HW 10 CMSC 452
Morally Due TUES April 15 11:00AM
Dead-Cat Due THU April 17 11:00AM

1. (50 points)

   Def Let $G = (V, E)$ be a graph. A *vertex cover for G of size k* is a set $U \subseteq V$ such that

   - $|U| = k$, and
   - For every $(a, b) \in E$ either $a \in U$ or $b \in U$ (or both).

   $\mathrm{VC} = \{(G, k) \colon G$ has a Vertex Cover of size $\le k\}$.

   It is known that VC is NP-complete.

   (a) (10 points) Give a connected graph G on 1000 vertices that has a vertex cover of size 1.

   (b) (10 points) Give a connected graph G on 1000 vertices so that the smallest vertex cover for it has size 999.

   (c) (10 points) Give a connected graph G on 1000 vertices so that the following are true.

   - G has a vertex cover of size 500.
   - G does not have a vertex cover of size 499.

   (d) (20 points) Let
   $\mathrm{VC}_{1000} = \{G \colon G$ has a Vertex Cover of size $1000\}$.
   Show that $\mathrm{VC}_{1000} \in \mathrm{P}$.

   (e) (0 points) Think about: Your algorithm in Part d ran in time $O(n^d)$ for some d. Do you think there is an algorithm with a substantially lower value of d?

   (f) (0 points) Think about but do not hand in: Consider the greedy algorithm to find a Vertex Cover: Remove the vertex of highest degree put it in set X. When you remove the vertex also remove the edges attached to it (obviously). Repeat: remove the vertex of highest degree put it in set X. Keep doing this until the graph no longer has any edges. X is a vertex cover.
   Find a graph where this algorithm DOES NOT give the min sized vertex cover.

2. (50 points)

   Def Let $G = (V, E)$ be a graph. A *dominating set for G of size k* is a set $D \subseteq V$ such that

   - $|D| = k$, and
   - For every $v \in V$ either $v \in D$ or some neighbor of v is in D.

   DS $= \{(G, k) \colon G$ has a Dominating Set of size $k\}$.

   It is known that DS is NP-complete.

   (a) (10 points) Give a connected graph G on 1000 vertices that has a dom set of size 1.

   (b) (10 points) Give a graph G on 1000 vertices so that the smallest dom set for it has size 1000. (Hint: It will NOT be connected.)

   (c) (10 points) Give a graph G on 1000 vertices so that the following are true.

   - G has a dom set of size 500.
   - G does not have a dom set of size 499.

   (Hint: It will NOT be connected.)

   (d) (20 points) Let
   $\mathrm{DS}_{1000} = \{G \colon G$ has a Dominating Set of size $1000\}$.
   Show that $\mathrm{DS}_{1000} \in \mathrm{P}$.

   (e) (0 points) Think about: Your algorithm in Part d ran in time $O(n^d)$ for some d. Do you think there is an algorithm with a substantially lower value of d?

   (f) (0 points) Think about but do not hand in: Consider the greedy algorithm to find a Dom Set: Remove the vertex of highest degree put it in set X. When you remove the vertex also remove the edges attached to it (obviously) AND the vertices on those edges. Repeat: remove the vertex of highest degree put it in set X. Keep doing this until the graph no longer has any edges. X is a dom set.

   Find a graph where this algorithm DOES NOT give the min sized dom set.