Rado's Thm

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What about other equations?

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(We can modify the proof to get a d-mono sol.)

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Thm $x + y = z$ is regular. (Can also show d-regular.)

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Take $x = y = z = 1$. Or any $x = y = z$.

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Could do $W = 1$, $X = 1$, $Y = 1$. But this causes $x = y = z$.

Will do $W = 0$, $X = 5$, $Y = 3$.

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So get $x = a \qquad x = a + 5d \qquad y = a + 3d$.

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What is it about

$$2w + 3x = 5y$$

that made all of the a's drop out? Discuss.

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We won't prove this but you have seen most of the ideas needed to prove it.

Other Equations

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So take EVDW with $k = 5$.

How We Get What We Want

Want

$2w + 3x = 5y + z$

$2(a + Wd) + 3(a + Xd) = 5(a + Yd) + (Zd)$

$2a + 2Wd + 3a + 3Xd = 5a + 5Yd + Zd$ **WOW** The a's drop out.

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We won't prove. You have seen most of the ideas needed to prove it.

An Equation Where Rado Fails

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Similar to $e_2 < e_1, 3_3$.

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$3b \equiv 0 \pmod{5}$ implies $b \equiv 0 \pmod{5}$. Contradiction.

Case $e_1 = e_3 < e_2$ and $e_2 = e_3 < e_1$

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Take this mod 5 to get $3b \equiv 4b$ so $b \equiv 0 \pmod 5$ Contradiction.

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1. The proof used that NO subset of $1, 2, -4$ sums to 0.
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   2.2 We needed $3b \equiv 0 \pmod{p}$ implies $b \equiv 0 \pmod{p}$
   5 is the lowest such prime.

Rado's Thm (Other Half of it)

Thm Let $a_1, \ldots, a_k \in \mathbb{Z}$ be st no subset of the a_i's sums to 0.
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Research Question

1. For $x + 2y = 4z$ what about 4-coloring? 3-coloring? 2-coloring?
2. More generally one can take an equation where no sum of the coefficients is 0 and look at colorings with a small number of colors.

Full Rado

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Full Rado Thm A linear equation $\sum_{i=1}^{n} a_i x_i = 0$ is regular iff some subset of the coefficient sum to 0.

(For most equations with the coefficients sum to 0 you actually get d-regular.)

Misc

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(Some is known about some of these.)
Prove or disprove that the equations below are regular.

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1. $\sum_{i=1}^{n} a_i x_i = A$ for some A.
2. Higher degree equations (seems hard).

Other Equations

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1. There is a matrix form of Rado which we omit.
2. **Folkman's Thm** For all k, c there exists $N = N(k, c)$ st for all $\mathrm{COL}\colon [N] \rightarrow [c]$ there exists $a_1, \ldots, a_k$ st ALL non-empty sums of the a_i's are the same color.

Other Equations

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   (This equation has certain properties that make it work, so there is really a more general thm here.) `http://fourier.math.uoc.gr/~ergodic/Slides/Host.pdf`

$x^2 + y^2 = z^2$ Result by Heule&Kullmann&Marek

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3) (Might be Hard) Obtain a human-readable proof with perhaps a much bigger N, but which can be generalized to $c = 3$ and beyond.