BILL AND NATHAN, RECORD LECTURE!!!!

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NPC SAT-type Problems

Exposition by William Gasarch—U of MD

NPC Problems on Boolean Formulas

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Bounding
(1) Literals Per Clause
(2) Occurrences of a Var

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Two Types of SAT

1. **kSAT-b**: Clauses have $\le k$ literals, each var occurs $\le b$ times.
2. **EU-kSAT-b**: Clauses have k literals, each var occurs $\le b$ times.

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SAT means no bound on number of literals-per-clause.
We will look at all four of these for various values of k, b.

No Bound on b

1. 1SAT:

No Bound on *b*

1. $\mathrm{1SAT}$: P,
   $\phi \in \mathrm{1SAT}$ iff there is no x such that both x and $\neg x$ occur.
2. $\mathrm{2SAT}$:

No Bound on b

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2. $\mathrm{2SAT}$: P. Known result. Sketch: Convert every clause $L_1 \vee L_2$ into $(\neg L_1 \rightarrow L_2) \wedge (\neg L_2 \rightarrow L_1)$. Make a directed graph with literals as vertices and the $\rightarrow$ as edges. $\phi \in \mathrm{2SAT}$ iff there is no path from an x to a $\neg x$.
3. $\mathrm{3SAT}$: NPC by Cook.

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The $k = 1$ and $k = 2$ cases are of course still in P if you bound b. Hence we look at $k = 3$ and bound on b.

$k = 3$ and $b = 1, 2$

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$\mathrm{3SAT\text{-}1}$: P. Always satisfiable, just set all literals that appear to T. EU version would still be in P.

$\mathrm{3SAT\text{-}2}$: P? NPC? Work on in Breakout Rooms.

3SAT, all vars occur $\le$ 2. P

1) Input ϕ in $3\mathrm{CNF}$, all vars occurs $\le$ 2.

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1) Input ϕ in $3\mathrm{CNF}$, all vars occurs ≤ 2.
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These operations may solve problem.

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3) Every clause has 2 or 3 literals, every literal occurs as pos and neg. We show SAT.

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(Numb NEG-clauses) + (Numb of clauses) DECREASES.

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(Numb NEG-clauses) + (Numb of clauses) DECREASES.
Eventually satisfy all clauses.
Moral This was a clever trick! To prove $\mathrm{P} \ne \mathrm{NP}$ would need to show that no clever trick will get SAT into P. Hard!

3SAT, all vars occur ≤ 3

3SAT-3: There are ≤ 3 clauses per literal and every var occurs ≤ 3 times.

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In P? NPC? Breakout Rooms!

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b) Add the clauses $x_1 \rightarrow x_2$, $x_2 \rightarrow x_3$, …, $x_{m-1} \rightarrow x_m$, $x_m \rightarrow x_1$.
(Formally $x_1 \rightarrow x_2$ is $(\neg x_1 \vee x_2$.))

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Clearly $\phi \in \mathrm{3CNF}$ and all variables occur ≤ 3 times.
Clearly $\phi \in \mathrm{SAT}$ iff $\phi' \in \mathrm{SAT}$
Moral Going from $b \le 2$ to $b \le 3$ matters!

EU-3SAT-3?

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Go to breakout rooms to work on this.

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This needs a known Theorem and its Corollary.
For this slide $G = (A, B, E)$ is a bipartite graph.
A **Matching of A into B** is a set of disjoint edges so that every element of A is an endpoint of some edge. View as an injection of A into B.
$X \subseteq A$. $E(X) = \{y \in Y : (\exists x \in X)[(x, y) \in E]\}$.

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Hall's Matching Theorem If, for all $X \subseteq A$, $|E(X)| \geq |X|$ then there exists a matching from A to B.

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Corollary If there exists k such that (1) for every $x \in A$, $\deg(x) \ge k$, and (2) for every $y \in B$, $\deg(y) \le k$, then there is a matching from A to B.

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We will use these on the next slide.

Every EU-3CNF-3 fml is Satisfiable

Let ϕ be EU-3CNF-3. $\phi = C_1 \vee \cdots \vee C_m$.
Form a bipartite graph:

1. Clauses on the left, variables on the right.
2. Edge from C to x if either x or $\neg x$ is in C.

Every clause has degree 3.

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By Corollary there is a matching of C's to V's. This gives a satisfying assignment.
Moral The algorithm used a THEOREM in math that perhaps you did not know. To prove $\mathrm{P} \neq \mathrm{NP}$ would need to say this can't happen. Hard!

A Variant of SAT

Exposition by William Gasarch—U of MD

1-in-3-SAT

Def **1-in-3-SAT** **(**1-in-3-SAT**)** is the problem of, given a formula $D_1 \wedge \cdots \wedge D_m$ find an assignment that satisfies **exactly** one literal-per-clause. We will show that 1-in-3-SAT is NPC.

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My Opinion The problem is **not** natural.

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Its a means to an end We will show natural problems NPC by using reductions from 1-in-3-SAT. We will do a reduction from a variant of 1-in-3-SAT.

1-in-3-SAT is NPC

Given $\phi = C_1 \wedge \cdots \wedge C_m$ in $3\mathrm{CNF}$ create the ϕ' as follows:

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Given $\phi = C_1 \wedge \cdots \wedge C_m$ in $3\mathrm{CNF}$ create the ϕ' as follows: Replace clause $(L_1 \vee L_2 \vee L_3)$ with

$$(\neg L_1 \vee a \vee b) \wedge (b \vee L_2 \vee c) \wedge (c \vee d \vee \neg L_3).$$

where a, b, c, d are new variables.

1-in-3-SAT is NPC

Given $\phi = C_1 \wedge \cdots \wedge C_m$ in $\mathrm{3CNF}$ create the ϕ' as follows:
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where a, b, c, d are new variables.
Leave it to the reader to prove

$$\phi \in \mathrm{3SAT} \text{ iff } \phi' \in \text{1-in-3-SAT}.$$

Mono 1-in-3-SAT

Mono 1-in-3-SAT (mono-1-in-3-SAT): Given a formula $E_1 \wedge \cdots \wedge E_p$ where all vars occur positively, is there an assignment that satisfies **exactly** one literal-per-clause.

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Thm $\text{1-in-3-SAT} \le \text{mono-1-in-3-SAT}$
Given 3CNF form $\phi(x_1, \ldots, x_n) = C_1 \vee \cdots \vee C_k$ want ϕ' such that $\phi \in \text{1-in-3-SAT}$ iff $\phi' \in \text{mono-1-in-3-SAT}$.

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1) New Vars t, f and new clause $E = (t \vee f \vee f)$. Any 1-in-3-SAT assignment of ϕ will set t to T and f to F.

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1) New Vars t, f and new clause $E = (t \vee f \vee f)$. Any 1-in-3-SAT assignment of ϕ will set t to T and f to F.
2) For each x_j have new var x_j' and clause $D_j = (f \vee x_j \vee x_j')$. Any 1-in-3-SAT assignment for ϕ will set x_j, x_j' to opposites.

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Thm 1-in-3-SAT $\le$ mono-1-in-3-SAT
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1) New Vars t, f and new clause $E = (t \vee f \vee f)$. Any 1-in-3-SAT assignment of ϕ will set t to T and f to F.
2) For each x_j have new var x_j' and clause $D_j = (f \vee x_j \vee x_j')$. Any 1-in-3-SAT assignment for ϕ will set x_j, x_j' to opposites.
3) For each C_i let C_i' be obtained by replacing every $\overline{x_j}$ with x_j'.

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1) New Vars t, f and new clause $E = (t \vee f \vee f)$. Any 1-in-3-SAT assignment of ϕ will set t to T and f to F.
2) For each x_j have new var x_j' and clause $D_j = (f \vee x_j \vee x_j')$. Any 1-in-3-SAT assignment for ϕ will set x_j, x_j' to opposites.
3) For each C_i let C_i' be obtained by replacing every $\overline{x_j}$ with x_j'.

$$\phi' = C_1' \wedge \cdots \wedge C_k' \wedge D_1 \wedge \cdots \wedge D_n \wedge E.$$

Leave it to the reader to show $\phi \in \text{1-in-3-SAT}$ iff $\phi' \in \text{mono-1-in-3-SAT}$.

A Puzzle we Prove Hard Using mono-1-in-3-SAT

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Why is mono-1-in-3-SAT Important?

We care about the mono-1-in-3-SAT problem!

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We care about the mono-1-in-3-SAT problem! **NOT!**

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We will use it to show that a puzzle we DO care about is NPC

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	S	E	N	D
+	M	O	R	E
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The SEND MORE MONEY Cryptarithms

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We will use it to show that a puzzle we DO care about is NPC

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	9	5	6	7
+	1	0	8	5
1	0	6	5	2

The Solution to The SEND MORE MONEY Cryptarithms

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CRYPTARITHM

Input $B, m \in \mathbb{N}$. Σ is alphabet of B letters.

$x_0, \ldots, x_{m-1}$. Each $x_i \in \Sigma$.

$y_0, \ldots, y_{m-1}$. Each $y_i \in \Sigma$.

$z_0, \ldots, z_m$. Each $z_i \in \Sigma$. The symbol z_m is optional.

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Question Does there exists injection $\Sigma \to \{0, \ldots, B-1\}$ so that the arithmetic below is correct in base B?

$$\begin{array}{cccc} & x_{m-1} & \cdots & x_0 \\ + & y_{m-1} & \cdots & y_0 \\ \hline z_m & z_{m-1} & \cdots & z_0 \end{array}$$

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We do the reduction in three parts, so three more slides!
We call the parts **gadgets**.

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We leave it to the reader to show that this ensures 0 maps to 0 and 1 maps to 1.

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Note Do this for all vars v, using a different a, b, c for each one.

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