \rceil$ and apply the Uniformity Lemma to G (we can clearly assume that the number of vertices is very large).

The number of edges not contained in pairs with density at least $\frac{\beta}{6}$ is clearly at most

$$k\binom{m/k}{2} + \frac{\beta}{6}\binom{k}{2}\left(\frac{m}{k}\right)^2 + \epsilon\binom{k}{2}\left(\frac{m}{k}\right)^2 + \epsilon m^2 < \frac{\beta}{3}\binom{m}{2}$$

After omision of these edges we get a graph which still contains a triangle T (as there are $\frac{\beta}{3}\binom{m}{2}$ edge disjoint traingles in G). Moreover all the edges of this triangle T are contained in pairs which are ϵ-uniform and have density at least $\frac{\beta}{6}$. Let C_p, C_q, C_r be three partition classes corresponding to these pairs. We prove the following:

2.9 Claim. *Let all pairs $(C_q, C_p), (C_p, C_r), (C_q, C_r)$ be ϵ-uniform with density at least $\frac{\beta}{6}$. Then there is a vertex contained in at least $\frac{\beta^3}{10^3}|C_p| \cdot |C_q|$ traingles.*

This Claim already implies that A contains an arithmetic progression of length 3:

We have $\left(\frac{\beta}{10}\right)^3 |C_q| \cdot |C_p| > |C_p| = |C_q|$ triangles containing a vertex x. As no two triangles of the form $(*)$ (see 2.7 above) share an edge there are not more than $|C_p| = |C_q|$ such triangles containing a vertex x. Thus there exists a triangle different from $(*)$ and hence by Claim 2.7 A contains an arithmetic progression of length three.

Proof of Claim: As both pairs $(C_p, C_r), (C_q, C_r)$ are ϵ-uniform there are at least $(1 - 2\epsilon)|C_r|$ points in C_r which are joined to at least $(\frac{\beta}{6} - \epsilon)|C_i|$ points of C_i for both $i = p$ and $i = q$. Fix one such vertex x and let N_x^i be a set of the neighbours of x in C_i. As $\frac{\beta}{6} - \epsilon = \frac{\beta}{10} > \epsilon$ we infer that there are at least

$$\left(\frac{\beta}{10}\right)^3 |C_p|\,|C_q|$$

edges joining vertices of N_x^p and N_x^q. Each such edge give rise to a triangle that contains x. □

2.10 Remark Rusza and Szemerédi investigated the triple systems (X, M) satisfying property that every 6-tuple contains at most 2 triples. They proved for such triple systems $|M| = o(n^2)$ necessary holds. This implies similarly as above that $\nu_3(n) = o(n)$ holds.

Similarly the truth of the following combinatorial conjecture (considered by P. Frankl and V. Rödl) implies Szemerédi's theorem.

Conjecture. *Let (M, X) be a simple k-uniform hypergraph not containing the following hypergraph depicted on Fig. 0 with k edges and $(k-1)^2 + 2$ vertices. Then $M = o(|X|^2)$.*

Let us close this section by stating two questions which relate Van der Waerden theorem to Pisier type theorems:

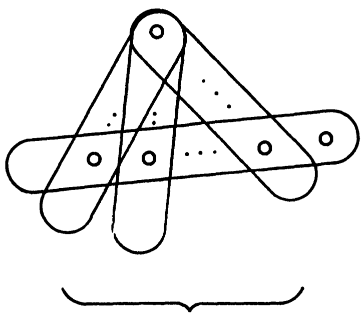

Fig. 0.

Problem 1. *Fix positive* $\epsilon < 1$. *Are the following two statements for a set* X *equivalent?*

1) X *is a finite union of arithmetic progressions*

2) For every finite subset $Y \subseteq X$ *there exists* $Z \subseteq Y$ *with at least* $\epsilon\,|Y|$ *terms contained in an arithmetical progression in* X.

Problem 2. *Fix positive* $\epsilon < 1$. *Are the following two statements for a set* X *equivalent?*

1) X *is a finite union of sequences without arithmetic progression with three terms.*

2) For every finite subset $Y \subseteq X$ *there exists a subset* $Z \subseteq Y$, $|Z| \geq \epsilon\,|Y|$ *such that* Z *contains no arithmetic progression with 3 terms.*

Whereas one can easily see that Problem 1 has a negative solution, Problem 2 appears to be interesting.

3. Graphical Sequences and Examples of Their Use

3.1 Let $G = (V, E)$ be a graph, let $v_1, \ldots, v_n, \ldots$ be its vertices (V may be either finite or countable set). For each v_i let a weight $w(v_i)$ be given; we assume that $w(v_i)$ is a positive integer.

For an edge $e = \{v_i, v_j\}$, $i < j$, define its weight $w(e)$ by

$$(*) \qquad w(e) = \sum_{k=i+1}^{j} w(v_k)$$

We always assume that w is chosen so that $w(e) \neq w(e')$ whenever $e \neq e'$.

Finally put

$$w(E) = \{w(e); e \in E\}.$$

A set of this type is called a *graphical sequence.*

By chosing appropriately labelling w we can transform graph-properties to number-theoretical properties of the graphical sequences. We shall mention three such examples.

Example I.

3.2 Put $w(v_i) = 2^i$. Then the mapping $w : E \longrightarrow w(E)$ is a bijection. Moreover the edges e, e', e'' form a triangle G iff

$$w(e) + w(e') = w(e'').$$

One can use this as a starting point for a proof of the following result:

A set $a_1, \ldots, a_n$ of integers is called *complete* in X if X contains all $2^n - 1$ sums of type

$$\sum_{i \in I} a_i, \quad \emptyset \neq I \subseteq \{1, \ldots, n\}.$$

3.3 Theorem *(Nešetřil, Rödl 1986). Let n be a fixed positive integer. Then there exists a set X with the following properties:*

1) *For every finite partition $X = X_1 \cup \ldots \cup X_k$ one of the classes X_i contains a set of n elements which is complete in X_i.*
2) *X does not contain a complete subset of size $n + 1$.*

For $n = 2$ one can put $X = w(E)$ where E is the edge set of countable complete graph and apply Ramsey theorem.

Note that this proof (Nešetřil, Rödl 1986) yields also a proof of Folkman-Rado-Sanders Theorem (or Finite Union Theorem, cf. Graham, Rothschild, Spencer 1980) which does not involve a use of Van der Waerden theorem and which guarantees a primitive recursive upper bound for the corresponding finite theorem, see also Taylor (1981), Graham, Rödl (1987).

The following infinitary version of the above theorem is presently open:

Problem 3. *Let n be a fixed positive integer. Does there exist a sequence $X = x_1 < x_2 < \ldots$ with the following properties:*

1) *For every finite partition $X_1 \cup \ldots \cup X_k$ of X one of the classes X_i contains an infinite subsequence $Y = y_1 < y_2 \ldots$ of X together with all sums of at most n members of Y.*
2) *X does not contain a complete subset of size $n + 1$.*

For $n = 2$ this follows easily by Ramsey theorem.

Example II.

3.4 The same weight function as in Example I may be used for a construction of locally sparse Schur-sets.

A set X of positive integers is called a *Schur set* if for every finite partition $X = X_1 \cup \ldots \cup X_k$ one of the classes contains two distinct numbers together with their sum.

The (local) density $q(A)$ of a subset A of X is the size of the following set:

$$\{\{x,y\}; x,y \in A, x+y \in X\}$$

Let $q_k(X)$ be the maximal density of a k-element subset of X. It is easy to see that for every Schur set X $q_k(X) \geq k$ holds for all sufficiently large k. We have the following:

3.5 Theorem *(Nešetřil, Rödl 1986). For every positive integer k_0 there exists a set X with the following properties:*
1) X is a Schur set;
2) $q_k(X) = k-1$ for every $k \leq k_0$.

This result is a consequence of the graphical sequence method when applied to the following Ramsey theoretic result:

3.6 Theorem *(Nešetřil, Rödl 1979). For every integer $l \geq 3$ there exists a graph $G = (V,E)$ with the following properties:*
1) $G \longrightarrow (K_3)^2_k$ for every k;
2) The hypergraph

$$(E, \{\{e,e',e''\}; e,e',e'' \text{ form a triangle in } G\})$$

has no cycle of length $\leq l$.

(An example of a cycle of length 13 is indicated on Fig. 1)

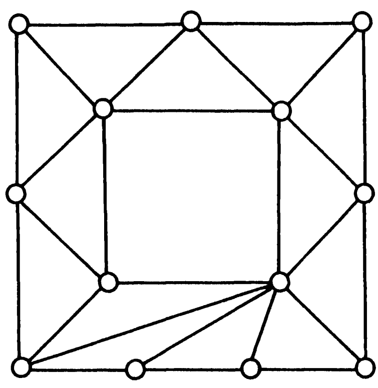

Fig. 1.

The existence of graphs G has been proved in Nešetřil, Rödl (1979).

Example III.

Recall that a sequence of integers $x_1 < x_2 < \ldots$ is called a $B_l^{(r)}$ sequence if the number of representations of every integer n as the sum of l distinct a_i's is at most r while some integer n actually has r representations.

In response to a P. Erdös, D.J. Newman problem (Erdös 1980) the following is proved in Nešetřil, Rödl (1985):

3.7 Theorem. *For every r, l there exists a sequence X with the following properties:*

1) X is a $B_l^{(r)}$ sequence

2) X fails to be a finite union of $B_l^{(r-1)}$ sequences.

This result is obtained by means of graphical sequences and the following technical result:

A ***k*-prism** is the graph with vertices $0, 1, 2, \ldots, k, k+1$ and edges $\{0, i\}, \{i, k+1\}, i = 1, \ldots, k$. A k-prism will be always considered with the ***standard ordering*** of its vertices:

$$0 < 1 < 2 < \ldots < k+1.$$

A (k, l)-prism is a subdivision of the k-prism where each edge is subdivided by $l-1$ vertices, see Fig. 2

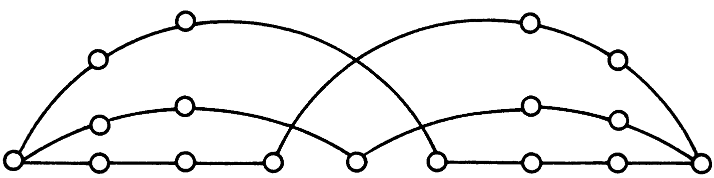

Fig. 2. (3, 3)-prism

3.8 Theorem. *(Nešetřil, Rödl 1981, 1985): There exists a graph $G = (V, E)$ and an ordering of V with the following properties:*

1) No $(k+1, l)$-prism in G has a standard ordering;

2) for every partition $E = E_1 \cup E_2$ there exists a (k, l)-prism in G (with a standard ordering) with all its edges belonging to E_i for either $i = 1$ or $i = 2$.

In Section 6 we shall indicate a proof which gives a result of this type.

4. Pisier Type Theorems

We make no attempts to provide a general scheme for Pisier type problems. Instead, we list several examples which may be treated uniformly. First, we shall consider two local variants of the Pisier problem introduced in Section 1. The following theorem answers an Alon-Erdös problem (Alon, Erdös 1985):

4.1 Theorem. *Let l, r be positive integers. There exists a sequence X with the following properties:*

1) *X fails to be a finite union of $B_l^{(r)}$ sequences.*
2) *For every finite subset Y of X there exists a $B_l^{(r)}$ subset Z of Y such that*

$$|Z| \geq \frac{l-1}{2l}\,|Y|.$$

In the other words $B_l^{(r)}$-sequences do not posses Pisier property. Similar situation occurs for the following concept: We call a sequence $X = x_1 < x_2 < \ldots$ *sum free* if no $x \in X$ is a sum of distinct x_i's different from x.

4.2 Theorem. *Let $\epsilon > 0$. There exists a sequence X with the following properties:*
1) *X fails to be a finite union of sum-free sequences.*
2) *For every finite subset Y of X there exists a sum-free subset Z of Y such that $|Z| \geq \frac{1}{2}(1+\epsilon)\,|Y|$*

Both of these results may be proved in the same vein by the graphical sequence method.

Proof of Theorem 4.2. Let $G = (V, E)$ be a graph with its vertex set (linearly) ordered by $\leq$. A special circuit is a circuit of the form $v_1, e_1, v_2, \ldots, v_l, e_l, v_1$, where $v_1 < v_2 < \ldots < v_l$. See Fig. 3

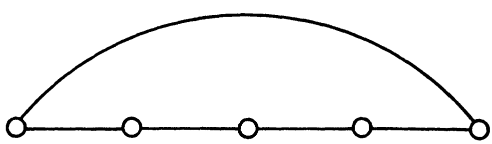

Fig. 3.

Let $\epsilon > 0$ and a positive integer k be fixed. Put $l = \epsilon^{-1}$. Let $G = (V, E)$, $V = \{1, 2, \ldots\}$ be a graph with the following properties:

1) For every positive k and for every partition $E = E_1 \cup \ldots \cup E_k$ one of the classes contains a special circuit (with respect to natural ordering of V)
2) G does not contain a special circuit of length $< l$ (again with respect to the standard ordering of V).

The existence of G is non-trivial and follows from the amalgamation technique developed in Nešetřil, Rödl (1981). A proof will be sketched in Section 6.

Put $w(i) = 2^{2^i}$ and consider the graphical sequence $S = w(E)$. Again $w : E \rightarrow S$ is a bijection. Also if $s_1 + \ldots + s_m = s$ for distinct $s, s_1, \ldots, s_m \in S$

then the edges $w^{-1}(s_1), \ldots, w^{-1}(s_m), w^{-1}(s)$ form a special cycle. Thus it suffices to prove that for every subset F of edges of E there exists a set $F' \subseteq F$ which contains no special cycle and for which $|F'| \geq \frac{1}{2}\frac{l-2}{l-1}|F|$.

To do so we proceede as follows:

1. Find $F'' \subseteq F$ such that F'' induces an $l-1$ partite graph and $|F''| \geq \frac{l-2}{l-1}|F|$.

(This is well known: find a partition $V = V_1 \cup \ldots \cup V_{l-1}$ such that for every $v \in V_i$ and $i \neq j$, $(l-2)|\{e \in E; v \in e \& e \subseteq V_i\}| \leq |\{e \in E; v \in e \& e \cap V_j \neq \emptyset\}|$.)

Let $V_1, \ldots, V_{l-1}$ be a partition of V which corresponds to F''.

2. Define partition $F'' = F_1 \cup F_2$ by $\{x, y\} \in F_1$ if ($x \in V_i$, $y \in V_j$, $i < j$ and $x < y$). Put $F_2 = F - F_1$. We may assume $|F_1| \geq |F_2|$. Finally put $F' = F_1$.

It is a routine to check that F' does not contain a special cycle. It is here where we use the fact that G has no special cycle of length $< l$. □

Proof of Theorem 4.1. The main change in the proof is that we consider prism graphs instead of special circuits. The subtle difference is that the Ramsey graph has to be rectangle free. This is guaranteed by a non-trivial result of Nešetřil and Rödl (1987). Appart from these changes the proof follows from graphical sequence method and thus we state only the key Ramsey result: □

4.3 Theorem. *Let l, r be positive integers, $l \geq 2$. There exists a graph $G = (V, E)$ and an ordering of V with the following properties:*
1) G contains no (r, l)-prism in a standard ordering;
2) G is rectangle free;
3) For every partition $E = E_1 \cup E_2$ there exists a (r, l)-prism in G in a standard ordering with all its edges belonging to E_i for either $i = 1$ or $i = 2$.

Similar Pisier type problems may be considered also for graphs and hypergraphs. Let us state three typical examples:

4.4 Theorem. *For every positive integer r there exists a graph $G = (V, E)$ with the following properties:*
1) G is a Ramsey graph for K_r with respect to any finite partition of the edge set E of ; symbolically

$$G \longrightarrow (K_r)^2_{<\omega}$$

2) For every subset $F \subseteq E$ there exists a subset $F' \subseteq F$ such that F' is K_r-free and $|F'| \geq \frac{(r-2)}{2(r-1)}|F|$.

4.5 Theorem. *There exists a graph $G = (V, E)$ with the following properties:*
1) G is a Ramsey graph for C_4 (=the rectangle) with respect to every finite partition of the edge set of G;
2) For every finite subset $F \subseteq E$ there exists a rectangle-free subset $F' \subseteq F$ such that $|F'| \geq \frac{1}{3}|F|$.

It is not known whether the constants in the above theorems are best possible for the negative solution of the corresponding Pisier-type result. This is not the case with the following result which is perhaps the earliest example of such statement contained in Erdős, Hajnal, Szemerédi (1979):

4.6 Theorem. *For every $\epsilon < 1/2$ there exists a graph G with the following properties:*
1) *The chromatic number of G is infinite;*
2) *Every set A of n vertices of G contains a subset B which is independent (in G) and which has at least $\epsilon \cdot n$ vertices.*

Examples of such a graph G may be obtained as a disjoint union of Kneser graphs $K\left(\binom{2n_i+k_i}{n_i}\right)$ where $k_i \longrightarrow \infty$ and $n_i >> k_i$. Related problems and some extensions are contained in Rödl (1982).

One may formulate every Pisier type problem as a statement about chromatic number and (suitably defined) independent subsets of a hypergraph. Theorem 4.6 (and its analogy for hypergraphs) then show that, in the full generality, the Pisier problem has negative solution in a very strong sense.

5. Pisier Problem – Positive Results

The results of Section 4 may be viewed as local variations of the Pisier problem. The conclusion which may be drawn from these results may be misleading as indicated by the result 5.3 below. We find it useful to introduce the following concept:

5.1 Definition. *A sequence X is called Pisier sequence if there exists a positive $\epsilon > 0$ such that every subsequence $Y \subseteq X$ contains a free subsequence with at least $\epsilon\,|Y|$ terms.*

Using this one can formulate the Pisier problem as follows:

5.2 Pisier problem. *Is it true that for a sequence X of positive integers the following two statements are equivalent:*
1) *X is a finite union of independent sequences;*
2) *X is a Pisier sequence?*

5.3 Theorem. *Let X be a graphical sequence. Then X is a Pisier sequence iff it is a finite union of free sequences.*

Proof. Let $X = w(E)$ be a graphical sequence. Put $E \subseteq \binom{V}{2}$, where $V = \{v_1 < v_2 < \ldots\}, w : V \longrightarrow \{1, 2, \ldots\}$. The key observation is that $X = w(E)$ is an independent sequence iff E is a forest (i.e. a graph without circuits). This is easy and it may be proved by induction. Thus the condition that X is a Pisier sequence is equivalent to the following:

Every subset $F \subseteq E$ of n edges contains a subforest of size $\epsilon \cdot n$. It follows from a theorem of Nash-Williams (1964) that every finite subset of E is a union of $\lceil \frac{1}{\epsilon} \rceil$ forests. For the infinite graphs we may apply compactness. □

We have been informed by M. Piccardello (Roma) that a related result was obtained as early as 1955 by A. Horn in a response to a problem of K.F. Roth and R. Rado: He proved that linear independent sets in a vector space have

Pisier property. We know just a few more examples of Pisier type problems with a positive solution. It seems that a certain regularity (e.g. a matroidal structure or perfect-graph property) is needed. Let us list without proof one more example:

5.4 Theorem. *For every positive $\epsilon > 0$ there exists k such that for every set X in $\mathbb{R}^n$ the following two conditions are equivalent:*
1) *For every finite set $Y \subseteq X$ there exists a set Z of colinear points of size $|Z| \geq \epsilon \cdot |Y|$;*
2) *X may be covered by k lines.*

6. Special Ramsey Graphs – the Partite Construction

In this section we sketch a proof of the missing part of the above proof of Theorem 4.2. Namely, we prove:

6.1 Theorem. *For every positive integers $k, l \geq 3$, there exists a graph $G = (V, E)$ and a linear ordering $\leq$ of V with the following properties:*
1) *For every partition $E = E_1 \cup \ldots \cup E_k$ one of the classes E_i contains a special cycle of length l.*
2) *G does not contain a special cycle of length $< l$.*

The proof which follows is a modification of the main construction – the partite construction, see Nešetřil, Rödl (1979, 1981), and we include it here for the completeness:

6.2 Preliminaries. Let $(V_i)_{i=1}^r$ be a system of pairwise disjoint sets, $V = \bigcup_{i=1}^r V_i$, and let $E \subseteq \binom{V}{2}$ such that $E \cap \binom{V_i}{2} = \emptyset$ for all $i = 1, 2, \ldots, r$. Then the couple $G = ((V_i)_{i=1}^r, E)$ is called a *r-partite graph.* It will be convenient to write $V_i = V_i(G)$. A *standard ordering* of G is any ordering of V which satisfies $V_i < V_j$ whenever $i < j$.

Let $G = ((V_i)_{i=1,E}^r), H = ((W_i)_{i=1}^r, F)$ be two r-partite graphs. We say that G is an induced subgraph of H if $V_i \subseteq W_i$ for every $i = 1, \ldots, r$ and the graph (V, E) is an induced subgraph of (W, F) (here $W = \bigcup_{i=1}^r W_i$). We denote this by $G \leq H$.

In the proof we shall make use of the following folkloristic lemma (see e.g. Graham, Rothschild, Spencer 1980, Nešetřil, Rödl 1981):

6.3 Lemma. *For every bipartite graph $B = (V_1, V_2, E)$ there exists a Ramsey bipartite graph $R(B) = (W_1, W_2, F)$. This means the following: For every partition $F = F_1 \cup F_2$ there exist $V_1' \subset W_1, V_2' \subset W_2$ and $i \in \{1, 2\}$ such that if we denote $E' = \{e \in F; e \subseteq V_1' \cup V_2'\}$ then $E' \subseteq F_i$ and (V_1', V_2', E') is isomorphic to (V_1, V_2, E).*

Proof of Theorem 6.1. Without loss of generality assume $k = 2, l \geq 3$. Let $H = K_r$ be the complete graph with r vertices where $r = r(2, 2, l)$ is the Ramsey number (for partition of pairs into two classes). Put $V(H) = \{1, \ldots, r\}$,

$E(H) = \{e_1, \ldots, e_R\}$, $R = \binom{r}{2}$. For each $A \in [V(H)]^l$ consider the special cycle of length l with vertex set A; let $C^1, C^2, \ldots, C^{\binom{r}{l}}$ be a system of such cycles. Define inductively an r-partite graph P^n for all $n \leq R$ as follows:

$$P^0 = ((V_i^0)_{i=1}^r, E^0)$$

where

$$V_i^0 = \{(v_i, j); j \leq R\}$$

and $\{(v_i, j), (v_i', j')\} \in E^0$ if and only if $j = j'$ and $\{v_i, v_i'\} \in E(C^j)$. In the induction step suppose we have defined the r-partite graph $P^n = ((V_i^n)_{i=1}^r, E^n)$. Put $e_{n+1} = \{v_1, v_2\}, v_1 < v_2$, and let B be a bipartite subgraph of P^n induced on a set $V_{v_1}^n \cup V_{v_2}^n$. Apply 6.3 to get a bipartite graph $R(B)$ which is Ramsey for B. Let $B_1, B_2, \ldots, B_q$ be all the induced subgraphs of $R(B)$ which are isomorphic to B. For each $i \leq q$ let $\varphi_i : B_i \to B$ be the natural inclusion. Put $V(P^{n+1}) = \bigcup_{i=1}^r V_i^{n+1}$ where

$$\begin{aligned} V_i^{n+1} &= \bigcup_{j \leq q} (V_i^n \times \{j\}) \quad \text{for} \quad i \neq v_1, v_2; \\ V_{v_i}^{n+1} &= V_i(R(B)) \text{ for } \quad i = 1, 2. \end{aligned}$$

Denote by $\Psi_j : V(P^n) \longrightarrow V(P^n)$ the $1-1$ mapping defined by

$$\begin{aligned} \Psi_j(v) &= \varphi_j(v) \text{ for } \quad v \in V(B), \\ \Psi_j(v) &= (v, j) \text{ for } \quad v \notin V(B). \end{aligned}$$

Finally, we define $E(P^{n+1})$ by $\{x_1, x_2\} \in E(P^{n+1})$ if and only if there exist $j \leq q$ and $\{y_1, y_2\} \in E(P^n)$ such that $x_1 = \Psi_j(y_1)$ and $x_2 = \Psi_j(y_2)$.

Claim 1. *P^R is Ramsey for C_l. This follows by a backward induction from R to 1 repeatedly applying Lemma 6.3 to edges with the "projection" $e_R, e_{R-1}, \ldots, e_1$.*

This part is common to all applications of the partite construction.

Claim 2. *P^R does not contain a special cycle of length $< l$.*

Proof of Claim 2 follows by induction on $n = 0, 1, 2, \ldots R$. Clearly P^0 satisfies the claim as P^0 (viewed as a graph) is a disjoint union of cycles of length l. Moreover if P^i does not contain a special cycle then also P^{i+1} has this property as every special cycle in P^{i+1} belongs to exactly one copy of P^i. □

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Combinatorial Statements Independent of Arithmetic

Jeff Paris

1. Introduction

When Peano's first order axioms of arithmetic (P) were originally formulated it was generally felt that these axioms summed up all that was obviously true about the natural numbers ($\mathbb{N}$) with addition and multiplication and that any true first order statement of arithmetic would follow from these axioms. This belief held sway until in 1931 Gödel exhibited a first order statement of arithmetic (or as we shall now call it an arithmetic statement) Θ_G which was true but neither it nor its negation could be proved for P. That is Θ_G was an arithmetic statement *independent of arithmetic.*

Whilst Gödel's result had an enormous effect on mathematical logic and the philosophy of mathematics its effect on mainline mathematics was much less dramatic. The reason for this was that Θ_G, and the many varients of Θ_G which appeared at the same time, was, as a statement about numbers, extremely complicated. Certainly one cannot envisage Θ_G being studied in its own right.

Of course one might ask then how it was that Gödel found Θ_G if it was so complicated. Well simplifying matters considerably Gödel derived a method of coding a formal proof q as a number $\ulcorner q \urcorner$, now called the Gödel number of q. This coding can be carried out in such a way that simple statements about q become equivalent to arithmetic statements about $\ulcorner q \urcorner$. Under this translation Θ_G is the arithmetic statement equivalent to "there is no proof from P of $0 = 1$". Clearly most mathematicians would accept Θ_G as true although as Gödel showed, in that case neither Θ nor $\neg\Theta$ (its negation) can be proved from P.

Well that was how the situation stood until 1977 when the author noticed that some earlier work with Laurie Kirby could be used to produce independent arithmetic statements which were meaningful to main line mathematicians.

In this short paper we shall explain the original method for obtaining independent statements and give a number of combinatorial examples.

2. Notation

An *arithmetic term* is an expression built up using the binary function symbols $+, .$, variables $x, y, z, \ldots$ and the constant symbols $\underline{0}, \underline{1}$. We write $\underline{n}$ for the arithmetic term

$$\underbrace{\underline{1} + (\underline{1} + (\underline{1} + \ldots + (\underline{1} + \underline{1})\ldots))}_{n \text{ copies of } \underline{1}.}$$

An *arithmetic formula* is an expression built up from the atomic formulae $p(x) = q(x), p(x) < q(x)$ where $p(x), q(x)$ are arithmetic terms, using parenthesis, the connectives $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\rightarrow$ (implies), $\longleftrightarrow$ (if and only if), and the quantifiers $\forall x, \exists x$. So for example

$$\underline{1} < z \wedge \forall x \forall y (x.y = z \rightarrow (x = z \vee y = z))$$

is an arithmetic formula. An *arithmetic statement* is an arithmetic formula in which all the variables are quantified.

Peano's (first order) axioms consist of the arithmetic statements

$$\begin{aligned}
&\forall x(x + \underline{0} = x)\\
&\forall x \forall y(x + (y + \underline{1}) = (x + y) + \underline{1})\\
&\forall x(x.\underline{0} = \underline{0})\\
&\forall x \forall y(x.(y + \underline{1}) = (x.y) + \underline{1})\\
&\forall x \forall y(x < y \longleftrightarrow \exists z((x + z) + \underline{1} = y)
\end{aligned}$$

together with the induction schema

$$\forall x((\Theta(x, \underline{0}) \wedge \forall y(\Theta(x, y) \rightarrow \Theta(x, y + \underline{1})) \rightarrow \forall y \Theta(x, y))$$

where $\Theta(x, y)$ is an arithmetic formula.

For an arithmetic formula Θ we write $P \vdash \Theta$ if there is a proof of Θ in the predicate calculus using only axioms from P. So an arithmetic statement Θ is independent of arithmetic if $P \nvdash \Theta$ and $P \nvdash \neg\Theta$.

Arithmetic formulae can be "ranked" in terms of the complexity of their quantifiers. Let

$$\begin{aligned}
&\exists x < y\ \Theta \quad \text{stand for} \quad \exists x(x < y \wedge \Theta)\\
&\forall x < y\ \Theta \quad \text{stand for} \quad \forall x(x < y \rightarrow \Theta).
\end{aligned}$$

We call these bounded quantifiers. A formula is said to be $\sum_0$ (or $\prod_0$) if it can be built up from the atomic formulae using just the connectives and bounded quantifiers. A formula is $\sum_{n+1}$ if it is of the form $\exists x_1 \exists x_2 \ldots \exists x_k \Theta$ for some $\Theta \in \prod_n$ and is $\prod_{n+1}$ if it is of the form $\forall x_1 \forall x_2 \ldots \forall x_k \Theta$ for some $\Theta \in \sum_n$.

In practice we shall say that a formula χ is $\sum_n(\prod_n)$ if there is a $\sum_n(\prod_n)$ formula Θ such that

$$P \vdash \Theta \longleftrightarrow \chi.$$

With this broadening of the definition of $\sum_n(\prod_n)$ every arithmetic formula is $\sum_n(\prod_n)$ for some n.

A ***structure for arithmetic*** M consists of a set $dom(M)$ (the domain of M), together with binary functions $+_M, \cdot_M$ on $dom(M)$, a binary relation $<_M$ on $dom(M)$ and distinguished elements $0_M, 1_M$ from $dom(M)$. In practice we write $x \in M$ instead of $x \in dom(M)$ etc. For an arithmetic formula $\Theta(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in M$ we write $M \models \Theta(a_1, \ldots, a_n)$ if the statement obtained by replacing everywhere in $\Theta, x_1, \ldots, x_n, +, \cdot, <, \underline{0}, \underline{1}, \wedge, \vee, \forall\neg, \rightarrow, \leftrightarrow, \forall x, \exists x$ by $a_1, \ldots, a_n, +_M, \cdot_M, <_M, 0_M, 1_M$, and, or, not, implies, if and only if, for all $x \in dom(M)$, there exists $x \in dom(M)$, respectively is true. We say M is an arithmetic, or is a model of P, if $M \models \Theta$ for every $\Theta \in P$.

Of course the standard natural numbers with addition, multiplication etc gives an arithmetic, denoted by N and called the standard model. So as an example, for $a \in \mathbb{N}$,

$$N \models \underline{1} < a \wedge \forall x \forall y (x.y = a \rightarrow (x = a \vee y = a)) \Longleftrightarrow a \text{ is prime.}$$

On the face of it, in terms of combinatorics, arithmetic statements seem to be rather limited. For example the statement of the finite Ramsey Theorem

$$\forall b \forall c \forall d \exists a, a \rightarrow (b)_c^d$$

talks about finite sets and functions and is not, as it stands an arithmetic statement. However given a finite set $\{a_1, \ldots, a_n\} \subseteq \mathbb{N}$ we can, by using the Chinese remainder theorem, find numbers n, m, k such that for $i < n$,

$$k = a_{i+1} \ mod(m(i+1)+1)$$
$$a_{i+1} < m(i+1)+1.$$

Hence we can code the set $\{a_1, \ldots, a_n\}$ as the three numbers n, m, k, in such a way that we can find an arithmetic formula $\lambda(x, y, z, w)$ such that for $b \in \mathbb{N}$

$$b \in \{a_1, \ldots, a_n\} \Longleftrightarrow N \models \lambda(n, m, k, b).$$

Continuing in this way to code also finite maps we can find an arithmetic statement χ "coding Ramsey's Theorem" such that

$$[\forall b \forall c \forall d \exists a,\ a \rightarrow (b)_d^c] \Longleftrightarrow N \models \chi.$$

Now given an arbitrary structure M for arithmetic we say that Ramsey's Theorem holds in M (or that $M \models \forall b \forall c \forall d \exists a,\ a \rightarrow (b)_d^c$) just if $M \models \chi$. In a sense then to say that "Ramseys Theorem holds in M" is ambiguous since it depends on the choice of χ, there may have been several options for χ. However it is an empirical fact that if M is a model of arithmetic (indeed much less will do) then any two natural choices for χ will be equivalent in M. So in future when we write say

$$M \models \forall b \forall c \forall d \exists a,\ a \rightarrow (b)_d^c$$

we shall mean that $M \models \chi$ for some such natural fixed arithmetic statement χ.

3. Arithmetic

In order to explain the origins of these independence results we need to investigate a little more deeply the properties of models of P.

Let M be an arithmetic. Then $<_M$ is a discreet linear ordering of *dom* (M). The least element is 0_M the next 1_M, the next $1_M +_M 1_M$ and so on. Thus M starts of looking like $\mathbb{N}$. Furthermore the axioms ensure that $+_M, \cdot_M$ act like the standard addition and multiplication on here. Thus M has a subarithmetic isomorphic to N and we shall agree to identify this subarithmetic with N. Notice then that, with respect to $<_M, N$ forms an initial segment of M, so every element of $M - N$ lies above every standard natural number.

We call a non–empty, proper initial segment I of M which is closed under $+_M$ and $\cdot_M$ and contains 1_M a *cut*, denoted $I \subset_e M$. We treat such a cut I as a structure for arithmetic by setting $+_I$ etc to be the restriction of $+_M$ to I etc. So if M is a model of P and $M \not\cong N$ (we can show such arithmetics exist) then there is a cut in M which is itself a model of P, namely N. Indeed by results of Friedman (1973) and Gaifman (1972) such an M has many cuts which are arithmetics.

Of course we cannot define a cut by an arithmetic formula since if say $I \subset_e M$ and

$$a \in I \Leftrightarrow M \models \Theta(a)$$

then

$$M \models \Theta(\underline{0}) \wedge \forall x(\Theta(x) \longrightarrow \Theta(x + \underline{1})) \wedge \neg\forall x \Theta(x)$$

so one of the induction axioms would fail in M. However whilst we cannot define any cut we do have methods for detecting cuts having certain properties. To be more precise let $\mathcal{P}$ be a property of cuts (a cut here is to be thought of as given along with its extending arithmetic). Then

Definition. A formula $Y(x, y, z)$ is an *indicator* for cuts satisfying $\mathcal{P}$ if

(i) $Y(x, y, z)$ is $\sum_1$
(ii) $P \vdash \forall x \forall y \exists! z Y(x, y, z)$

i.e. for every x, y there is a unique z such that $Y(x, y, z)$. For this reason in an arithmetic M we think of Y as defining a function Y^M such that

$$Y^M(a, b) = c <=> M \models Y(a, b, c),$$

and we often write $Y(x, y) = z$ for $Y(x, y, z)$.

(iii) For any countable arithmetic M and $a, b \in M$, $Y^M(a, b) \in' N \Leftrightarrow \exists I \subset_e M$ having property $\mathcal{P}$ such that $a \in I$ and $b \in' I$.

In other words $Y^M(a, b)$ is above every number in N just if there is a cut $I \subset_e M$ satisfying $\mathcal{P}$ and lying in the interval (a, b). So Y^M indicates the presence of cuts satisfying $\mathcal{P}$.

As an example just let $\mathcal{P}$ be vacuous. Then $\exists$ cut $I \subset_e M$, $a \in I$ and $b \in' I \Leftrightarrow (a + 2)^n < b$ for all $n \in N$ (since then $I = \{x \mid x < (a + 2)^n$

some $n \in N\}$ will do) $\Leftrightarrow Y^M(a,b) \in' N$ where $Y^M(a,b) = min.\ c$ such that $(a+2)^c > b$.

So the formula $Y(x,y,z)$ expressing

$$z \text{ is minimal such that } (x+2)^z > y$$

is an indicator for cuts satisfying $\mathcal{P}$. (Property (ii) is believable, property (i) is less obvious but is still true for a suitable formulation of exponentation). It turns out that most interesting properties $\mathcal{P}$ have indicators. In particular there are many indicators when $\mathcal{P}$ is the property of being a model of P. The importance of such indicators is given by the following result which is a corollary to a result of Kirby–Paris.

3.1 Theorem *(See lemma 1 of Paris 1978). Let Y be an indicator for models of P and let $Y(x,y) > z$ abbreviate $\exists t(Y(x,y,t) \wedge t > z)$. Then*
(i) $P \vdash \forall x \exists y Y(x,y) > \underline{n}$
(ii) $N \models \forall x \forall z \exists y Y(x,y) > z$,
(iii) $P \not\vdash \forall x \forall z \exists y Y(x,y) > z$. □

By the completeness theorem for the predicate calculus if $P \vdash \Theta$ and M is an arithmetic then $M \models \Theta$. Hence (ii) ensures that $P \not\vdash \neg\forall x \forall z \exists y Y(x,y) > z$ so $\forall x \forall z \exists y Y(x,y) > z$ is independent of arithmetic.

In practice most natural indicators also satisfy a convexity condition like

$$P \vdash \forall x \forall y (Y(\underline{0}, x+y) \leq \underline{1} + Y(\underline{0}, x) + Y(x, x+y)).$$

In such cases we can fix x to be zero in the above theorem to give

3.2 Theorem.

$$\forall z \exists y Y(\underline{0}, y) > z \textit{ is independent of arithmetic.}$$ □

If one wanted an explanation of why $\forall x \forall z \exists y Y(x,y) > z$ was independent of arithmetic we could say that it was because for given x, z the y must be so ineffably larger that in arithmetic there is no way of proving that it exists. To be more precise define for $m \in \mathbb{N}, g_m : \mathbb{N} \longrightarrow \mathbb{N}$ by

$$g_m(x) = \text{ the least } y \text{ such that } Y^N(x,y) > m$$

and

$$g(x) = g_x(x) = \text{ the least } y \text{ such that } Y^N(x,y) > x.$$

i.e. g is the diagonalization of the g_m's. Then the g_m's are recursive and provably total by Theorem 3.1 (i). Furthermore

3.3 Theorem *(See lemma 8 of Paris 1978). If f is recursive and provably total then*

$$\exists m \forall x f(x) < g_m(x).$$ □

Since g eventually dominates each g_m it follows from this that g is not provably total and furthermore that g is increasing inconceivably fast.

What we need now is to find suitable indicators for models of P such that the corresponding independent statements are saying something interesting. Our first example was found by L. Harrington (see Paris, Harrington 1977, Paris 1978).

Example 1

Let $[a_1, a_2] \underset{*}{\rightarrow} (b)^c_d$ be the statement that whenever F maps the c element subsets of $[a_1, a_2] (= \{a_1, a_1 + 1, \ldots, a_2\})$ into $\{1, 2 \ldots, d\}$ then there is $X \subseteq [a_1, a_2]$ homogeneous for F such that $|X| \geq b, min\ (X)$.

Now let $Y(x, y, z)$ be the natural formulation of the statement

$$z \text{ is maximal such that } [x, y] \underset{*}{\rightarrow} (z+1)^z_z.$$

Then $Y(x, y, z)$ is an indicator for models of P (for a proof see Lascar, Paris 1978). Hence, identifying y with its set of predecessors.

3.4 Theorem *(Harrington–Paris 1978).*

$$\forall z \exists y, y \underset{*}{\rightarrow} (z+1)^z_z \textit{ is independent of arithmethic.} \qquad \Box$$

This result can be refined a little since the lower z can be fixed. The best result known at present is

3.4' Theorem *(J. Quinsey).*

$$\forall z \exists y, y \underset{*}{\rightarrow} (z+1)^z_3 \textit{ is independent of arithmetic.} \qquad \Box$$

It is not known if 3 can be replaced by 2. Clearly then it is the top z which is really important one.

There are now many variations on Theorem 4. The following one is due to H. Friedman (unpublished). Say that $F : [b]^n \longrightarrow N$ is *decreasing* if for all $a_1 < a_2 < \ldots < a_n < b,\ F(a_1, a_2, \ldots, a_n) \leq a_1$

3.4" Theorem *(H. Friedman). The statement $\forall w \forall z \exists y$ such that if $F : [y]^z \longrightarrow y$ is decreasing then $\exists X \subseteq y, |X| \geq w \wedge |F[X]^z| \leq w$ is true but independent of arithmetic.* $\Box$

It follows by Theorem 3.1 and some fairly easy properties of $\underset{*}{\rightarrow}$ that for each $n \in \mathbb{N}$,

$$P \vdash \forall z \exists y, y \underset{*}{\rightarrow} (n+1)^n_z.$$

In fact what happens is that as we increase n so these proofs use ever more complicated induction axioms. To be precise let $I\Sigma_n$ be the same as Peano's axioms but with the induction scheme only for the case when Θ is Σ_n. Then

3.5 Theorem *(Paris 1980).*
(i) $I\Sigma_n \nvdash \forall z \exists y, y \underset{*}{\rightarrow} (n+2)^{n+1}_z$.
(ii) For each $m \in \mathbb{N}, I\Sigma_n \vdash \forall x \exists y, [x,y] \underset{*}{\rightarrow} (n+2)^{n+1}_m$.
(iii) $I\Sigma_{n+1} \vdash \forall z \exists y, y \underset{*}{\rightarrow} (n+2)^{n+1}_z$. □

Theorems 3.3 and 3.4 together raise a slightly embarassing spectre. For consider the following common situation in combinatorics. We first show that a function, given by some combinatorial property, is a total function from $\mathbb{N}$ to $\mathbb{N}$. We then ask for an upper bound on f. What we want here is something like exponentiation, iterated exponentiation, etc. or maybe even some primitive recursive function. In other words we want to bound f in terms of functions built up from the basic functions of addition and multiplication by recursion and substitution. But all such functions are recursive and provably total. So, since we know that the function

$$g(n) = \text{ the least } m \text{ such that } [n,m] \underset{*}{\rightarrow} (n+1)^n_n$$

dominates all provably total recursive functions the problem "find an upper bound on the least m such that $[n,m] \underset{*}{\rightarrow} (n+1)^n_n$" has no *reasonable solution* in the sense in which the question is asked.

When Theorem 3.4 first became known Harrington and, independently, McAloon noticed a surprising connection between this independent statement and Gödel's original independence results. Using Gödel's coding trick it is possible to find an arithmetic statement $Con_1(P)$ saying, essentially, that "if there is a proof of a $\sum_1$ statement then that statement is true". Again it is a classical result that $Con_1(P)$ is independent of arithmetic. Now what Harrington and McAloon noticed was that for the indicator Y as above,

$$P \vdash Con_1(P) \leftrightarrow \forall x \forall z \exists y Y(x,y,z).$$

So whilst both these statements are independent of P they are actually equivalent in P. Such a connection between pure logic and combinatorics is quite surprising.

Furthermore this is not an isolated occurence. The same result holds for all the indicators mentioned in examples 2–5 below (although this is not a general theorem about indicators for models of P).

One consequence of this is that since $Con_1(P)$ implies Θ_G so does $\forall x \forall z \exists y, [x,y] \underset{*}{\rightarrow} (z+1)^z_z$. Indeed in Paris, Harrington (1977) this result is obtained directly in order to show that $\forall x \forall z \exists y, [x,y] \underset{*}{\rightarrow} (z+1)^z_z$ is independent.

In view of this result one might hope that the method of indicators would give an independent statement equivalent within arithmetic to Θ_G. However without some major reinforcements this cannot be since we know that Θ_G is $\prod_1$ whilst the form of the independent statement obtained from an indicator is $\prod_2$ and not $\prod_1$.

Example 2

This was the first example of a meaningful statement independent of arithmetic and was arrived at much more indirectly than example 1, as we shall now explain.

In early work of Kirby–Paris (1977) it was observed that many properties of cuts (given *with* the extending arithmetic) were symbiotic with the property of being a model of P, where properties $\mathcal{P}_1, \mathcal{P}_2$ of cuts are *symbiotic* if for any countable arithmetic M and $a, b \in M$,
$\exists I \subset_e M$ having property $\mathcal{P}_1$ (as a subset of M), $a \in I$ and $b \in' I, \Leftrightarrow \exists I \subset_e M$ having property $\mathcal{P}_2$ (as a subset of M), $a \in I$ and $b \in' I$.

In other words properties $\mathcal{P}_1, \mathcal{P}_2$ are symbiotic if arbitrarily close to cuts satisfying one there are cuts satisfying the other.

Now if two properties are symbiotic then they have the same indicators. This observation yields new indicators based on combinatorial rather than semantic properties of cuts. In particular strong cuts were known to be symbiotic with cuts which are models of P where we can define what it means for a cut to be strong as follows.

Let $I \subset_e M$. We say $X \subseteq I^n$ is coded in M if there are $c, d, e \in M$ such that for all $x_1, \ldots, x_n \in I$,

$$< x, \ldots, x_n > \in X \Leftrightarrow c = 0 mod\ [d(x_1 + x_2 e + \ldots + x_n e^{n-1} + 1) + 1]$$

(Compare this with the coding of finite sets when $n = 1$.) Notice that if M is countable then there will be $2^{\aleph_O}$ subsets of I but only $\aleph_O$ subsets of I coded in M. We say $I \subset_e M$ is *strong* if whenever $\beta \in I$ and $F : [I]^3 \to \beta$ is coded in M then there is a homogeneous set $A \subseteq I$ which is coded in M and unbounded in I. Notice that the property of being strong also depends on the extending arithmetic M.

Now a direct attempt to produce an indicator for strong cuts produces the following train of definitions.

Let $S \subseteq \mathbb{N}$ be finite. Define any such S to be 0–dense. Define S to be 1–dense if $S \neq \emptyset$ and $3 + min\ (S) < |S|$. Define S to be $(n+2)$–dense if whenever $f : [S]^3 \longrightarrow 2$ then $\exists T \subseteq S$ such that T is $(n+1)$–dense and homogeneous for f. Now let $Y(x, y, z)$ be the natural formulation in arithmetic of the statement

"z is maximal such that $[x, y]$ is z-dense."

Then Y is an indicator for strong cuts and hence, by symbiosis, for models of P. Hence

3.6 Theorem *(Paris 1978).*

$(\forall z \exists y[0, y]$ *is* $z-$*dense) is independent of arithmetic.* □

It is an interesting open problem as to what happens if we replace $f : [S]^3 \longrightarrow 2$ by $f : [S]^2 \longrightarrow 2$ in the above definition.

Example 3

There are many equivalent formulations of the definition of a strong cut. Using an alternative definition and a natural indicator corresponding to this definition P. Pudlák gave an independence result which we now describe.

Let $A = \{a_1, a_2, \ldots, a_n\}$. We say that A is an approximation to the partial function f, where $dom\ (f) \subseteq [0, a_n]$, if for $1 \leq i < n, f''[0, a_i) \subseteq [0, a_{i+1}) \bigcup (a_n, \infty)$. Now let $X \subseteq \mathbb{N}$ be finite. Define X to be 0–dense for any such X, to be 1–dense if $|X| \geq 3$ and to be $(n+2)$–dense if for every partial function f with $dom\ (f) \subseteq [0, max(X)]$ there is an approximation A of f such that $A \subseteq X$ and A is $(n+1)$–dense.

Then as in example 2 the natural formulation $Y(x, y, z)$ of the statement

"z is maximal such that $[x, y]$ is z-dense"

is an indicator for strong cuts and hence for models of P. So we obtain an analogous result to Theorem 3.6.

Example 4

Following the discovery of the independent statement in example 1 Ketonen and Solovay (1981) set out to prove by *purely combinatorial* means that the function

$$\sigma(n) = \text{ the least } m \text{ such that } m \underset{*}{\rightarrow} (n+1)^n_n$$

eventually dominated every provably total recursive function. The independence result of Theorem 3.4' would then follow. Their approach was as follows. For each ordinal $\alpha < \epsilon_0$ (recall ϵ_o is the least solution of $\alpha = \omega^\alpha$) and each $n \in \mathbb{N}$ define $\{\alpha\}(n)$ as follows.

$$\begin{aligned}
&\{0\}(n) = 0,\\
&\{\beta + 1\}(n) = \beta,\\
&\{\omega^{\beta+1}(\gamma + 1)\}(n) = \omega^{\beta+1}\gamma + \omega^\beta n,\\
&\{\omega^\lambda(\gamma + 1)\}(n) = \omega^\lambda\gamma + \omega^{\{\lambda\}(n)} \text{ for limit } \lambda < \epsilon_0,\\
&\{\epsilon_0\}(0) = \omega,\ \{\epsilon_0\}(n+1) = \omega^{\{\epsilon_0\}(n)}
\end{aligned}$$

Now for $\alpha \leq \epsilon_0$ define $F_\alpha : \mathbb{N} \longrightarrow \mathbb{N}$ by

$$\begin{aligned}
&F_0(x) = x + 1\\
&F_{\beta+1}(x) = F_\beta^{x+1}(x)\\
&F_\lambda(x) = max\ \{F_{\{\lambda\}(n)}(x) \mid n \leq x\} \text{ for limit } \lambda.
\end{aligned}$$

S. Wainer (1970) has shown that if $f : \mathbb{N} \longrightarrow \mathbb{N}$ is recursive and provably total then f is eventually dominated by F_α for some $\alpha < \epsilon_0$. Also if $\alpha < \beta \leq \epsilon_0$ then F_β eventually dominates F_α.

What Solovay and Ketonen show directly is that σ has approximately the same rate of growth as F_{ϵ_0} and hence dominates all provably total recursive functions.

In the course of their paper they introduce the notion of α–large sets for $\alpha < \epsilon_0$. Precisely for a finite set $\{s_0, s_1, \ldots, s_n\}$ in ascending order and $\alpha < \epsilon_0$ set

$$\{\alpha\}(\emptyset) = \alpha$$
$$\{\alpha\}(s_0, s_1, \ldots, s_n) = \{\{\alpha\}(s_0)\}(s_1, \ldots, s_n).$$

Define $\{s_0, s_1, \ldots, s_n\}$ to be α–large if $\{\alpha\}(s_0, s_1, \ldots, s_n) = 0$. For $k \in \mathbb{N}$ let

$$H_\alpha(k) = \text{ the least } m \text{ such that } [k, m] \text{ is } \alpha- \text{ large.}$$

Then Solovay and Ketonen show that H_{ω^α} is essentially the same as F_α. From this it follows that

3.7 Theorem *(Ketonen–Solovay 1981).*

$(\forall \alpha < \epsilon_0 \exists y, [0, y]$ *is* α*–large) is independent of arithmetic.* □

Their proof of this result does not mention indicators. However a direct proof of Theorem 3.7 can be given which does use indicators. Such a proof is given in Paris (1980). The idea is to define for $n, m \in \mathbb{N}$,

$$\omega_n^m = \omega^{\omega^{\cdot^{\cdot^{\omega^m}}}} \Big\} n\text{–times}$$

and let $Y(x, y, z)$ be a suitable formulation of

"z is maximal such that $[x, y]$ is $\omega_z^0 -$ large".

Then as shown in Paris (1980) Y is an indicator for cuts which are n–extendible for all $n \in \mathbb{N}$ where n–extendible cuts are defined as follows:
$I \subset_e M$ is 1– extendible if there is an arithmetic K extending M such that
(i) $I \subset_e K$ and $\exists \alpha \in K - I$ such that for all $\beta \in M - I, \alpha < \beta$.
(ii) Whenever $\Theta(x)$ is an arithmetic formula and $a \in M$ then $M \models \Theta(a) \Leftrightarrow K \models \Theta(a)$.

We express (i) and (ii) by $M <_I K$. We say $I \subset_e M$ is n–extendible (as a cut in M) if $\exists K_1, \ldots, K_n, M <_I K_1 <_I K_2 <_I \ldots <_I K_n$. The property of being n–extendible for all n is symbiotic with the property of being a model of P. Hence Y is also an indicator for models of P and so we obtain

3.7' Theorem *(Paris 1980).*

$(\forall z \exists y [0, y]$ *is* ω_z^0*–large) is independent of arithmetic.* □

Theorem 3.7 follows from this since $\epsilon_0 = lim_n\, \omega_n^0$.

As with the first example we can tell just how much induction is needed to show the existence for each x of y such that $[x, y]$ is α–large.

3.8 Theorem *(Paris 1980).*
(i) $I\Sigma_n \nvdash \forall x \forall z \exists y, [x, y]$ *is* ω_n^z*–large.*
(ii) For each $m \in \mathbb{N}, I\Sigma_n \vdash \forall x \exists y, [x, y]$ *is* ω_n^m*–large.*
(iii) $I\Sigma_{n+1} \vdash \forall x \forall z \exists y, [x, y]$ *is* ω_n^z*–large.* □

Example 5

Whilst Theorem 3.7 is perhaps a little technical and its known proofs rather unpleasant it is a very useful result in that it allows us to directly compare other "ordinal valued measures on finite sets" with the notion of α–largeness to obtain meaningful independent statements. We give an example.

For $m, n \in \mathbb{N}, n > 1$ define the *base n* representation of m as follows. First write m as the sum of powers of n. (E.g. if $m = 266, n = 2$ write $266 = 2^8 + 2^3 + 2^1$.) Now write each exponent as the sum of powers of n. (E.g. $266 = 2^{2^3} + 2^{2+1} + 2^1$.) Repeat with exponents of exponents and so on until the representation stabilizes. (E.g. 266 stabilizes at the representation $2^{2^{2+1}} + 2^{2+1} + 2^1$.) Now define the number $G_n(m)$ as follows. If $m = 0$ set $G_n(m) = 0$. Otherwise set $G_n(m)$ to be the number produced by replacing every n in the base n representation of m by $n + 1$ and then subtracting 1. (E.g. $G_2(266) = 3^{3^{3+1}} + 3^{3+1} + 2$.)

Now define the Goodstein sequence for m starting at 2 by, $m_0 = m, m_1 = G_2(m_0), m_2 = G_3(m_1), m_3 = G_4(m_2), \ldots$

$$\begin{aligned} \text{E.g } 266_0 &= 266 = 2^{2^{2+1}} + 2^{2+1} + 2, \\ 266^1 &= 3^{3^{3+1}} + 3^{3+1} + 2 \sim 10^{38}, \\ 266_2 &= 4^{4^{4+1}} + 4^{4+1} + 1 \sim 10^{616}, \\ 266_3 &= 5^{5^{5+1}} + 5^{5+1} \sim 10^{10,000}. \end{aligned}$$

3.9 Theorem.
(i) (Goodstein 1944.) $N \models \forall x \exists y, x_y = 0$
(ii) (Kirby–Paris 1982.) $\forall x \exists y, x_y = 0$ *is independent of arithmetic.* □

To see the connection between Theorems 3.7 and 3.9 associate with each pair $n, m \in \mathbb{N}, n > 1$ the ordinal $O_n(m)$, in Cantor normal form, formed by replacing n everywhere in the base n representation of m by ω.

By comparing the ordinals

$$\begin{aligned} &O_2(m_0), O_3(m_1), O_4(m_2), \ldots \\ \text{and } \quad &O_2(m_0), \{O_2(m_0)\}(2), \{O_2(m_0)\}(2,3), \ldots \end{aligned}$$

we can show that termwise the top sequence dominates the bottom sequence. Also since $O_{k+2}(m_k) = 0$ just if $m_k = 0$ it follows that if $m_k = 0$ then $\{O_2(m_0)\}(2, 3, \ldots, 2 + k) = 0$ so $[2, 2 + k]$ is $O_2(m_0)$–large. Since this proof can be carried out in arithmetic, by considering $m = 2^{2^{\cdot^{\cdot^{2}}}} \Big\}(n-times)$ we see that the Theorem 3.9 (ii) follows from Theorem 3.7. The first part of Theorem 3.9 follows by noticing that whilst the $O_{k+2}(m_k)$ are non–zero,

$$O_2(m_0) > O_3(m_1) > O_4(m_2) > \dots$$

and recalling that there cannot be an infinite decreasing sequence of ordinals.

Of course we know that as we increase m the least k such $m_k = 0$ must grow phenomenally quickly. Some early values of this k are

$$\begin{array}{ll} m = 1 & k = 1 \\ m = 2 & k = 3 \\ m = 3 & k = 5 \\ m = 4 & k = 3 \times 2^{402,653,211} - 3. \end{array}$$

Example 6

The fact that the independence results introduced as far were all equivalent to $Con_1(P)$ suggests obtaining meaningful independence result by relating combinatorial assertions to known independent statements obtained via proof theoretic methods, (as in Paris, Harrington 1977.)

Of course such an approach had been available for many years but had remained untapped. A considerable body of work along these lines has been carried out by H. Friedman. The following theorem is a particularly fine example of the marque.

3.10 Theorem.

(i) *(Finite Kruskal Theorem.) For every non–zero $k \in \mathbb{N}$ there exists a non–zero $n \in \mathbb{N}$ such that for every sequence of trees $T_1, \dots, T_n$ such that T_i has at most $k + i$ vertices there is a homeomorphism of one tree into a later tree.*

(ii) *(H. Friedman.) The Finite Kruskal Theorem is independent of arithmetic.* □

Indeed Friedman shows much more, that the Finite Kruskal Theorem is independent of predicative analysis.

Recently Theorem 3.10 was sharpened with a surprising exactness. In order to state the corresponding result which is due to Loebl and Matoušek let us state first the following:

Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be a function. Denote by FKT_f the statement:

For every non–zero $k \in \mathbb{N}$ there exists a non–zero $n \in \mathbb{N}$ such that for every sequence of trees $T_1, \dots, T_n$ such that T_i has at most $k + f(i)$ vertices there is a homeomorphism of one tree into another tree.

Then we have:

3.11 Theorem *(Loebl, Matoušek 1987).*

(i) *FKT_f is a true statement for every f.*

(ii) $FKT_{\frac{1}{2}\ \log}$ *is provable in arithmetic.*
(iii) $FKT_{4\ \log}$ *is independent of arithmetic.*

These results are also related to long games played on graphs. Nešetřil and Thomas (1987) give another survey of this recent development.

4. Conclusions

In this paper we have given a number of statements which are independent of arithmetic. Certainly such results are interesting and certainly more such results will be obtained in the future by these methods. However the long term benefit to cominatorics will come from a deeper analysis of this independence and the broadening of available techniques – as for example, in the papers of Solovay and Ketonen (1981) and the author (1980).

As an example of this we site the following theorem which is proved in Paris (1981).

4.1 Theorem *(Paris 1981. Also (ii) independently by H. Friedman).*
(i) *For any arithmetic statement* Θ *and* $n > 0$, $B\Sigma_{n+1} \vdash \Theta \Leftrightarrow I \models \Theta$ *for every* n*–extendible cut* $I \subset_e M$ *and countable arithmetic* M.
(ii) *For any* $\Theta \in \Pi_{n+2}$,

$$B\Sigma_{n+1} \vdash \Theta \Leftrightarrow I\Sigma_n \vdash \Theta \qquad \square$$

Here $B\Sigma_{n+1}$ is Peano's axioms with the induction axiom replaced by the Σ_{n+1}–collection schema

$$\forall x \forall z [\forall y < z \exists w \Theta(x,y,z,w) \longrightarrow \exists t \forall y < z \exists w < t \Theta(x,y,z,w)]$$

where $\Theta \in \Sigma_{n+1}$.

Now in practise for combinatorial Θ it may be much easier to show that Θ holds in all n–extendible cuts rather than to show $B\Sigma_{n+1} \vdash \Theta$. The reason, for combinatorialists, of being interested in the amount of Peano's axioms need to prove a result is that this may give information on the size of functions implicit in the result. For example to show a function has a primitive recursive bound it is enough to show that the totally of the function is a theorem of $I\sum_1$.

But by (i), (ii) to show this we only need to show that this theorem holds in all 1–extendible cuts.

So it is to be hoped that the better understanding we are beginning to aquire of these independence results will in the future make a positive as well as negative contribution to combinatorics.

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Boolean Complexity and Ramsey Theorems

Pavel Pudlák

1. General Remarks

The aim of this paper is to bring attention to some connections between these two fields. In the complexity theory there are difficult open problems most of which are essentially of combinatorial character. It is generally believed that some interaction between complexity theory and combinatorics may help to solve these problems.

An n-dimensional *Boolean function* is any mapping $f : \{0,1\}^n \to \{0,1\}$. Thus a Boolean function can be also viewed as a partition of the n-cube $\{0,1\}^n$. A Boolean function is called symmetric if $f(a_1, \ldots, a_n)$ depends only on the number of 1's among $a_1, \ldots, a_n$. We call the set of all vectors with exactly k ones the k-th *level* of the n-cube. Hence a symmetric function is a function which is constant on every single level. Given a complete basis of connectives, we define *the formula size complexity* $L(f)$ of a function f to be the size of the smallest formula realizing f, where the size of a formula is conveniently defined to be the number of all the occurrences of variables in it. (E.g. $x_1 \wedge (\neg x_1 \vee x_2)$ has size 3).

Theorem A. *For every basis there exists $\epsilon > 0$ such that if f is n-dimensional and*

$$L(f) \leq \epsilon \cdot n(\log\log n - \log r),$$

then there exists an interval $I = (\mathbf{0}, \mathrm{a})$ of length r in the n-cube such that

(1) *$f \mid I$ is symmetric;*

(2) in $f \mid I$ all the even levels, with a possible exeption of the 0-th level, are of the same color, and all the odd levels are of the same color. (It is not excluded that f is constant on I.)

Theorem B. *For the basis of all at most binary connectives there exists $\epsilon > 0$ such that it f is n-dimensional and*

$$L(f) \leq \epsilon \cdot n \cdot (\log n - \log r),$$

then there exists an interval $I = (\mathbf{a}, \mathbf{b})$ of length r in the middle of the n-cube such that

(1) $f|I$ is symmetric;

(2) in $f|I$ all the even levels are of the same color, and all the odd levels are of the same color. (The exact meaning of "the middle" is that the number of 1's in $\mathbf{a}$ equals to the number of 0's in $\mathbf{b}$ possibly -1.)

Theorem A is a reformulation of the Hodes-Specker theorem (Hodes, Specker 1968) with the bound proved in (Pudlák 1984), the second theorem is a reformulation of the main theorem of Fischer, Meyer, Paterson (1982). The bounds are known to be of the best growth rate. I see at least three connections of these theorems to Ramsey theory.

1. The general form of the statement is:"If an object is of small complexity, then it is locally very simple". If we consider e.g. the number of colors as the complexity of a coloration (say of a complete graph), then the Ramsey theorem is of this form.
2. Using Ramsey theorem one can prove e.g. that there exists a function $r(n)$, with $\lim_{n\to\infty} r(n) = \infty$, such that for every n-dimensional Boolean function f there exists an interval $I = (\mathbf{0}, \mathbf{a})$ of length $r(n)$ such that $f \mid I$ is symmetric. In case f is of small complexity Theorem A extends the information about $f|I$ in two ways: gives us a larger interval I and the condition (2).
3. The original proof of the Hodes-Specker theorem and the proof of Fischer-Meyer-Paterson theorem use the standard heuristic "divide and take the largest one" used also for Ramsey theorems. Ramsey theorem was also used in the proof of a generalization of Hodes-Specker theorem by Vilfan (1976). Ramsey theorem is the corner-stone of the proof of the bound of Theorem A in Pudlák (1984). Roughly speaking the proof goes as follows. Given a Boolean formula $\alpha(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are the propositional variables of the formula, we define *the induced formula* αX for every $X \subseteq \{x_1, \dots, x_n\}$ in a suitable way. The formula is called ***homogeneous*** if for every $X, Y, |X| = |Y| = 2$, αX is isomorphic to αY, (which means that if we substitute the first variable of X for the first variable of Y and the second variable of X for the second variable of Y in αY, then we obtain αX). Given r, if the complexity of $\alpha(x_1, \dots, x_n)$ is small (i.e. $\leq \epsilon \cdot n \cdot (\log\log n - \log r)$), then using the Ramsey theorem one can find a subset of variables H of cardinality r such that αH is homogeneous. Then it is shown (and this is the difficult part of the proof) that every homogeneous formula determines a Boolean function which satisfies (1), (2) of Theorem A.

The theorem of Ajtai (1983) and Furst, Saxe and Sipser (1981) can be stated also in a form resembling the Ramsey theorem. A theorem of Hodes-Specker type for branching programs was announced in Pudlák (1984).

During the preparation of this book several new lower bounds to the complexity of Boolean functions have been obtained. A large part of these results uses some version of the Ramsey theorem. The lower bound of this paper

$n \cdot (\log\log n)^\epsilon$, $\epsilon > 0$, can now be improved to $\Omega(n \cdot \log n)$ using the results of Babai et al. (to appear). These results are based on a Ramsey-type combinatorial lemma, which was discovered independently also by Alon and Maass (1986). This technique was further developed in Babai et al. (1989); they obtained a bound $\Omega(n \cdot (\log n)^2)$ which is currently the largest lower bound obtained using Ramsey-type arguments. The lower bounds of Ajtai (1983) and Fischer et al. (1982) have been also significantly improved by Yao (1985) and Hestad (1986). In a recent paper Razborov (to appear) uses also Ramsey theorem to obtain nonlinear bounds for the directed switching networks computing symmetric Boolean functions.

2. An Example of a Lower Bound to Formula Size Complexity

In this section we shall present a nonlinear lower bound to formula size of some Boolean functions. The proof uses a quite different approach, "the graph theoretical method", which is the approach to prove lower bounds using only graph theoretical properties of the Boolean formula (which is essentially a labelled tree), the Boolean circuit (which is a labelled directed acyclic graph) etc. The reason for including this result in this paper is that the proof uses the Ramsey theorem. However it is quite likely that a better bound can be proved without it.

The idea of the proof is to show that any circuit, in particular formula, which realizes some Boolean function must contain a special graph called superconcentrator. Then we show that the superconcentrators which are embeddable in formulas must have a nonlinear size, (Lemma 2.5). The idea to use superconcentrators for lower bounds is not new, but we think that it has not been used for formula size complexity. (Better lower bounds follow from recent results of Babai, Pudlák, Rödl and Szemerédi.)

It should be stressed that the idea does not work for circuit size, since there are superconcentrators of linear size. For formula size we cannot get a much better bound too, since there is an upper bound for a larger class of superconcentrators, namely superconcentrators of depth 2, of the form $O(n \cdot (\log n)^2)$, see Pippenger (1982). (I owe for this observation to Ravi Boppana.) We do not know, if this can be used to show such an upper bound for the formula size of the functions that we consider.

For an overview of results on superconcentrators see Dolev et al. (1983).

For $\mathbf{a}, \mathbf{b} \in \{0,1\}^n$ define

$$\mathbf{a}|\mathbf{b} = (a_{i_1}, \ldots, a_{i_r}),$$

where $i_1 < \ldots < i_r$ and

$$i \in \{i_1, \ldots, i_r\} \leftrightarrow b_i = 1.$$

Theorem 2.1. *There exists $\epsilon > 0$ such that if $f(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t})$ is a $4n$-dimensional Boolean function with $|\mathbf{x}| = |\mathbf{y}| = |\mathbf{z}| = |\mathbf{t}| = n > 1$, and*

$$\sum c_i = \sum d_i \to (f(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = 1 \leftrightarrow \mathbf{a} \mid \mathbf{c} = \mathbf{b} \mid \mathbf{d}),$$

then in the basis of all at most binary connectives

$$L(f) \geq n \cdot (\log\log n)^{\epsilon}.$$

We need some lemmas. If G is a graph and $A_1, \ldots, A_n, B_1, \ldots, B_n$ are pairwise disjoint subsets of vertices of G then we say that there are *vertex* (resp. *edge*) *disjoint paths between* $\{A_1, \ldots, A_n\}$ *and* $\{B_1, \ldots, B_n\}$ if for some permutation $(i_1, \ldots, i_n)$ there are pairwise vertex (resp. edge) disjoint paths $P_1, \ldots, P_n$ such that P_k connects some point of A_k with some point of B_{i_k}, for $k = 1, \ldots, n$.

Lemma 2.2. *Let $f(\mathbf{x}, \mathbf{y})$ be a Boolean function such that for every $\mathbf{a}$, $\mathbf{b} \in \{0,1\}^n$*

$$f(\mathbf{a}, \mathbf{b}) = 1 \leftrightarrow \mathbf{a} = \mathbf{b},$$

then in any circuit realizing f there exist edge disjoint paths between $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$, where X_i, resp. Y_i, is the set of occurrences of x_i, resp. y_i, in the circuit.

Proof. Suppose not. Then we can choose $< n$ vertices which separate inputs $x_1, \ldots, x_n$ from $y_1, \ldots, y_n$, (by Menger's theorem). For every $\mathbf{a} \in \{0,1\}^n$ we assign to each of the chosen edges the value which appears on it during the computation of $f(\mathbf{a}, \mathbf{a})$. Then for some $\mathbf{a}$, $\mathbf{b} \in \{0,1\}^n$, $\mathbf{a} \neq \mathbf{b}$ we have the same value on all the chosen edges. We assign the values of the computation of $f(\mathbf{a}, \mathbf{a})$, resp. of $f(\mathbf{b}, \mathbf{b})$, to the part of the circuit containing. $x_1, \ldots, x_n$, resp. $y_1, \ldots, y_n$. This gives us the computation of $f(\mathbf{a}, \mathbf{b})$, but we get 1 on the output which is a contradiction.

If we consider formulas with at most binary connectives, then every vertex has degree at most 3, hence two loop-free paths are edge disjoint iff they are vertex disjoint.

Lemma 2.3. *Let C_q be the cycle $(\{0, 1, \ldots, q-1\} \mid \{\{i, j\} \mid j = i + 1 \bmod q\})$ and $0 = p_0 < p_1 < \ldots < p_5 < q$, $p_6 = 0$. Then there is no homomorphism F mapping C_q into a tree such that for $j = 0, 1, 2$,*

$$F(\{p_j, p_j + 1, \ldots, p_{j+1}\}) \cap F(\{p_{j+3}, p_{j+3} + 1, \ldots, p_{j+4}\}) = \emptyset.$$

Proof. Use induction over q. □

Lemma 2.4. *Let G be a graph, $k > 1$ and let $A_1, \ldots, A_s, B_1, \ldots, B_m$ be pairwise disjoint sets such that*

(i) $|A_1|, \ldots, |A_s|, |B_1|, \ldots, |B_m| \leq k$;

(ii) $s = 2k + 1$;

(iii) $m \geq R_l^3(s)$, *where* $l = s^3k^4 + 1$;
(iv) for every $1 \leq i_1 < \ldots < i_s \leq m$ *there are vertex disjoint paths from* $\{A_1, \ldots, A_s\}$ *to* $\{B_{i_1}, \ldots, B_{i_s}\}$.

Then G *is not a tree.*

Proof. We shall find a mapping of some cycle C_q into G satisfying the condition of Lemma 2.3. Suppose G satisfies the assumptions of Lemma 2.4. In order to simplify the notation, we shall assume that equality holds in (i). Choose some enumeration of the sets $A_u, B_v : A_u = \{a_{ui}\}_{i \leq k}, B_v = \{b_{vi}\}_{i \leq k}, u = 1, \ldots, s, v = 1, \ldots, m$. If $a_{ui} \longrightarrow b_{vj}$ is a path, then we say that j is its type. For every three element set $\{u, v, w\} \subseteq \{1, 2, \ldots, m\}$, whenever it is possible, choose a triple of vertex disjoint paths of the same type $a_{u'g} \longrightarrow b_{ui}$, $a_{v'h} \longrightarrow b_{vi}$, $a_{w'j} \longrightarrow b_{wi}$ and assign to it the color (u', v', w', g, h, j, i). If such a triple of paths does not exist, we assign to $\{u, v, w\}$ a fixed different color, say 0. By (iii) there exists a homogeneous set $H \subseteq \{1, 2, \ldots, m\}$, $|H| = s = 2k + 1$. By (iv) there are vertex disjoint paths from $A_1, \ldots, A_s$ to $\{B_t \mid t \in H\}$. Since there are only k types of paths and by (ii), at least three of these paths are of the same type. Hence the color of the triples of H is different from 0; say it is (u', v', w', g, h, j, i). Now we take arbitrary four elements $u < v < w < t$ of H. Then we have the following paths:

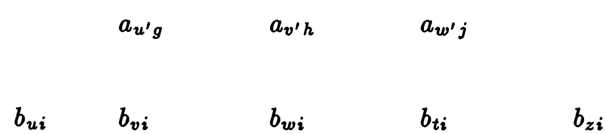

Fig. 1.

where every two paths denoted by the same kind of a line are disjoint. Hence the cycle in Figure 2 is the required cycle. □

Fig. 2.

Lemma 2.5. *There exists a constant* $\epsilon > 0$ *such that every tree* T *which satisfies the following condition has at least* $n \cdot (\log \log n)^\epsilon$ *vertices:*

(*) There exist pairwise disjoint sets $A_1, \ldots, A_n$, $B_1, \ldots, B_n$, *of vertices of* T *such that for every* $r \leq n, 1 \leq i_1 < \ldots < i_r \leq n, 1 \leq j_1 < \ldots < j_r \leq n$, *there are vertex disjoint paths from* $\{A_{i_1}, \ldots, A_{i_r}\}$ *to* $\{B_{j_1}, \ldots, B_{j_r}\}$.

Proof. Recall that
$$R_l(s) < l^{l^{2ls}},$$
(see e.g. Lovász 1979). Hence, for some $\delta > 0$, condition (iii) of Lemma 2.4 is implied by $k \leq (\log\log m)^\delta$. Let T satisfy the condition of Lemma 2.5. Suppose $|T| \leq 1/2 \cdot n \cdot (\log\log n)^\delta$. Then at least for one half of the indices $i \leq n, |A_i| \leq (\log\log n)^\delta$ and the same is true about B_i's. W.l.o.g. we can assume that it is for $i \leq \lceil n/2 \rceil$. Hence if n is sufficiently large, then T satisfies the assumptions of Lemma 2.4 with
$$m = \lceil n/2 \rceil, k = \lceil (\log\log n)^\delta \rceil, s = 2k+1.$$
Thus by Lemma 2.4, if T is a tree then
$$|T| > 1/2 \cdot n \cdot (\log\log n)^\delta,$$
for every sufficiently large n. If $\epsilon > 0$ is sufficiently small, then
$|T| > n \cdot (\log\log n)^\epsilon$ for every $n > 1$. □

Proof of Theorem 2.1. For every two subsets $\{i_1, \ldots, i_r\}, \{j_1, \ldots, j_r\}$ of $\{1, \ldots, n\}$ take $\mathbf{c}, \mathbf{d}$ such that
$$f(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = 1 \leftrightarrow (a_{i_1}, \ldots, a_{i_r}) = (b_{j_1}, \ldots, b_{j_r}).$$

Hence by Lemma 2.2, in every formula realizing f there are vertex disjoint paths between the occurrences of $x_{i_1}, \ldots, x_{i_r}$ and $y_{j_1}, \ldots, y_{j_r}$. Thus by Lemma 2.5 every formula realizing f must have at least $n(\log\log n)^\epsilon$ vertices; here we count also the inputs, but we do not have to count the vertices with degree 2. The complexity of a formula is bigger than 1/2 of the number of such vertices. □

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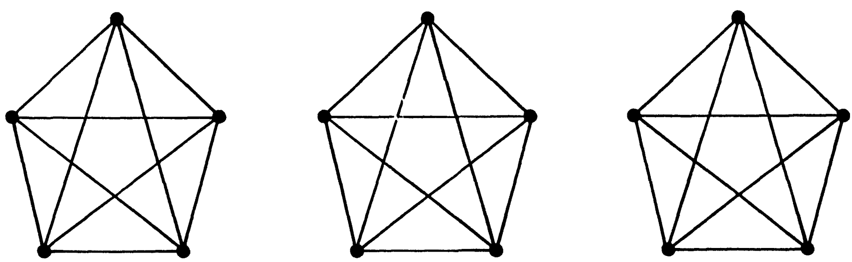
Uncrowded Graphs

Joel Spencer

1. Graphs

Turán's Theorem provides a relation between the number of vertices n, the number of edges e, and the independence number α of a graph G. In the simplest case, when $e = tn/2$ and $(t+1)|n$, Turán's Theorem yields $\alpha \geq n/(t+1)$. This inequality is best possible. An extremal graph is given by letting G be the union of $n/(t+1)$ vertex disjoint cliques, each on $t+1$ vertices.



TURÁN GRAPH WITH $n = 15$, $t = 4$

Fig. 1.

These extremal graphs are unusual in the imprecise sense that their edges are crowded together. In recent years a number of results have given strengthenings of Turán's Theorem when one requires that the graph be "uncrowded," in various precise formulations. These results have seen application to Sidon Sequences (Ajtai, Komlós, Szemerédi 1981) the Heilbronn Conjecture (Komlós, Pintz, Szemerédi 1982) and to bounds (Ajtai, Komlós, Szemerédi 1980) on the Ramsey Function $R(k,t)$. In this article we outline the basic Graph–Theoretical ideas and some of the applications.

Notation. For any graph G we let $n = n(G)$ denote the number of vertices of G, $e = e(G)$ the number of edges of G, $t = t(G)$ the average degree, and $\delta = \delta(G)$ the density of G. These parameters are related by the equations

$$t = 2e/n$$
$$\delta = 2e/n(n-1)$$

We let $\alpha = \alpha(G)$ be the independence number of G, i.e., the size of the largest independent set in G. We let $girth(G)$ denote the length of the shortest cycle in G.

Let G be a graph with n vertices and average degree t. Turán's Theorem implies $\alpha \geq n/(t+1)$. We shall be mainly interested in the case when t is large absolutely but small relative to n. The results will often be stated with rough caveats to that effect though more precise formulations may be found in the literature. The fundamental strengthening of Turán's Theorem is given by the following result.

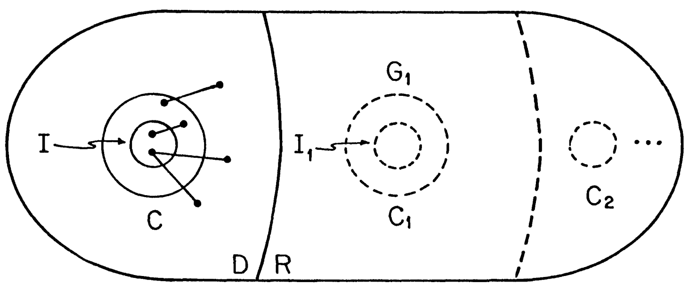
Theorem 1. *Let G have n vertices and average degree $t \geq 100$ and assume that G is trianglefree. Then*
$$\alpha(G) \geq c(n/t)(\ln t)$$
where c is an absolute constant.

For ease of exposition we shall also deal with the following logically weaker statement.

Theorem 2. *Let G have n vertices and average degree $t \geq 100$ and assume that $girth(G) > 4$. Then*
$$\alpha(G) \geq c(n/t)(\ln t)$$
where c is an absolute constant.

We approach these results with the use of probabilistic methods. (A second approach, completely constructive in nature, is described later.) First we prove a weak form of Turán's Theorem. Let G be given with n vertices and average degree t. Set, with foresight, $p = 1/t$. Let $\mathbf{C}$ be a randomly chosen subset of $V = V(G)$ where each vertex $v \in V$ is placed in $\mathbf{C}$ independently with probability p. We call $\mathbf{C}$ the chosen points. On average, $np = n/t$ points are chosen. $\mathbf{C}$ will, in general, not be independent itself but it will have few edges. Each edge $\{x, y\}$ of G lies in $\mathbf{C}$ with probability p^2. As there are $nt/2$ such potential edges, $\mathbf{C}$ has, on average, $(nt/2)p^2 = n/2t$ edges. Thus, on average, $\mathbf{C}$ has $n/2t$ more vertices than edges. Therefore there is a particular C which does have at least $n/2t$ more vertices than edges. Select one point from each edge arbitrarily and remove it from this particular C. The remaining set I will be independent and have at least $n/2t$ vertices. Thus $\alpha(G) \geq n/2t$, which is Turán's result within a factor of two. Generally, we will not be concerned with constant factors.

We now indicate the argument for Theorem 2. Let G be given with n vertices and average degree t and suppose further (skipping some Lemmas) that G is regular of degree t. Assume G has girth at least five. We then know exactly what the 2–neighborhood of any point P looks like. P is adjacent to $Q_1, \ldots, Q_t$ and Q_i is adjacent to $R_{i1}, \ldots, R_{i,t-1}$ and all these points are distinct. As before, we set $p = 1/t$ and randomly select the set of chosen points $\mathbf{C}$, inside of which lies an independent set $\mathbf{I}$. We *discard* a vertex $y \in V$ if it is adjacent to any chosen point x. Let $\mathbf{D}$ be the set of discarded points. The set of remaining points (neither chosen nor discarded) is denoted by $\mathbf{R}$. A vertex P lies in $\mathbf{R}$ if and only if none of the points, $P, Q_1, \ldots, Q_t$ have been chosen and this occurs with probability $(1-p)^{t+1} \sim e^{-1}$. On average, n/e points remain. Let $\mathbf{deg}^+(\mathrm{P})$ denote the degree of P in $\mathbf{R}$, i.e., the number of Q_i that remain. If we condition on the statement "$P \in \mathbf{R}$ " then $Q_i \in \mathbf{R}$ if and only if none of $R_{i1}, \ldots, R_{i,t-1}$ are chosen and that occurs with probability $(1-p)^{t-1} \sim e^{-1}$. Thus the average value of $\mathbf{deg}^+(\mathrm{P})$ is approximately t/e. Thus far we have not used the fact that G has no 4–cycle. (We have used that G is trianglefree since we have needed that none of the R_{is} are another Q_j.) Since G has no 4–cycle the events "$Q_i \in \mathbf{R}$ " (conditional on $P \in \mathbf{R}$) are *mutually independent* as all the R_{ij} are distinct points. Thus $\mathbf{deg}^+(\mathrm{P})$ has binomial distribution approximately $B(t, e^{-1})$. By the Law of Large Numbers the probability that $\mathbf{deg}^+(\mathrm{P})$ differs from e^{-1} by a factor of $(1+\epsilon)$ asymptotically negligible for any fixed $\epsilon > 0$.



THE TRANSFORMATION

Fig. 2.

With the above information there exists a specific set C of chosen points with the following properties. Inside C lies a set I of independent points of size at least $n/2t$. The remaining set R has approximately n/e points and almost every $P \in R$ has $\mathbf{deg}^+(\mathrm{P})$ approximately t/e. Delete the points with abnormal $\mathbf{deg}^+$ from R and let G_1 be the restriction of the original graph G to the still remaining points. Then G_1 has approximately n/e points and every point has degree approximately t/e. We now begin the entire procedure all over again with G_1! Inside G_1 we chose C_1, inside of which lies set I_1 of independent points of size at least $(n/e)/2(t/e)$. That is, I_1 has the same size $n/2t$ as I. We have already discarded all points adjacent to I (in fact, to C) so we may add the independent set I_1 to I to form a larger independent set $I \cup I_1$ of size at least $2(n/2t)$. But we also have new discarded points and remaining points and a new graph G_2 with approximately n/e^2 points, each with degree approximately t/e^2. Inside G_2 we find yet another independent set I_2, still of size at least $n/2t$. This procedure can be continued (with some careful looking at the approximations) for t times (at that point t becomes small) until an independent set of size $c(n/t)(\ln t)$ is found.

Two proofs are available of Theorem 1. We skip the probabilistic proof given in Ajtai, Komlós, Szemerédi (1981) and discuss instead the constructive proof given in Ajtai, Komlós, Szemerédi (1980). Call a point P in a graph G a *groupie* if the average degree of its neighbors is at least as large as the average degree t over the entire graph. (In colloquial usage a groupie is one who seeks the friendship of celebreties, particularly entertainers. Equating vertices with people and adjacency with friendship a groupie is a person whose friends, on average, have many friends. Note that the groupie him/her/itself may or may not have many friends/adjacencies.) *Every graph has a groupie.* For let G be any graph with n vertices and average degree t and let the adjacency relation be denoted by I. Then

$$\sum_{x\in G}\sum_{xIy} d_y = \sum_{z\in G} d_z^2 \geq nt^2$$

The first equality holds because each summand d_z occurs precisely d_z times, once for each x such that xIz. The inequality is simply Cauchy–Schwartz. If there were no groupie then

$$\sum_{xIy} d_y < td_x$$

for every x so that

$$\sum_{x\in G}\sum_{xIy} d_y < \sum_{x\in G} td_x = nt^2$$

a contradiction.

Let $G = G_0$ be a trianglefree graph with $n = n_0$ vertices and average degree $t = t_0$. We find an independent set roughly as follows. Take a groupie P and place it in the independent set. Remove P, its neighbors, and all their edges from G_0 yielding G_1 and continue. Suppose P has s neighbors $Q_1, \ldots, Q_s$.Then $\sum \deg(Q_i) \geq st$ since P is a groupie. Since G is trianglefree the edges from the Q_i are distinct (i.e. there is no edge from Q_i to Q_j) so that at least st edges and precisely $(s+1)$ vertices are removed. Ignoring the "+1" we have removed nonadjacent points of, on average, at least average degree and so the density δ_1 of the remaining graph G_1 should be at most the original density $\delta = t_0/n_0$ of G. An idea of the calculation may be seen by assuming that at every stage a groupie of average degree is found and that the density remains constant. Letting n_u, t_u be the values of n, t at stage (or "time") u we would have

$$n_{u+1} = n_u - t_u - 1 \sim n_u(1-\delta)$$

and

$$t_u = \delta n_u$$

so that

$$n_u \sim n_0(1-\delta)^u$$
$$t_u \sim t_0(1-\delta)^u$$

The procedure halts when t_u becomes small. Since $\delta = t_o/n_0$ this occurs at $u \sim (n_0/t_0)\ln t_0$ and so there is an independent set of approximately this size.

In fact, the details are more complicated. For one thing, the groupie may not have the average degree t. In the original proof (Ajtai, Komlós, Szemerédi 1980) if the groupie had degree more than ten times the average degree then it was simply discarded. It was then shown that $\alpha \geq (1/K)(n/t)\ln t$ for $K = 100$. Jerrold Griggs and Lih–Hsing Hsu were able, independently, to reduce K to approximately 2.4. Recent work of Jim Shearer has reduced K to 1 and given the following precise result.

Theorem 3. *If G has n vertices, average degree $t \geq 1$ and is trianglefree then*

$$\alpha(G) \geq n[t\ln t - t + 1]/(t-1)^2.$$

An immediate application of these results is to the Ramsey Function $R(3,t)$. Recall $R(3,t)$ is the least n such that all graphs on n vertices contain either a triangle or an independent set of size t. Suppose G is a trianglefree graph with $n = R(3,t+1) - 1$ vertices and $\alpha(G) < t+1$. The neighbors of any vertex P form an independent set (as G is trianglefree) so all degrees are at most t and hence the average degree is at most t. Thus

$$t \geq \alpha(G) \geq \frac{1}{K}\frac{n}{t}\ln t$$

and so

$$R(3,t+1) = n+1 \leq K\frac{t^2}{\ln t} + 1$$

The more precise result of Shearer yields

$$R(3,t+1) \leq t(t-1)^2/[t\ln t - t + 1] + 1$$

or, asymptotically,

$$R(3,t) < (t^2/\ln t)(1+o(1)).$$

The previously best known upper bound on $R(3,t)$ had been

$$R(3,t) < ct^2 \ln\ln t/\ln t$$

due to Graver and Yackel (1968). Their proof had used considerably different techniques. The lower bound

$$R(3,t) > ct^2/(\ln t)^2$$

due to Erdös (1961) has not been improved on since its discovery though this author (1977) has provided an alternative proof.

Extending the "groupie" method it was shown (Ajtai, Komlós, Szemerédi 1980) that for any fixed k

$$R(k,t) < c_k t^{k-1}/(\ln t)^{k-2}$$

for t sufficiently large. Note that this bound is stronger than what would be obtained by simply plugging the new bound on $R(3,t)$ into the standard recursion $R(k,t) \leq R(k-1,t) + R(k,t-1)$. For $k \geq 4$, however, the appropriate exponent of t is not known. For example, when $k = 4$ the best known lower bound (Spencer 1977) is

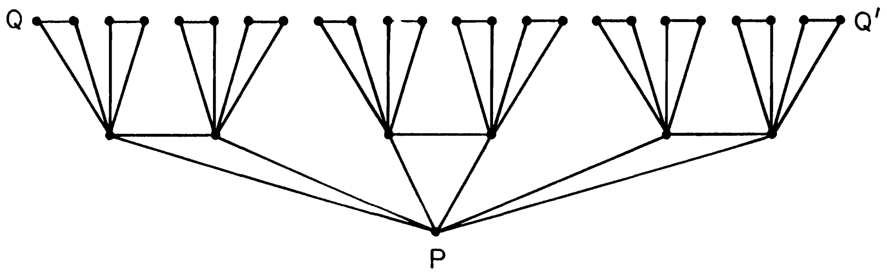
$$R(4,t) > t^{2.5+o(1)}$$

The positive results on trianglefree graphs have led to a spate of questions that meld the concerns of Ramsey Theory and Extremal Graph Theory. Suppose that a graph contains no K_4. (Results for excluding K_k for fixed $k \geq 4$ are similar.) What can one say about the independence number α in terms of n and t. Paul Erdös conjectured that when $1 \ll t \ll n$ one may bound $\alpha > (n/t)f(t)$ where $f(t)$ approaches infinity with t. In Ajtai, Erdös, Komlós, Szemerédi (1981) it was shown that $\alpha > c(n/t)(\ln\ln t)$, thus answering Erdös question in the affirmative. Still, the appropriate order of α remains unkown. In particular, it is not known if $\alpha > c(n/t)(\ln t)$.

2. Hypergraphs

Let G be a k–graph on n vertices (i.e., G is a collection of k–element subsets) with average degree t^{k-1}. We think of k fixed ($k = 3$ is a representative case) and $k \ll t \ll n$. What bounds can be made on the independence number $\alpha = \alpha(G)$? This analogue of Turán's Theorem for hypergraphs may be answered within an constant factor. Given such a G we imitate the random construction for graphs. Let **C** be a subset of vertices where each vertex x is placed in **C** with probability $p = 1/t$. G has nt^{k-1}/k edges, each of which are in **C** with probability p^k. Thus on average **C** vertices and n/kt edges. There exists a particular C which has at least $n/t - n/kt$ more vertices than edges. Deleting one vertex from each edge in this C leave an independent set of size at least $n/t - n/kt$. Thus $\alpha \geq c_k n/t$. On the other hand, let G be the union of n/t vertex disjoint complete k–graphs, each on t points. Each vertex has degree $\binom{t-1}{k-1} \sim t^{k-1}/(k-1)!$ and $\alpha(G) = (k-1)n/t$. Absorbing the factors $(k-1)!$ and $(k-1)$ into the constant we see that a k–graph G with n vertices and average degree t^{k-1} may have $\alpha \leq c'_k n/t$. Unlike the situation for graphs, the precise maximum of α is unknown and we do not contribute to that problem here.

Let $girth(G)$ be, as before, the length of the minimal cycle of G. A cycle of length s in a k–graph is defined technically, as a set of s edges whose union contains at most $s(k-1)$ vertices. Assume $girth(G) > 4$. Then the 2–neighborhood



UNCROWDED REGULAR 3-GRAPH OF DEGREE 3

Fig. 3.

of any point P has no intersections. Figure 3 gives the 2–neighborhood of a 3–graph where each point has degree 3 and the girth is at least 4. If Q, Q' are identified then a cycle of length 4 is created. In Ajtai et al. (1982) it was shown that if G is a k–graph with n vertices, average degree t^{k-1}, $girth(G) > 4$ then

$$\alpha > c_k(n/t)(\ln t)^{1/(k-1)}$$

The proof follows the lines of the probabilistic proof given earlier. A random set $\mathbf{C}$ of n/t vertices is selected, inside of which lies an independent set $\mathbf{I}$ of cn/t vertices. Having chosen $\mathbf{C}$ we are forced to discard those vertices x for which exist $y_1, \ldots, y_{k-1} \in \mathbf{C}$ with $\{x, y_1, \ldots, y_{k-1}\}$ an edge of G. Furthermore, if $\{x_1, \ldots, x_s, y_1, \ldots, y_t\}$ is an edge ($x_i \in \mathbf{C}$, $y_j \notin \mathbf{C}$) then it is transformed to a new edge $\{y_1, \ldots, y_t\}$. The remaining graph $\mathbf{G_1}$ is such that any independent set in G_1 may be added to $\mathbf{I}$. (Note that $\mathbf{G_1}$ is not only the restriction of G to the remaining points $\mathbf{R}$ but also contains s–edges for $2 \leq s < k$.) With the assumption $girth(G) > 4$ the degrees in $\mathbf{G_1}$ are tightly controlled and the above bound on α (after some technical effort) is achieved.

Suppose we require $girth(G) > 3$ but allow cycles of length 4. Once again let G be a k–graph with n vertices and average degree t^k, $k \ll t \ll n$. We conjecture that the independence number still satisfies

$$\alpha > (n/t)f(t)$$

for some function $f(t)$ approaching infinity with t.

3. Heilbronn's Conjecture

The results of this section are given in detail in Komlós, Pintz, Szemerédi (1982).

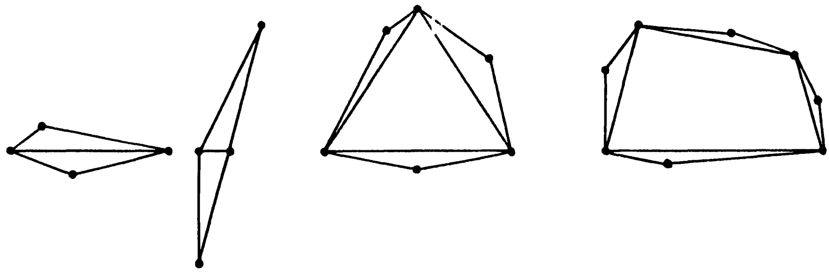


Fig. 4.

Define $H(n)$ has the maximal real such that given any n points $P_1, \ldots, P_n$ in the unit square S some triangle $P_iP_jP_k$ has area at least $H(n)$. (Three collinear points are considered to form a triangle of area zero and so are never used.) We shall be concerned with lower bounds on $H(n)$. Thus we wish to find $P_1, \ldots, P_n$ containing no "small" triangle. Heilbronn conjectured that $H(n)$ was of order Kn^{-2} for some constant K.

Let $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$ be independently and uniformly chosen from S. Then $area(\mathbf{PQR})$ is a random variable and we are interested in its values near zero. It is a nice probability/calculus exercise (which we omit) to show that

$$Pr[area(\mathbf{PQR}) < \epsilon] < 100\epsilon$$

Here "100" is certainly not the best constant.

Let $\mathbf{P}_1, \ldots, \mathbf{P}_{2n}$ be independently and uniformly chosen from S. Call a triangle small if its area is less than $10^{-4}n^{-2}$. For each i, j, k the triangle $\mathbf{P}_i, \mathbf{P}_j, \mathbf{P}_k$ is small with probability less than $10^{-2}n^{-2}$. There are $\binom{2n}{3} < 2n^3$ such triples. Hence the average number of small triangles is less than $2n^3(10^{-2}n^{-2}) = .02n$. Thus there exist specific $P_1, \ldots, P_{2n}$ forming fewer than $.02n$ small triangles. Delete one point from each triangle. What remains is a collection of more than n points with no small triangle. Thus $H(n) \geq 10^{-4}n^{-2}$.

Let K be an arbitrarily large constant and call a triangle small if its area is less than Kn^{-2}. To improve the above method we let $t \gg K$ and $n \gg t$ and choose tn points $\mathbf{P}_1, \ldots, \mathbf{P}_{tn}$ independently and uniformly from S. We define a 3–graph $\mathbf{G}$ on $\{1, \ldots, tn\}$, letting $\{i, j, k\}$ be an edge if and only if $\mathbf{P}_i\mathbf{P}_j\mathbf{P}_k$ is small. There are $\binom{tn}{3} < t^3n^3$ such triples and each $\{i, j, k\}$ is an edge with probability less than $100Kn^{-2}$ so $\mathbf{G}$ has, on average, fewer than $(100Kn^{-2})(t^3n^3) = nt(10\sqrt{Kt})^2$ edges. Now we examine the small cycles (see Figure 4) in $\mathbf{G}$. A further probability calculus calculation shows that the probability $\mathbf{P}_i\mathbf{P}_j\mathbf{P}_k$ and $\mathbf{P}_i\mathbf{P}_j\mathbf{P}_m$ are both small (i, j, k, m unequal) is less than

$(100n^{-2})^2(100\ln n)$. Thus the average number of 2–cycles in **G** is less than $(tn)^4 10^6 K^2 n^{-4} \ln n$ which is $o(n)$ since $K, t \ll n$. The calculations for 3–cycles and 4–cycles are progressively more complex but the results are the same. **G** has on average $o(n)$ cycles of length $2, 3$, or 4. Thus there exists a specific choice of tn points $P_1, \ldots, P_{tn}$ for which G has tn vertices, roughly $nt(10\sqrt{K}t)^2$ edges and $o(n)$ small cycles. Deleting one point from each small cycle gives a graph G' with roughly the same number of vertices and edges and girth at least five. Applying the result of the previous section on uncrowded k–graphs (for $k = 3$) we see

$$\alpha(G) \geq \alpha(G') \geq c\frac{nt}{10\sqrt{K}t} \ln[10\sqrt{K}t]^{1/2}$$

where c is an absolute constant. Let t be so large that $c\ln[10\sqrt{K}t]^2 \geq 10\sqrt{K}$. Then $\alpha(G) \geq n$. Among the points $P_1, \ldots, P_{tn}$ we may find n points which are independent in G, i.e., such that no triangle formed by them is small. Hence $H(n) \geq Kn^{-2}$. That is, Heilbronn's Conjecture is false.

The following construction (published in Roth 1951) was first made by Erdős. Let n be prime. Let a mod n denote that unique b, $0 \leq b < n$, such that $a \equiv b$ mod n. Set

$$V = \{(i, i^2 \bmod n) : 0 \leq i < n\}$$

(For example, when $n = 5$, $V = \{(0,0), (1,1), (2,4), (3,4), (4,1)\}$.) We claim that no three points of V are collinear. Suppose a line $y = mx + b$ did intersect V at x–coordinates i, j, k. Then the slope m could be written as a fraction N/D where the denominator D was strictly less than n. The intercept b could similarly be written. Then we could interpret m, b as elements of Z_n and in Z_n the equation $x^2 = mx + b$ would have three solutions i, j, and k. Since n is prime this is not possible. It is a theorem of recreational mathematics that any triangle with lattice points as vertices has area $s/2$ where s is an integer. Hence all triangles in V have area at least $1/2$. Normalizing by dividing both coordinates by n gives a set $V' = V/n$ contained in the unit square which has n points and no triangle with area less than $1/2n^2$. Thus $H(n) \geq 1/2n^2$. The uncrowded graph techniques have made this result obsolete but the construction is too pretty to be forgotten.

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Subject Index

STORY OF THE PARTITION ARROW

$$n \longrightarrow \left(k, \begin{bmatrix} u \\ v \end{bmatrix}\right)^r$$

$$n \longrightarrow (G_1^{(r)}, \ldots, G_l^{(r)})_l^r$$

$$(2^\alpha)^+ \longrightarrow (\alpha^+)^2_\alpha$$

$$G \xrightarrow{ind} H$$

$$F \not\longrightarrow (G, H)$$

$$R(n; l_1, \ldots, l_k) \to (l_1, \ldots, l_k)^n_k$$

$$\omega \longrightarrow (\omega)^n_k$$

$$\omega^2 \longrightarrow (\omega^2, n)$$

$$c \longrightarrow (b)^a_t$$

$$\mathsf{c} \longrightarrow (\mathsf{B})^{\mathsf{A}}_t$$

$$\mathsf{E}^{k+1} \xrightarrow[y]{} (\mathsf{D}^{k+1})^{\mathsf{A}}_t$$

$$\mathsf{C} \xrightarrow{\mathcal{B}} (\mathsf{B})^{\mathsf{A}}_t$$

$$A \xrightarrow[k]{\mathcal{B}} C$$

$$X \longrightarrow [Y]^n_\lambda$$

$$\mathbb{R} \not\longrightarrow (top\ \mathbb{Q})^2_2$$

$$\omega^2 + \omega \longrightarrow (top\ \omega^2 + \omega + 1)^1_1$$

$$n \longrightarrow (h)^e_c\ mod\langle \mathcal{I}, p_1 \leq \rangle$$

$$G \longrightarrow (K_r)^2_{<\omega}$$

$$[n, m] \xrightarrow[*]{} (n+1)^n_n$$