Due: February 20th, 2025
Date: February 6th, 2025

Assignment 1

Please submit it electronically to ELMS. This assignment is 8% in your total points. For the simplicity of the grading, the total points for the assignment are 80. Note that we will reward the use of Latex for typesetting with bonus points (an extra 5% of your points).

Problem 1.

1. *(3 points)* Express each of the three Pauli operators,
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
using Dirac notation in the computational basis.
2. *(2 points)* Find the eigenvalues and the corresponding eigenvectors of each Pauli operator. Express the eigenvectors using Dirac notation.
3. *(2 points)* Write the operator $X \otimes Z$ as a matrix and using Dirac notation (in both cases using the computational basis).
4. *(3 points)* What are the eigenspaces of the operator $X \otimes Z$? Express them using Dirac notation.
5. *(2 points)* Using the Spectral Decomposition show that $HXH^\dagger = Z$, where
$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Problem 2. *Product and entangled states.* Determine which of the following states are entangled. If the state is not entangled, show how to write it as a tensor product; if it is entangled, prove this.

1. *(2 points)* $\frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle$
2. *(2 points)* $\frac{1}{2}(|00\rangle - i|01\rangle + i|10\rangle + |11\rangle)$
3. *(2 points)* $\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)$

Problem 3. *Unitary operations and measurements.* Consider the state
$$|\psi\rangle = \frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle - \frac{2}{3}|11\rangle.$$

1. *(2 points)* Let $|\phi\rangle = (I \otimes H)|\psi\rangle$, where H denotes the Hadamard gate. Write $|\phi\rangle$ in the computational basis.

2. *(2 points)* Suppose the first qubit of $|\phi\rangle$ is measured in the computational basis. What is the probability of obtaining 0, and in the event that this outcome occurs, what is the resulting state of the second qubit?

3. *(2 points)* Suppose the second qubit of $|\phi\rangle$ is measured in the computational basis. What is the probability of obtaining 0, and in the event that this outcome occurs, what is the resulting state of the first qubit?

4. *(2 points)* Suppose $|\phi\rangle$ is measured in the computational basis. What are the probabilities of the four possible outcomes? Show that they are consistent with the marginal probabilities you computed in the previous two parts.

Problem 4. Let θ be a fixed, known angle. Suppose someone flips a fair coin and, depending on the outcome, either gives you the state

$$|0\rangle \qquad \text{or} \qquad \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$$

(but does not tell you which).

1. *(4 points)* Consider measuring the given state in the orthonormal basis consisting of

$$|\phi\rangle = \cos\phi\,|0\rangle + \sin\phi\,|1\rangle\,, \qquad |\phi^{\perp}\rangle = \sin\phi\,|0\rangle - \cos\phi\,|1\rangle$$

   Find the probabilities of all the possible measurement outcomes for each possible value of the given state

2. *(5 points)* Calculate the probability of correctly distinguishing the two possible states using the above measurement (In terms of ϕ)

3. *(Bonus: 5 points)* Calculate the optimal value of ϕ in order to best distinguish the states. What is the optimal success probability?

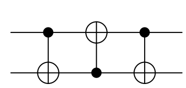
Problem 5. *Quantum Circuits*

1. *(10 points)* The CCCNOT (triple-controlled NOT) gate is a four-bit reversible gate that flips its fourth bit if and only if the first three bits are all in the state 1. Show how to implement a CCCNOT gate using Toffoli gates. You may use additional workspace as needed. You may assume that bits in the workspace start with a particular value, either 0 or 1, provided you return them to that value.

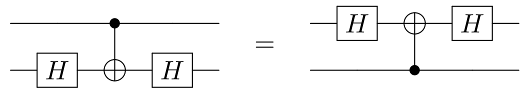
2. *(Bonus: 10 points)* Show that a Toffoli gate cannot be implemented using any number of CNOT gates, with any amount of workspace. Hence the CNOT gate alone is not universal. (Hint: It may be helpful to think of the gates as performing arithmetic operations on integers mod 2.)

Problem 6. *Circuit identities.*

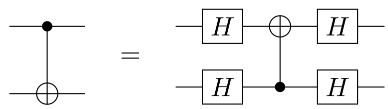
1. *(4 points)* Show that the following circuit swaps two qubits:



2. *(4 points)* Verify the following circuit identity:



3. *(4 points)* Verify the following circuit identity:



   Give an interpretation of this identity.

Problem 7. *Density matrices.* Consider the ensemble in which the state $|0\rangle$ occurs with probability 3/5 and the state $(|0\rangle + |1\rangle)/\sqrt{2}$ occurs with probability 2/5.

1. *(2 points)* What is the density matrix ρ of this ensemble?

2. *(2 points)* Write ρ in the form $\frac{1}{2}(I + r_x X + r_y Y + r_z Z)$, and plot ρ as a point in the Bloch sphere.

3. *(3 points)* Suppose we measure the state in the computational basis. What is the probability of getting the outcome 0? Compute this both by averaging over the ensemble of pure states and by computing $\mathrm{tr}(\rho\,|0\rangle\,\langle 0|)$, and show that the results are consistent.

4. *(3 points)* How does the density matrix change if we apply the Hadamard gate? Compute this both by applying the Hadamard gate to each pure state in the ensemble and finding the corresponding density matrix, and by computing $H\rho H^\dagger$.

Problem 8. *Local operations and the partial trace.*

1. *(2 points)* Let $|\psi\rangle = \frac{\sqrt{3}}{2}\,|00\rangle + \frac{1}{2}\,|11\rangle$. Let ρ denote the density matrix of $|\psi\rangle$ and let ρ' denote the density matrix of $(I \otimes H)\,|\psi\rangle$. Compute ρ and ρ'.

2. *(2 points)* Compute $\mathrm{tr}_B(\rho)$ and $\mathrm{tr}_B(\rho')$, where B refers to the second qubit.

3. *(4 points)* Let ρ be a density matrix for a quantum system with a bipartite state space $A \otimes B$. Let I denote the identity operation on system A, and let U be a unitary operation on system B. Prove that $\mathrm{tr}_B(\rho) = \mathrm{tr}_B((I \otimes U)\rho(I \otimes U^\dagger))$.

4. *(3 points)* Show that the converse of part (c) holds for pure states. In other words, show that if $|\psi\rangle$ and $|\phi\rangle$ are bipartite pure states, and $\mathrm{tr}_B(|\psi\rangle\,\langle\psi|) = \mathrm{tr}_B(|\phi\rangle\,\langle\phi|)$, then there is a unitary operation U acting on system B such that $|\phi\rangle = (I \otimes U)\,|\psi\rangle$.

5. *(2 points)* Does the converse of part (c) hold for general density matrices? Prove or disprove it.
