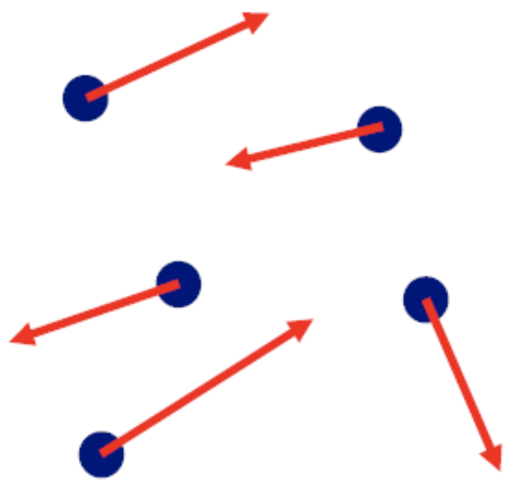
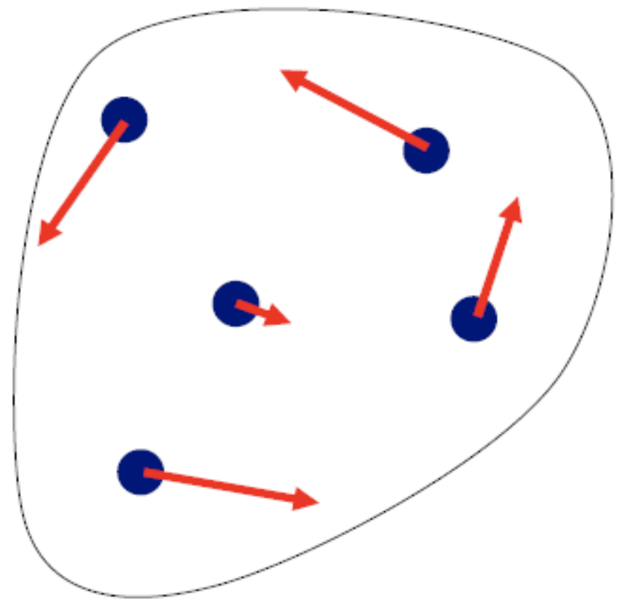
From Particles to Rigid Bodies



- Particles
  - No rotations
  - Linear velocity v only
  - 3N DoFs

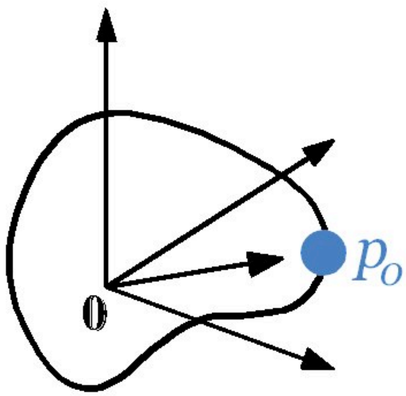


- Rigid bodies
  - 6 DoFs (translation + rotation)
  - Linear velocity v
  - Angular velocity ω

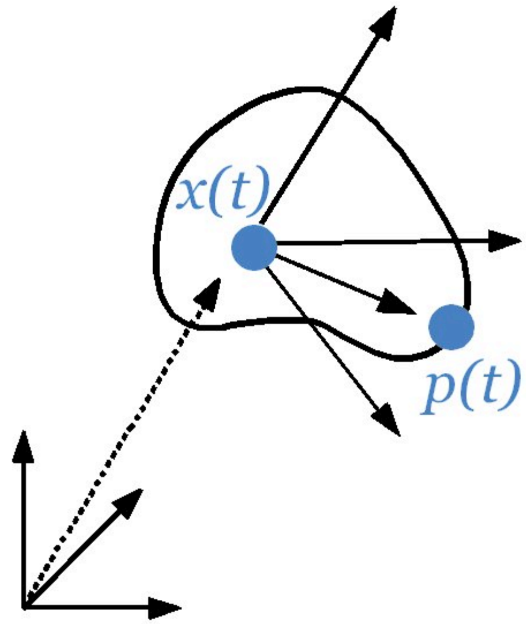
Outline

- Rigid Body Representation
- Kinematics
- Dynamics
- Simulation Algorithm
- Collisions and Contact Response

Coordinate Systems

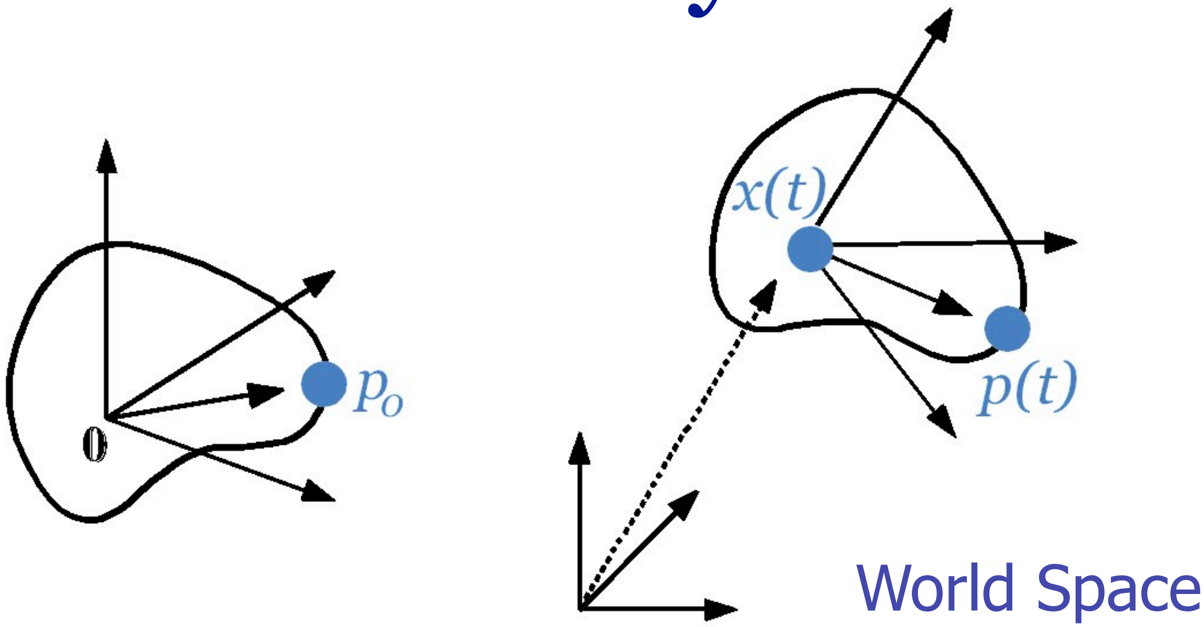


Body Space



- Body Space (Local Coordinate System)
  - Rigid bodies are defined relative to this system
  - Center of mass is the origin (for convenience)
    - We will specify body-related physical properties (inertia, …) in this frame

Coordinate Systems



- World Space:
  rigid body transformation to common frame

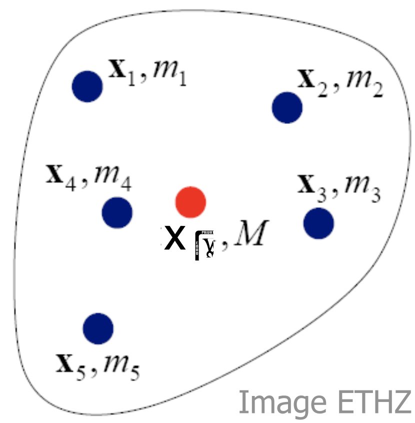
$$\mathbf{p}(t) = \underbrace{\mathbf{x}(t)}_{\text{translation}} + \underbrace{\mathrm{Rot}(\mathbf{p}_0)}_{\text{rotation}}$$

Center of mass

- Definition

$$\mathbf{x}_0 = \frac{\sum m_i \mathbf{x}_i}{\sum m_i} = \frac{\sum m_i \mathbf{x}_i}{M}$$

$$M\mathbf{x}_0 = \sum m_i \mathbf{x}_i$$



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- Motivation: forces

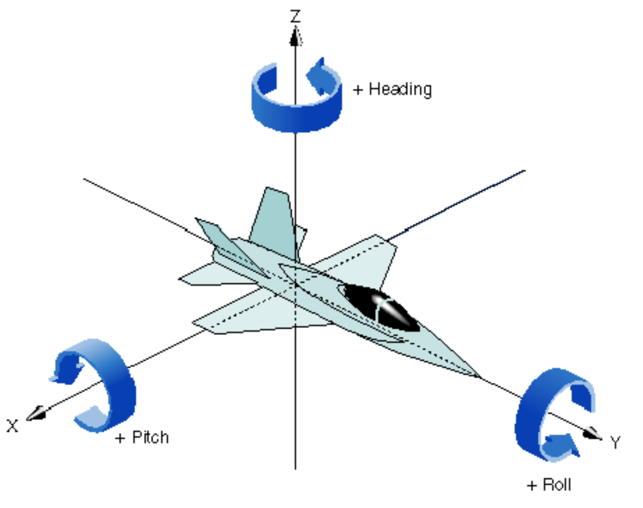
(one mass particle:) $\mathbf{f}_i = m_i \ddot{\mathbf{x}}_i$

(entire body:) $\mathbf{F} = \sum \mathbf{f}_i = \frac{d^2}{dt^2} \sum m_i \mathbf{x}_i$

$$\mathbf{F} = M\ddot{\mathbf{x}}_0$$

Rotations

- Euler angles:
  - 3 DoFs: roll, pitch, heading
  - Dependent on order of application
  - Not practical



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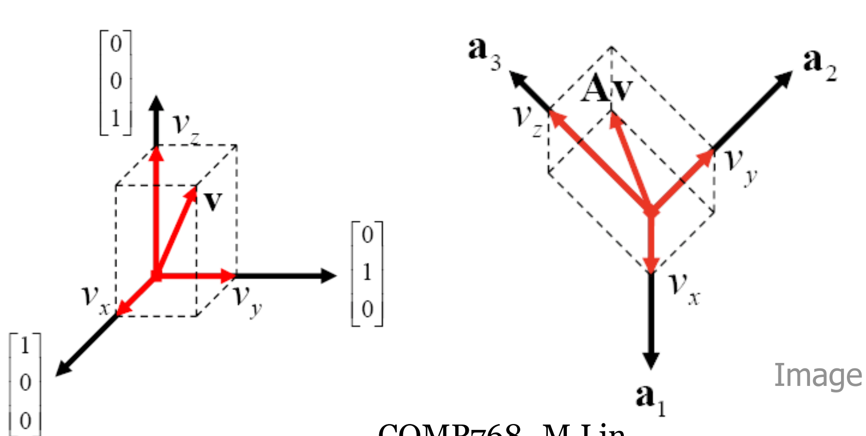
Rotations

- Rotation matrix
  - 3x3 matrix: 9 DoFs
  - Columns: world-space coordinates of body-space base vectors
  - Rotate a vector: $\mathtt{Rot}(\mathbf{v}) = R\mathbf{v} = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_1 & \mathbf{a}_1 \end{pmatrix} \mathbf{v}$



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Rotations

- Problem with rotation matrices: numerical drift

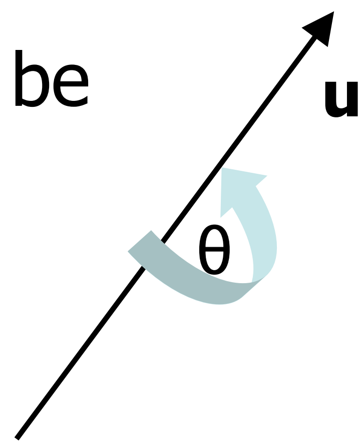
$$R(t_k) = \Delta t^k \dot{R}(t_k)\dot{R}(t_{k-1})\dot{R}(t_{k-2})\ldots R(t_0)$$

- Fix: use Gram-Schmidt orthogonalization
- Drift is easier to fix with quaternions

Unit Quaternion Definition

- $\mathbf{q} = [s, \mathbf{v}] : s$ is a scalar, $\mathbf{v}$ is vector
- A rotation of θ about a unit axis $\mathbf{u}$ can be represented by the unit quaternion:

$$[\cos(\theta/2), \sin(\theta/2)\ \mathbf{u}]$$



- Rotate a vector: $\mathrm{Rot}(\mathbf{v}) = \mathbf{q}\mathbf{a}\mathbf{q}^{\star}$
- Fix drift:
  - 4-tuple: vector representation of rotation
  - Normalized quaternion always defines a rotation in $\Re^3$

Unit Quaternion Operations

- Special multiplication:

$$[s_1, v_1][s_2, v_2] = [s_1 s_2 - v_1 \cdot v_2, s_1 v_2 + s_2 v_1 + v_1 \times v_2]$$

$$\frac{dq(t)}{dt} = \tfrac{1}{2}\omega(t)\mathbf{q}(t) = \tfrac{1}{2}\begin{bmatrix} 0 & \omega(t) \end{bmatrix}\mathbf{q}(t)$$

- Back to rotation matrix

$$R = \begin{pmatrix} 1 - 2v_y^2 - 2v_z^2 & 2v_x v_y - 2s v_z & 2v_x v_z + 2s v_y \\ 2v_x v_y + 2s v_z & 1 - 2v_x^2 - 2v_z^2 & 2v_y v_z - 2s v_x \\ 2v_x v_z - 2s v_y & 2v_y v_z + 2s v_x & 1 - 2v_x^2 - 2v_y^2 \end{pmatrix}$$

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Kinematics: Velocities

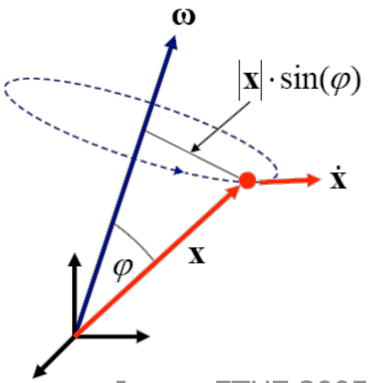
$$\dot{\mathbf{p}}(t) = \underbrace{\dot{\mathbf{x}}(t)}_{\text{Linear velocity}} + \underbrace{\dot{R}(t)}_{\text{Angular velocity}}\mathbf{p}_0$$

- How do $\mathbf{x}$(t) and $\mathbf{R}$(t) change over time?
- Linear velocity $\mathbf{v}$(t) describes the velocity of the center of mass **x** (m/s)

Kinematics: Velocities

- Angular velocity, represented by $\omega(t)$
  - Direction: axis of rotation
  - Magnitude $|\omega|$: angular velocity about the axis (rad/s)

$$\dot{\mathbf{x}} = \omega \times \mathbf{x}$$



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- Time derivative of rotation matrix:
  - Velocities of the body-frame axes, i.e. the columns of R

$$\dot{R} = \left(\omega(t) \times \begin{pmatrix} r_{xx} \\ r_{xy} \\ r_{xz} \end{pmatrix} \quad \omega(t) \times \begin{pmatrix} r_{yx} \\ r_{yy} \\ r_{yz} \end{pmatrix} \quad \omega(t) \times \begin{pmatrix} r_{zx} \\ r_{zy} \\ r_{zz} \end{pmatrix} \right)$$

Angular Velocities

- $\mathbf{R}(t)$ and $\omega(t)$ are related by:

$$\frac{d}{dt}\mathbf{R}(t) = \begin{pmatrix} 0 & -\omega_z(t) & \omega_y(t) \\ \omega_z(t) & 0 & -\omega_x(t) \\ -\omega_y(t) & \omega_x(t) & 0 \end{pmatrix} \mathbf{R}(t)$$

$$= \omega(t)^* \mathbf{R}(t)$$

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Dynamics: Accelerations

- How do $v(t)$ and $\omega(t)$ change over time?
- First we need some more machinery
  - Forces and Torques
  - Linear and angular momentum
  - Inertia Tensor
- Simplify equations by formulating accelerations in terms of momentum derivatives instead of velocity derivatives

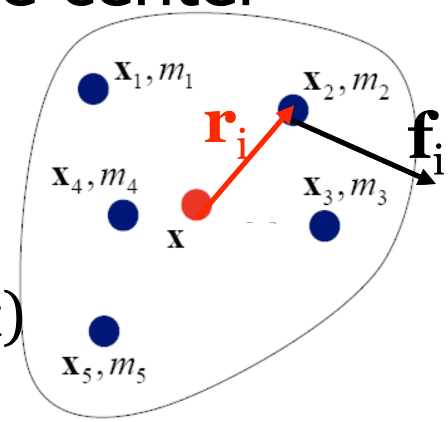
Forces and Torques

- External forces $\mathbf{f}_i(t)$ act on particles
  - Total external force $\mathbf{F} = \sum \mathbf{f}_i(t)$
- Torques depend on distance from the center of mass:

$$\tau_i(t) = (\mathbf{r}_i(t) - \mathbf{x}(t)) \times \mathbf{f}_i(t)$$

  - Total external torque

$$\tau(t) = \sum ((\mathbf{r}_i(t) - \mathbf{x}(t)) \times \mathbf{f}_i(t)$$



- $\mathbf{F}(t)$ doesn't convey any information about where the various forces act
- $\tau(t)$ does tell us about the distribution of forces

Linear Momentum

- Linear momentum P(t) lets us express the effect of total force F(t) on body (due to conservation of energy): $F(t) = \frac{dP(t)}{dt}$
- Linear momentum is the product of mass and linear velocity
  - P(t) $= \sum m_i dr_i(t)/dt$
    $= \sum m_i \mathbf{v}(t) + \omega(t) \times \sum m_i(\mathbf{r}_i(t) - \mathbf{x}(t))$
    $= \sum m_i \mathbf{v}(t) = M\ \mathbf{v}(t)$
  - Just as if body were a particle with mass M and velocity v(t)
  - Time derivative of **v**(t) to express acceleration:

$$\dot{\mathrm{v}}(t) = M^{-1}\frac{dP(t)}{dt} = M^{-1}F(t)$$

- Use P(t) instead of v(t) in state vectors

Angular momentum

- Same thing, angular momentum L(t) allows us to express the effect of total torque τ(t) on the body:

$$\dot{L}(t) = \tau(t)$$

- Similarily, there is a linear relationship between momentum and velocity:

$$L(t) = I\omega(t)$$

  – I(t) is inertia tensor, plays the role of mass

- Use L(t) instead of ω(t) in state vectors

Inertia Tensor

- 3x3 matrix describing how the shape and mass distribution of the body affects the relationship between the angular velocity and the angular momentum $L(t)$
- Analogous to mass – rotational mass
- We actually want the inverse $I^{-1}(t)$ to compute $\omega(t)=I^{-1}(t)L(t)$

Inertia Tensor

$$I = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}$$

Bunch of volume integrals:

$$I_{xx} = \int_V \rho(x,y,z)\left(y^2 + z^2\right) dV$$

$$I_{yy} = \int_V \rho(x,y,z)\left(x^2 + z^2\right) dV$$

$$I_{zz} = \int_V \rho(x,y,z)\left(x^2 + y^2\right) dV$$

$$I_{xy} = I_{yx} = \int_V \rho(x,y,z)\,(xy)\,dV$$

$$I_{xz} = I_{zx} = \int_V \rho(x,y,z)\,(zx)\,dV$$

$$I_{yz} = I_{zy} = \int_V \rho(x,y,z)\,(yz)\,dV$$

Inertia Tensor

- Avoid recomputing inverse of inertia tensor
- Compute I in body space I_{body} and then transform to world space as required
  - $I(t)$ varies in world space, but I_{body} is constant in body space for the entire simulation
- Intuitively:
  - Transform $\omega(t)$ to body space, apply inertia tensor in body space, and transform back to world space
  - $L(t)=I(t)\omega(t)= R(t)\, I_{body}\, R^T(t)\, \omega(t)$
  - $I^{-1}(t)= R(t)\, I_{body}^{-1}\, R^T(t)$

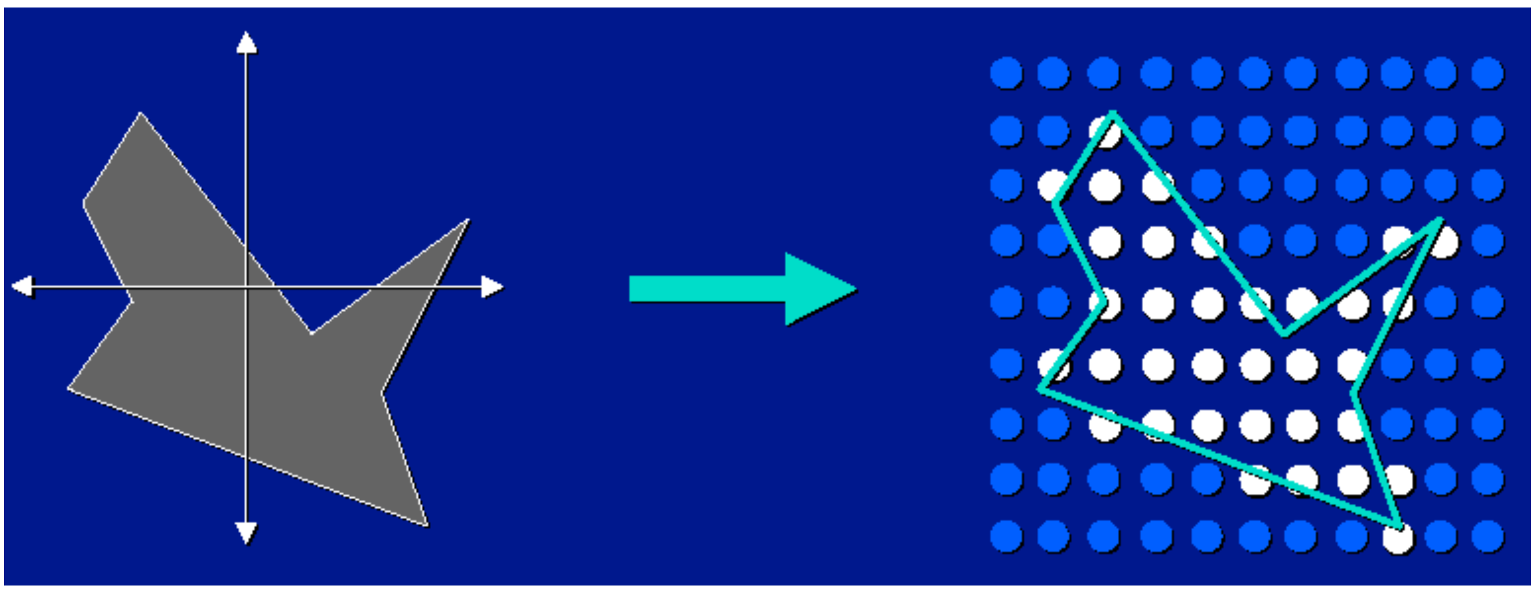
Computing I_{body}^{-1}

- There exists an orientation in body space which causes I_{xy}, I_{xz}, I_{yz} to all vanish
  - Diagonalize tensor matrix, define the eigenvectors to be the local body axes
  - Increases efficiency and trivial inverse
- Point sampling within the bounding box
- Projection and evaluation of Greene's thm.
  - Code implementing this method exists
  - Refer to Mirtich's paper at

    http://www.acm.org/jgt/papers/Mirtich96

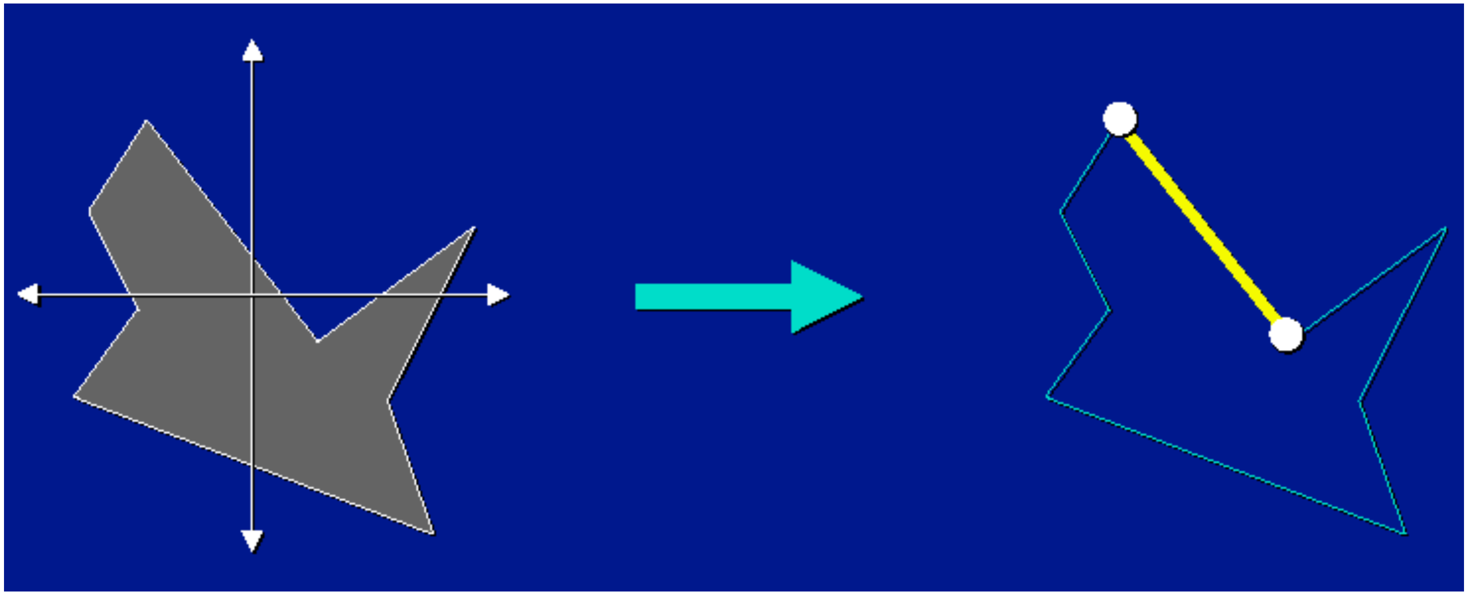
Approximation w/ Point

- Pros: Simple, fairly accurate, no B-rep needed.
- Cons: Expensive, requires volume test.



Use of Green's Theorem

- Pros: Simple, exact, no volumes needed.
- Cons: Requires boundary representation.



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Position state vector

$$\dot{\mathbf{X}}(t) = \frac{d}{dt}\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{q}(t)\\ P(t)\\ L(t)\end{pmatrix}\begin{matrix}\left.\begin{matrix}\vphantom{\mathbf{x}(t)}\\ \vphantom{\mathbf{q}(t)}\end{matrix}\right\}\longrightarrow \text{Spatial information}\\ \left.\begin{matrix}\vphantom{P(t)}\\ \vphantom{L(t)}\end{matrix}\right\}\longrightarrow \text{Velocity information}\end{matrix}$$

v(t) replaced by linear momentum P(t)

ω(t) replaced by angular momentum L(t)

Size of the vector: (3+4+3+3)N = 13N

Velocity state vector

$$\dot{\mathbf{X}}(t) = \frac{d}{dt}\begin{pmatrix} \mathbf{x}(t) \\ \mathbf{q}(t) \\ P(t) \\ L(t) \end{pmatrix} = \begin{pmatrix} \mathbf{v}(t) \\ \frac{1}{2}\omega(t)\mathbf{q}(t) \\ F(t) \\ \tau(t) \end{pmatrix} = \begin{pmatrix} \frac{P(t)}{M} \\ \frac{1}{2}I^{-1}L(t)\mathbf{q}(t) \\ F(t) \\ \tau(t) \end{pmatrix}$$

Conservation of momentum (P(t), L(t)) lets us express the accelerations in terms of forces and torques.

Simulation Algorithm

Pre-compute:

$$M \leftarrow \sum m_i$$

$$I_{\text{body}}$$

Initialize

$$\mathbf{x}, \mathbf{v}, R, \omega, \mathbf{X}, \dot{\mathbf{X}}$$

$$I^{-1} \leftarrow R I_{\text{body}} R^T$$

$$L \leftarrow I\omega$$

$$\tau \leftarrow \sum \mathbf{r}_i \times \mathbf{f}_i$$

$$\mathbf{F} \leftarrow \sum \mathbf{f}_i$$

Accumulate forces

$$(\mathbf{X}, \dot{\mathbf{X}}) \leftarrow \text{step}(\mathbf{X}, \dot{\mathbf{X}}, \mathbf{F}, \tau)$$

$$R \leftarrow \text{quat2mat}(\mathbf{q})$$

$$I^{-1} \leftarrow R I_{\text{body}} R^T$$

Your favorite ODE solver

Simulation Algorithm

Pre-compute:

$$M \leftarrow \sum m_i$$

$$I_{\text{body}}$$

Initialize

$$\mathbf{x}, \mathbf{v}, R, \omega$$

$$I^{-1} \leftarrow R I_{\text{body}} R^T$$

$$L \leftarrow I\omega$$

$$\tau \leftarrow \sum \mathbf{r}_i \times \mathbf{f}_i$$

$$\mathbf{F} \leftarrow \sum \mathbf{f}_i$$

Accumulate forces

$$P \leftarrow P + \Delta t \mathbf{F}$$

$$L \leftarrow L + \Delta t \tau$$

$$\omega \leftarrow I^{-1} L$$

$$\mathbf{x} \leftarrow \mathbf{x} + \Delta t \frac{\mathbf{P}}{\mathbf{M}}$$

$$\mathbf{q} \leftarrow \mathbf{q} + \Delta t \frac{\mathbf{1}}{\mathbf{2}} \omega \mathbf{q}$$

$$R \leftarrow \text{quat2mat}(\mathbf{q})$$

$$I^{-1} \leftarrow R I_{\text{body}} R^T$$

Explicit Euler step

Outline

- Rigid Body Representation
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- Simulation Algorithm
- Collision Detection and Contact Determination
  - Contact classification
  - Intersection testing, bisection, and nearest features

What happens when bodies collide?

- Colliding
  - Bodies bounce off each other
  - Elasticity governs 'bounciness'
  - Motion of bodies changes **discontinuously** within a discrete time step
  - 'Before' and 'After' states need to be computed
- In contact
  - Resting
  - Sliding
  - Friction

Detecting collisions and response

- Several choices
  - Collision detection: which algorithm?
  - Response: Backtrack or allow penetration?
- Two primitives to find out if response is necessary:
  - Distance(A,B): cheap, no contact information → fast intersection query
  - Contact(A,B): expensive, with contact information

Distance(A,B)

- Returns a value which is the minimum distance between two bodies
- Approximate may be ok
- Negative if the bodies intersect
- Convex polyhedra
  - Lin-Canny and GJK -- 2 classes of algorithms
- Non-convex polyhedra
  - Much more useful but hard to get distance fast
  - PQP/RAPID/SWIFT++
- Remark: most of these algorithms give inaccurate information if bodies intersect, except for DEEP

Contacts(A,B)

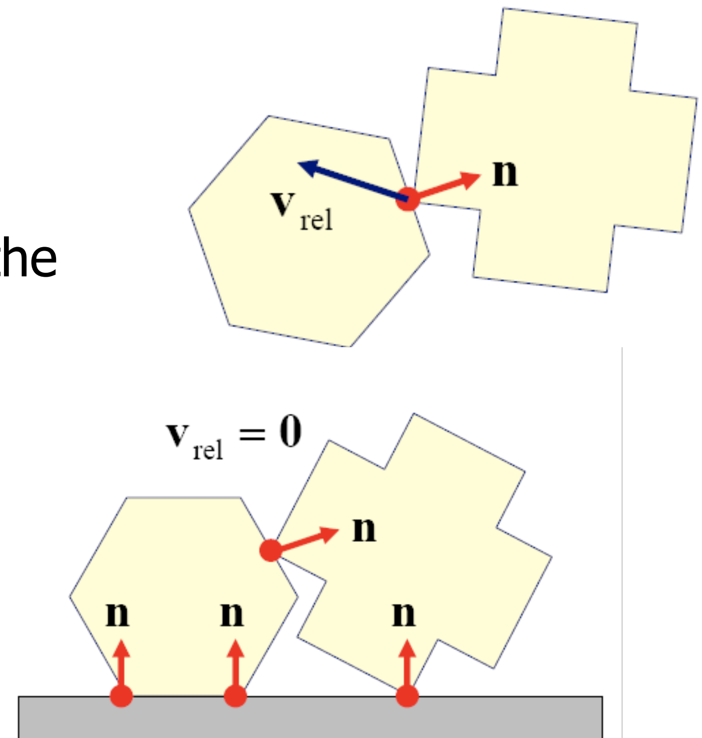
- Returns the set of features that are nearest for disjoint bodies or intersecting for penetrating bodies
- Convex polyhedra
  – LC & GJK give the nearest features as a bi-product of their computation – only a single pair. Others that are equally distant may not be returned.
- Non-convex polyhedra
  – Much more useful but much harder problem especially contact determination for disjoint bodies
  – Convex decomposition: SWIFT++

Prereq: Fast intersection test

- First, we want to make sure that bodies will intersect at next discrete time instant
- If not:
  - X_{new} is a valid, non-penetrating state, proceed to next time step
- If intersection:
  - Classify contact
  - Compute response
  - Recompute new state

Bodies intersect → classify contacts

- Colliding contact ('easy')
  - $v_{rel} < -\varepsilon$
  - Instantaneous change in velocity
  - Discontinuity: requires restart of the equation solver
- Resting contact (**hard!**)
  - $-\varepsilon < v_{rel} < \varepsilon$
  - Gradual contact forces avoid interpenetration
  - No discontinuities
- Bodies separating
  - $v_{rel} > \varepsilon$
  - No response required



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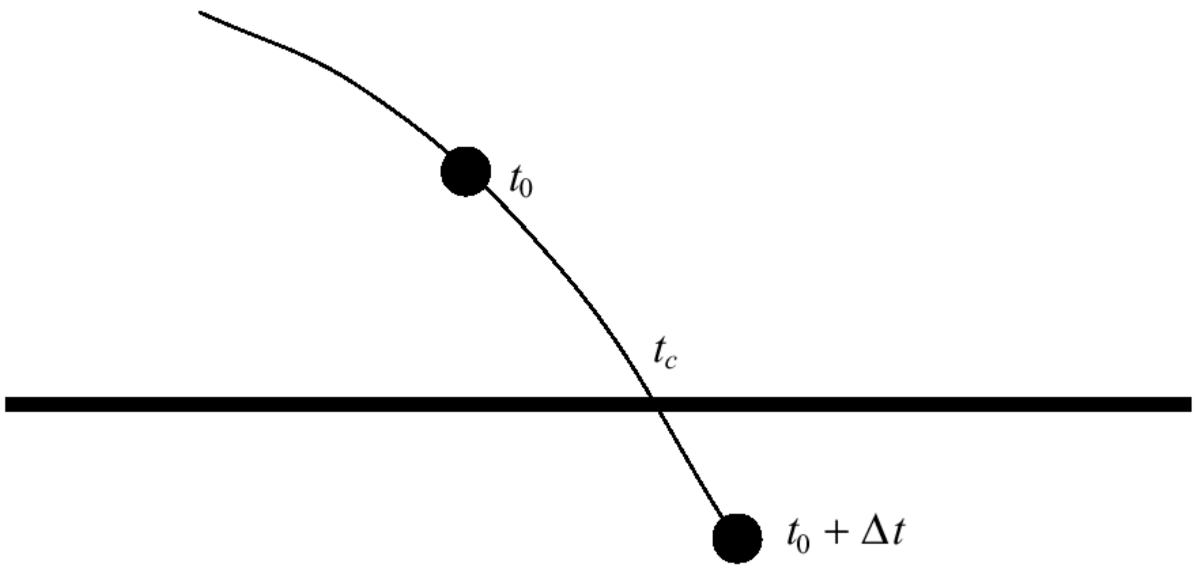
Colliding contacts

- At time t_i, body A and B intersect and

  $v_{rel} < -\varepsilon$
- Discontinuity in velocity: need to stop numerical solver
- Find time of collision t_c
- Compute new velocities $v^+(t_c) \rightarrow X^+(t)$
- Restart ODE solver at time t_c with new state $X^+(t)$

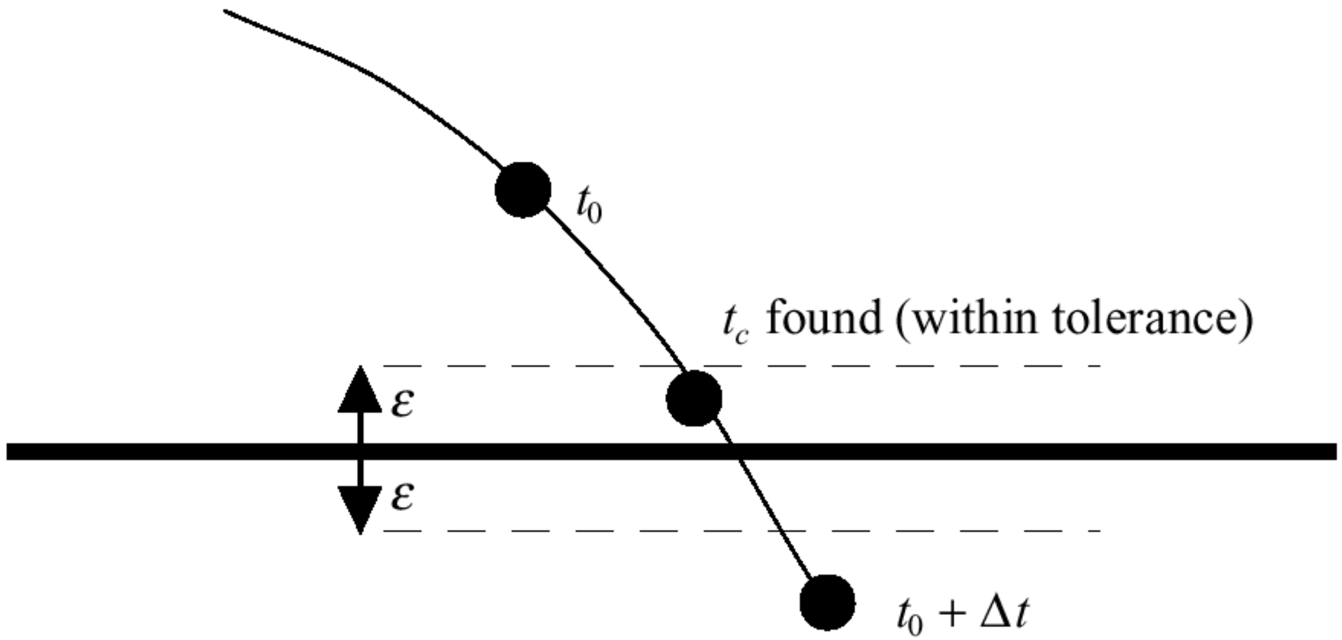
Time of collision

- We wish to compute when two bodies are "close enough" and then apply contact forces
- Let's recall a particle colliding with a plane



Time of collision

- We wish to compute t_c to some tolerance



Time of collision

1. A common method is to use **bisection search** until the distance is positive but less than the tolerance
2. Use **continuous collision detection**

3. $\mathbf{t_c}$ not always needed

   → **penalty-based methods**

Bisection

```
findCollisionTime(X,t,Δt)
    foreach pair of bodies (A,B) do
        Compute_New_Body_States(S_copy, t, Δt);
        hs(A,B) = Δt;    // H is the target timestep
        if Distance(A,B) < 0 then
            try_h = Δt /2;   try_t = t + try_h;
            while TRUE do
                Compute_New_Body_States(S_copy, t, try_t - t);
                if Distance(A,B) < 0 then
                        try_h /= 2;   try_t -= try_h;
                else if Distance(A,B) < ε then
                        break;
                else
                        try_h /= 2;   try_t += try_h;
            hs(A,B)->append(try_t – t);
        h = min( hs );
```

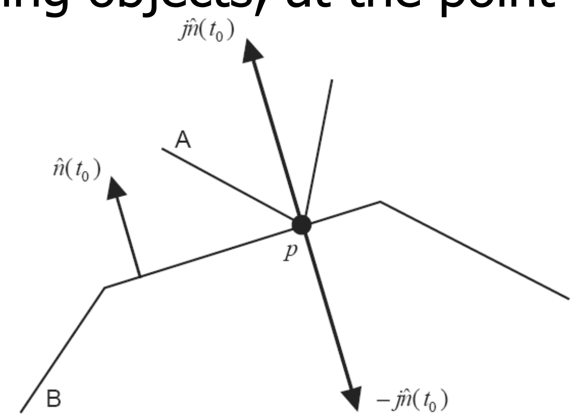
What happens upon collision

- Force driven
  - Penalty based
  - Easier, but slow objects react 'slow' to collision
- Impulse driven
  - Impulses provide instantaneous changes to velocity, unlike forces

$$\Delta(P) = J$$

  - We apply impulses to the colliding objects, at the point of collision
  - For frictionless bodies, the direction will be the same as the normal direction:

$$J = j\,\mathbf{n}$$



Colliding Contact Response

- Assumptions:
  - Convex bodies
  - Non-penetrating
  - Non-degenerate configuration
    - edge-edge or vertex-face
    - appropriate set of rules can handle the others
- Need a contact unit normal vector
  - Face-vertex case: use the normal of the face
  - Edge-edge case: use the cross-product of the direction vectors of the two edges

Colliding Contact Response

- Point velocities at the nearest points:

$$\dot{p}_a(t_0) = v_a(t_0) + \omega_a(t_0) \times (p_a(t_0) - x_a(t_0))$$

$$\dot{p}_b(t_0) = v_b(t_0) + \omega_b(t_0) \times (p_b(t_0) - x_b(t_0))$$

- Relative contact normal velocity:

$$v_{rel} = \hat{n}(t_0) \cdot (\dot{p}_a(t_0) - \dot{p}_b(t_0))$$

Colliding Contact Response

- We will use the empirical law of frictionless collisions: $v^+_{rel} = -\epsilon v^-_{rel}$
  - Coefficient of restitution ϵ [0,1]
    - $\epsilon = 0$ – bodies stick together
    - $\epsilon = 1$ – loss-less rebound
- After some manipulation of equations...

$$j = \frac{-(1+\epsilon)v^-_{rel}}{\frac{1}{M_a} + \frac{1}{M_b} + \hat{n}(t_0) \cdot \left(I_a^{-1}(t_0)\left(r_a \times \hat{n}(t_0)\right)\right) \times r_a + \hat{n}(t_0) \cdot \left(I_b^{-1}(t_0)\left(r_b \times \hat{n}(t_0)\right)\right) \times r_b}$$

Compute and apply impulses

- The impulse is an instantaneous force – it changes the velocities of the bodies instantaneously:

$$J = j\mathbf{n}$$

$$\Delta\mathbf{v} = \frac{J}{M}$$

$$\Delta L = (\mathbf{x}_{\text{impact}} - \mathbf{x}) \times J$$

Penalty Methods

- If we don't look for time of collision t_c then we have a simulation based on penalty methods: the objects are allowed to intersect.
- **Global** or **local** response
  - **Global**: The penetration depth is used to compute a spring constant which forces them apart (dynamic springs)
  - **Local:** Impulse-based techniques

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