Arithmetic Mean–Geometric Mean-Inequalities

AM and GM

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How do AM and GM compare when $x_1, \ldots, x_n \in \mathbb{R}^+$?

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How do $\frac{x+y}{2}$ and $\sqrt{xy}$ compare?

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The AM-GM Theorem

Thm For all $n \in \mathbb{N}$ and for all $x_1, \ldots, x_n \in \mathbb{R}^+$

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Equality happens iff $x_1 = \cdots = x_n$.

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$(\forall n)[P(2^{n-1}) \rightarrow P(2^n)]$
$(\forall n < m)[P(m) \rightarrow P(n)]$ (YES, $n < m$). NOT a typo!)
From these implications we easily obtain $(\forall n)[P(n)]$.

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Next Slide

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you can reach any $n \in \mathbb{N}$, then $(\forall n)[P(n)]$.