Free Response of a Spring Mass System

Consider a spring mass system with no forcing and arbitrary initial conditions,

$$mx''(t) = -Kx(t) - cx'(t), \qquad x(0) = x_0, \quad x'(0) = v_0.$$

This corresponds to an initial displacement of the mass by the amount x_0, and an initial velocity of v_0 for the mass. There are four cases we will consider:

1) The undamped case- $c = 0$

In this case $mx''(t) + Kx(t) = 0$ has as its characteristic equation, $mr^2 + K = 0$. The roots are the imaginary pair $r = \pm i\Omega$, where

$$\Omega = \sqrt{\frac{K}{m}} = \text{the natural frequency of the system.}$$

The independant solutions in this case are, $x_1(t) = \cos \Omega t \ and \ x_2(t) = \sin \Omega t$, and the general solution is then

$$x(t) = C_1 \cos \Omega t + C_2 \sin \Omega t.$$

Then $\quad x(0) = C_1 = x_0, \quad and \quad x'(0) = \Omega C_2 = v_0 \quad$ hence

$$x(t) = x_0 \cos \Omega t + \frac{v_0}{\Omega} \sin \Omega t.$$

This is a solution whose amplitude does not decrease with time and whose frequency, Ω, increases as the stiffness K of the spring increases and decreases as the mass m increases.

2) The Damped Case $c > 0$

In this case $mx''(t) + cx'(t) + Kx(t) = 0$ has as its characteristic equation, $mr^2 + cr + K = 0$. Then the roots are given by

$$r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4Km}}{2m}$$

and there are three distinct cases that can occur.

(a) the underdamped case $0 < c^2 < 4Km$

In this case the roots are a complex conjugate pair $\qquad r_{1,2} = -\alpha \pm i\omega \qquad$ where $\alpha = \frac{c}{2m}$

$$and \qquad \omega = \frac{\sqrt{4Km - c^2}}{2m} = \sqrt{\frac{K}{m} - \left(\frac{c}{2m}\right)^2} = \sqrt{\Omega^2 - \left(\frac{c}{2m}\right)^2} < \Omega$$

Note that one effect of the damping is to reduce the frequency at which the system oscillates from the natural frequency Ω, to a lower frequency, ω. In this case two independent solutions are

$$x_1(t) = e^{-\alpha t} \cos \omega t \qquad and \qquad x_2(t) = e^{-\alpha t} \sin \omega t,$$

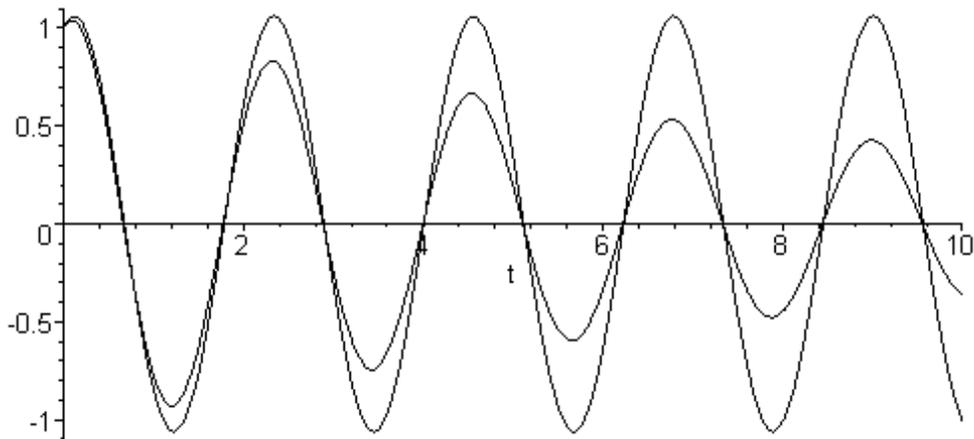and the general solution is $\qquad x(t) = C_1 e^{-\alpha t} \cos \omega t + C_2 e^{-\alpha t} \sin \omega t.$
Then
$$x(0) = C_1 = x_0 \qquad and \qquad x'(0) = -\alpha C_1 + \omega C_2 = v_0$$
and
$$x(t) = x_0 e^{-\alpha t} \cos \omega t + \frac{v_0 + \alpha x_0}{\omega} e^{-\alpha t} \sin \omega t.$$

Since $\alpha = -\frac{c}{2m} < 0,$ this is an oscillating solution whose amplitude decreases exponentially as time increases. The following figure shows the undamped and underdamped solutions plotted on the same axes.

(b) **the critically damped case** $\quad c^2 = 4Km$
As the damping is increased, the damped frequency, ω, decreases until finally reaching zero when $c^2 = 4Km$. At this point, the system no longer oscillates at all and the character of the solution changes. In this case, the roots of the characteristic equation are real and equal with the value $r_{1,2} = -\alpha = -\frac{c}{2m}$. Then $x_1(t) = e^{-\alpha t}$ and $x_2(t) = te^{-\alpha t}$ are two linearly independant solutions and the general solution is

$$x(t) = C_1 e^{-\alpha t} + C_2 t e^{-\alpha t}$$
The initial conditions are satisfied by the following solution

$$x(t) = x_0 e^{-\alpha t} + (v_0 + \alpha x_0) t e^{-\alpha t}$$

This solution decays to zero without oscillating.

(c) **the overdamped case** $c^2 > 4Km$

When the damping increases beyond the critical point $c^2 = 4Km$, the characteristic equation has distinct real roots

$$r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4Km}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}} = -\alpha \pm \beta, \qquad \beta = \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}} < \alpha$$

both of which are negative since
$$-\alpha - \beta < -\alpha < -\alpha + \beta < 0.$$

This leads to the general solution

$$x(t) = C_1 e^{-(\alpha-\beta)t} + C_2 e^{-(\alpha+\beta)t}$$

and

$$x(t) = \frac{v_0 + (\alpha + \beta)x_0}{2\beta} e^{-(\alpha-\beta)t} - \frac{v_0 + (\alpha - \beta)x_0}{2\beta} e^{-(\alpha+\beta)t}$$

is the solution that satisfies the initial conditions. This solution, like the critically damped solution, decays to zero without oscillating. Note that if $x_0 > 0$ and $v_0 = 0$, then the overdamped solution decays to zero less rapidly than the critically damped solution. This is shown in the figure below where both solutions are plotted on the same axes. This is because when the damping is increased beyond the critical value, not only is the oscillation supressed, the motion of the mass is further impeded so that a mass released from rest takes longer to return to the equilibrium position. Heavy doors are often equipped with a spring device that pulls the door shut but the motion is sufficiently damped that the door can not slam. For this to work properly, the damping must equal or exceed the critical value.
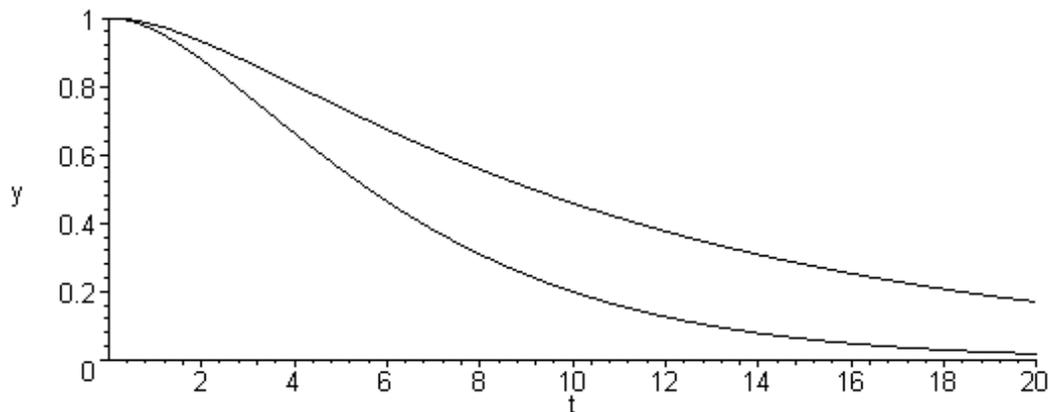
If the initial conditions are changed to $x_0 = 0$ and $v_0 > 0$, then both solutions begin from the equilibrium position, $x = 0$, and rise to a maximum displacement before decreasing back to $x = 0$. This is illustrated in the figure below. The critically damped solution reaches a higher maximum deflection than does the overdamped solution and therefore takes a longer time to decay to zero. This set of initial conditions is a reasonable approximation of the situation

that occurs when a shock absorber is actuated by a car hitting a bump. The shock absorber is compressed by the bump but returns to the uncompressed state without oscillating as long as the damping exceeds the critical value. If the damping is less than the critical value, then the car will bob up and down after hitting a bump. It is time then to replace the shock absorbers.
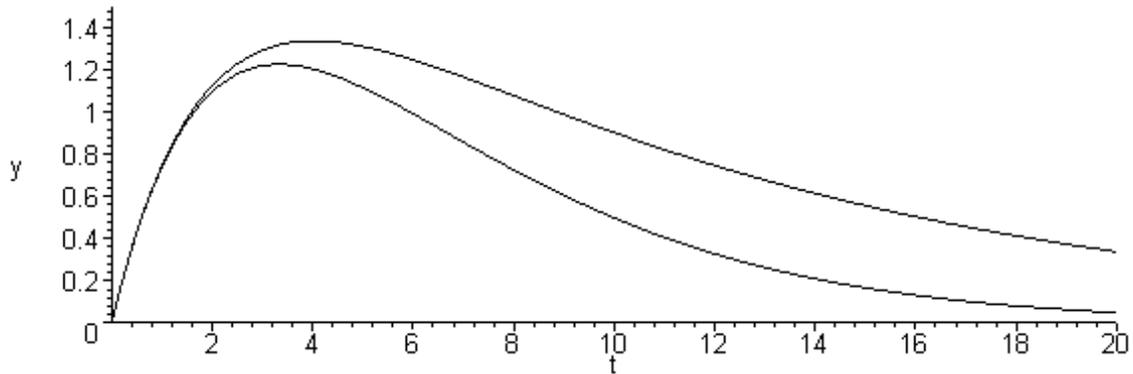
The unforced spring mass system is governed by the differential equation

$$m x''(t) + c x'(t) + K x(t) = 0$$

hence if we define the total energy in the system to be

$$E(t) = \tfrac{1}{2} m x'(t)^2 + \tfrac{1}{2} K x(t)^2 = kinetic\ energy\ + potential\ energy$$

then
$$E'(t) = \tfrac{1}{2} m\, 2 x'(t) x''(t) + \tfrac{1}{2} K\, 2 x(t) x'(t)$$

$$= x'(t)[m x''(t) + K x(t)] = -c x'(t)^2.$$

If $c = 0$ (the undamped case) then $E'(t) = 0$, so the the total energy is constant; the energy only changes from kinetic to potential and back. At the instant when the mass reaches its maximum deflection, the velocity is zero and the energy is all potential energy stored in the spring. At the instant when the mass passes through the equilibrium position at $x = 0$, the velocity is maximal and the energy is all kinetic since the potential energy is zero when the spring is in its equilibrium state. If $c > 0$, then $E'(t) = -c x'(t)^2 < 0$ so the energy is decreasing, which is why the friction is referred to as an energy dissipating mechanism. The rate at which energy is dissipated increases as the damping increases.