Goodstein Sequences

Exposition by William Gasarch-U of MD

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This is called **Hereditary Base *n* Notation**

Ackerman's Function and Goodstein Seq

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Note that the right most digit is 0. That will happen ∞ often.

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Now its a 2-digit number and use induction.

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Next Slide will indicate why am asking this.

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