452 MIDTERM SOLUTIONS

1. (25 points) Let the alphabet be $\Sigma = \{a, b\}$.

   Recall that $\#_a(w)$ is the number of a's in w.

   Recall that $\#_b(w)$ is the number of b's in w. Let

   $$L = \{w : \#_a(w) \equiv 1 \pmod 3 \text{ OR } \#_b(w) \equiv 2 \pmod 4\}.$$

   L is regular. Give either the transition table OR draw the DFA for L.

   SOLUTION ON NEXT PAGE

BEGIN SOL

We need to keep track of BOTH the number of a's mod 3 and the number of b's mod 4. So there will be 12 states

$Q = \{(i,j) : 0 \le i \le 2 \wedge 0 \le j \le 3\}$.

$s = (0,0)$

$F = \{(1,0),(1,1),(1,2),(1,3)\} \cup$

$\{(0,2),(1,2),(2,2)\} = \{(1,0),(1,1),(1,2),(1,3),(0,2),(2,2)\}$

And now δ, the transition table.

$\delta((i,j),a) = (i+1 \pmod 3), j)$

$\delta((i,j),b) = (i, j+1 \pmod 4)$

END SOL

2. (25 points) Give an example of a language L such that the following are true.

   - L is a subset of a^*.
   - Every DFA for L has ≥ 300 states. Give a DFA for it either with a picture or a transition table. You CAN use DOT DOT DOT. You DO NOT have to proof that it requires ≥ 300 states.
   - There is an NFA for L with ≤ 60 states. Give an NFA for it with ≤ 60 states with with a picture or a transition table. You may use DOT DOT DOT. You need to describe how many states it has. You DO NOT have to prove that it works.

   SOLUTION ON NEXT PAGE

BEGIN SOL

We give TWO solutions.

SOL ONE

Note that $19 \times 17 = 323$.

$L = \{a^i : i \not\equiv 0 \pmod{323}\}$

The DFA for L is the obvious one with a loop of size 323. Formally

$Q = \{0, \ldots, 322\}$.

$s = 0$

$F = \{1, 2, \ldots, 322\}$

$\delta(i, a) = i + 1 \pmod{323}$.

This DFA has 323 states. Any DFA for L has ≥ 323 states (we do not prove this).

There is an NFA with $1 + 17 + 19 = 37$ states.

We describe it in words and then formally. From the start state there are two e-transitions.

One is to a DFA that recognizes $\{a^i : i \not\equiv 0 \pmod{17}\}$.

The other is to a DFA that recognizes $\{a^i : i \not\equiv 0 \pmod{19}\}$.

Formally $Q = \{s\} \cup \{0, \ldots, 16\} \cup \{0', \ldots, 18'\}$.

s is of course s.

$F = \{1, ldots, 16\} \cup \{1', \ldots, 18'\}$.

Δ is

$\Delta(s) = \{0, 0'\}$

$\Delta(i) = i + 1 \pmod{17}$.

$\Delta(i') = i' + 1 \pmod{19}$.

For **Solution Two** see next page

SOL TWO

Note that $19 \times 18 - 19 - 18 = 305$. So by the Chicken McNugget Theorem there is no combination of 18 and 19 that adds up to 305, BUT for all $n \geq 306$ there is.

$L = \{a^i : i \neq 305\}$.

We omit the DFA for L since it is easy.

We describe the NFA informally but omit the formal description.

The NFA has an e-transition from the start state to the following:

(a) A loop of length 19 that has a shortcut at 18. The origin of the loop is the final state. Note that we don't need a chain.
(b) Since $305 \equiv 1 \pmod 2$:
A DFA for $\{a^i : i \not\equiv 1 \pmod 2\}$.
(c) Since $305 \equiv 2 \pmod 3$:
A DFA for $\{a^i : i \not\equiv 2 \pmod 3\}$.
(d) Since $305 \equiv 0 \pmod 5$:
A DFA for $\{a^i : i \not\equiv 0 \pmod 5\}$.
(e) Since $305 \equiv 4 \pmod 7$:
A DFA for $\{a^i : i \not\equiv 4 \pmod 7\}$.
(f) Since $305 \equiv 8 \pmod{11}$:
A DFA for $\{a^i : i \not\equiv 8 \pmod{11}\}$.

The number of states is

$1 + 19 + 2 + 3 + 5 + 7 + 11 = 48$

Grading Criteria

(a) If it was unreadable you lost points.
(b) If you tried to use

$$\{a^i : i \geq 300\}$$

Then you lost points since there is NO small NFA for this problem. If you use Chicken McNugget on this then you will get an NFA that recognizes that set UNION some elements of the form a^i with $i < 300$.

3. (25 points)

Definition Let $w \in \Sigma^*$. Then ISAAC(w) is the set of words that can be formed by removing any set of symbols from w. For example

$$\text{ISAAC}(abab) = \{e, a, b, aa, ab, ba, bb, aab, aba, abb, bab, abab\}$$

If L is a language (a subset of Σ^*) then

$$\text{ISAAC}(L) = \bigcup_{w \in L} \text{ISAAC}(w).$$

For example if $L = \{abab, bbbb\}$ then

$$\text{ISAAC}(L) = \{e, a, b, aa, ab, ba, bb, aab, aba, abb, bab, bbb, abab, bbbb\}$$

Show that if L is a **context free language** then

ISAAC(L) is a **context free language.**

Next page for **SOLUTION**

BEGIN SOL

We give three solutions.

SOL ONE

Let L be a CFL. Let G be a CHOMSKY NORMAL FORM GRAMMAR for it.

For every rule of the form

$A \rightarrow \sigma$

where A is a nonterminal and $\sigma \in \Sigma$ ADD the rule

$A \rightarrow e$.

SOL TWO

Let L be a CFL. Let G be a CFG for it (some people did not assume Chomsky Normal Form). Let $G = (V, \Sigma, S, R)$.

For every rule of the form

$A \rightarrow \alpha$

where A is a nonterminal and $\alpha \in (V \cup \Sigma)^*$ do the following:

Let α be of the form

$$\beta_1 \sigma_1 \beta_2 \sigma_2 \cdots \sigma_n \beta_n$$

where the $\beta \in V^*$ and $\sigma_i \in \Sigma$.

For every subset of $\{1, \ldots, n\}$ add a rule that replaces all $\{\sigma_i : i \in A\}$ with e. So note that for every rule with n nonterminals we add 2^n rules. Also note that this is a MUCH CLUNKIER solution then using Chomsky Normal Form.

Grading Criteria on next page

Grading Criteria

(a) If you did the first solution but did not mention that every CFL has a Chomsky Normal Form CFL then you DID NOT lose points because are nice people. We might not be nice on the final.

(b) If you did this with an infinite union, that is 0 points.

(c) If you used the Pumping Lemma that is 0 points. Note that the PL is used to show languages either not regular or not CF depending on which version. That has nothing to do with this problem.

(d) If you talked about regular languages or finite automata then 0 points. You were probably doing the problem on the HW from memory even though it has nothing to do with this problem. As I warned you- DO NOT MEMORIZE, instead UNDERSTAND.

END SOL

COMMENT FROM SPRING 2025: WE HAVE DONE P AND NP YET SO THIS KIND OF QUESTION WOULD NOT BE ON OUR MIDTERM.

4. (25 points) Let $\Sigma = \{a, b\}$. Let $X \subseteq \Sigma^*$ and $Y \subseteq \Sigma^*$.

   (a) (5 points) Define what $X \le Y$ means.

   (b) (5 points) Define what $X \in \mathrm{NP}$ means.

   (c) (15 points) Show that if $X \le Y$ and $Y \in \mathrm{NP}$ then $X \in \mathrm{NP}$

SOLUTION ON NEXT PAGE

BEGIN SOL

(a) Define what $X \leq Y$ means.

There exists a function f COMPUTABLE IN POLY TIME such that

for all x, $x \in X$ iff $f(x) \in Y$.

Grading Criteria If you left out that the function f is in poly time then you get 0 points.

(b) Define what $X \in \mathrm{NP}$ means.

There exists a polynomial q and a set $B \in \mathrm{P}$ such that

$$X = \{x : (\exists y, |y| = q(|x|))[(x, y) \in B].$$

Grading Criteria If you left out that there is a poly bound on y or that B is in poly time then you got 0 points.

For SOLUTION to Part c Next Page

(c) Show that if $X \le Y$ and $Y \in \mathrm{NP}$ then $X \in \mathrm{NP}$

Assume $X \le Y$ via f. Assume that f run in time $p(|x|)$ where p is a Polynomia. Note that the output of f is of length $\le p(|x|)$. We will take it to be exactly $p(|x|)$.

Assume $Y \in \mathrm{NP}$ via polynomial $q(|y|)$ and $B(x, y)$ which is in time $r(|x| + |y|)$.

Note that

$x \in X$ iff $f(x) \in Y$ iff

$$(\exists z, |z| = q(|y|))[(f(x), z) \in B]$$

We need to show that $q(|y|)$ is a poly in $|x|$. It is since $|y| = p(|x|)$ so $q(|y|) = q(p(|x|))$ which is a poly since its a composition of two poly.s

We need that $\{(x, y) : (f(x), z) \in B\}$ is in P. This is easy and we omit it.

Grading Criteria

i. If you had the right idea and used polynomials but made a few errors so that the proof is wrong, you got 10 out of 15 points.

ii. If you describe a procedure that is correct but did not mention anything about polynomials you got 5 points.

iii. If your procedure is not correct and you don't mention polynomials then you get 0 points.

END SOL