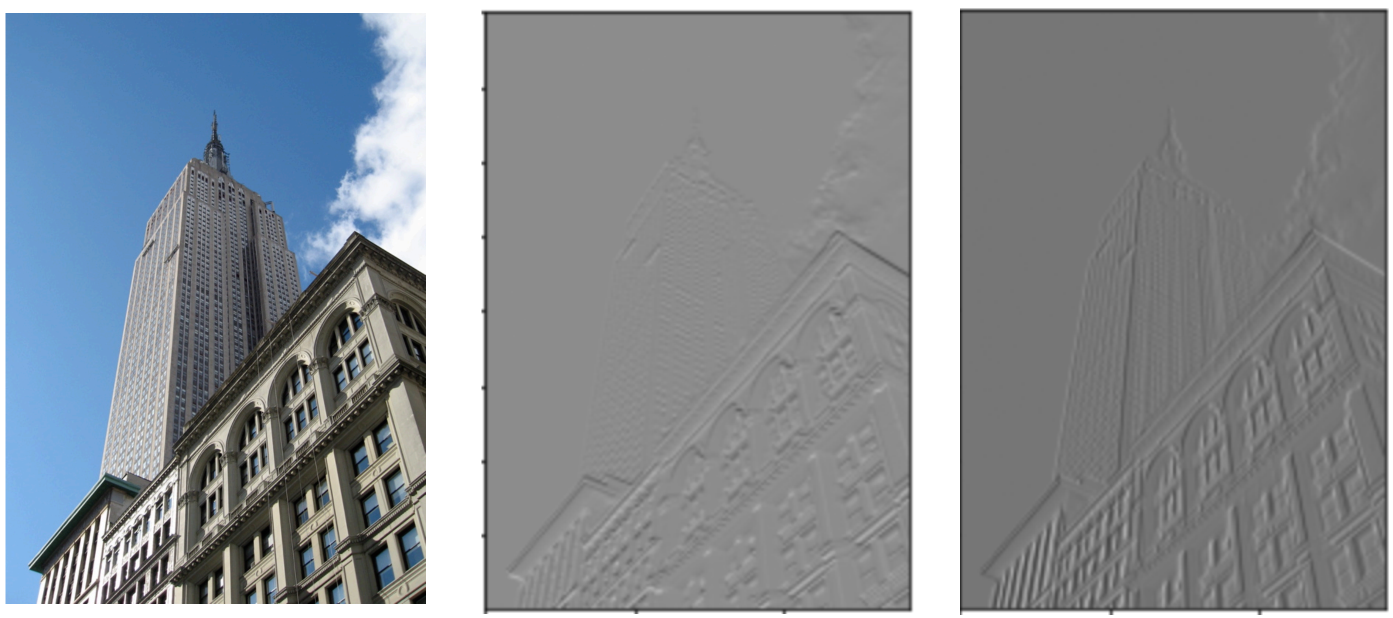
Edge Detection

Mohammad Nayeem Teli



Gaussian filters

- Remove "high-frequency" components from the image (low-pass filter)
- Convolution with self is another Gaussian
- Separable kernel
  - Factors into product of two 1D Gaussians

Separability of the Gaussian filter

$$G_\sigma(x,y) = \frac{1}{2\pi\sigma^2} \exp^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma} \exp^{-\frac{x^2}{2\sigma^2}}\right) \left(\frac{1}{\sqrt{2\pi}\sigma} \exp^{-\frac{y^2}{2\sigma^2}}\right)$$

The 2D Gaussian can be expressed as the product of two functions, one a function of *x* and the other a function of *y*

In this case, the two functions are the (identical) 1D Gaussian

Source: D. Lowe

Separability example

2D convolution
(center location only)

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 5 \\ 4 & 4 & 6 \end{bmatrix}$$

The filter factors into a product of 1D filters:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

Perform convolution along rows:

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 2 & 3 & 3 \\ 3 & 5 & 5 \\ 4 & 4 & 6 \end{bmatrix} = \begin{bmatrix} \; & 11 & \; \\ \; & 18 & \; \\ \; & 18 & \; \end{bmatrix}$$

Followed by convolution along the remaining column:

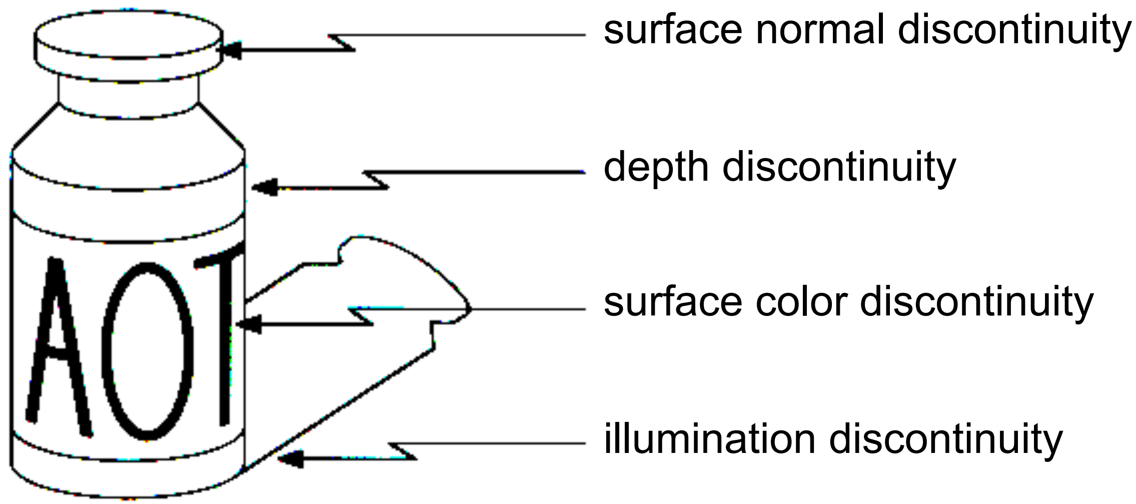
Source: K. Grauman

Efficient Implementation

Both, the BOX filter and the Gaussian filter are separable:

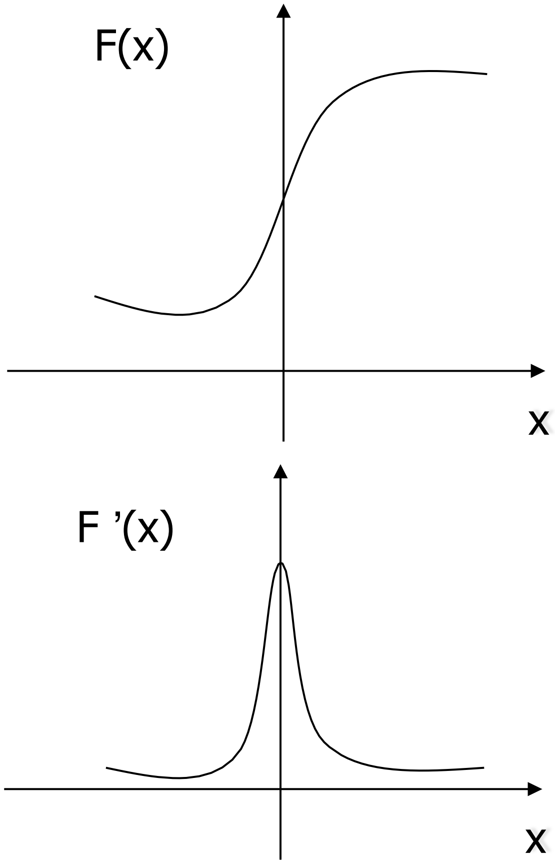
- First convolve each row with a 1D filter
- Then convolve each column with a 1D filter.

Origin of Edges

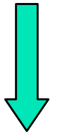


Edges are caused by a variety of factors

Edge detection (1D)



Edge= sharp variation



Large first derivative

Edge is Where Change Occurs

Change is measured by derivative in 1D

Biggest change, derivative has maximum magnitude

Or 2nd derivative is zero.

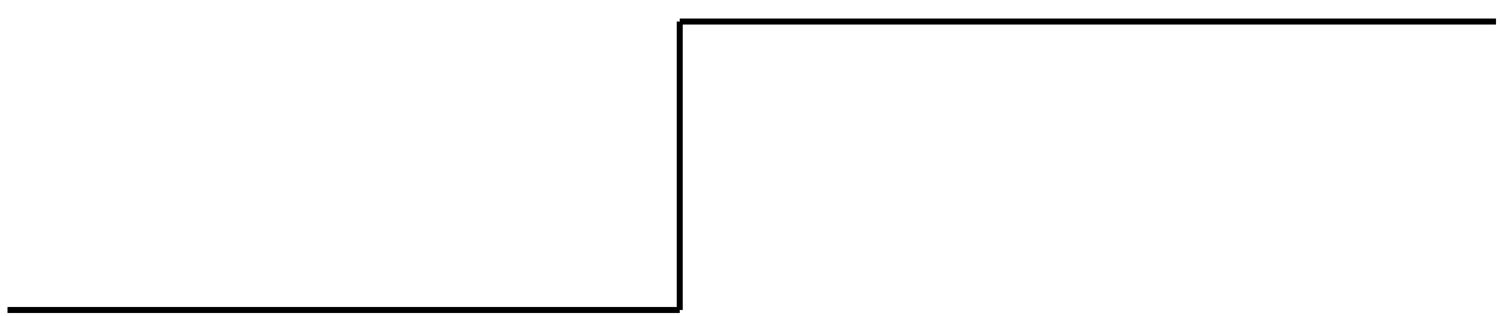
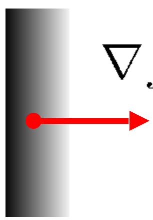


Image gradient

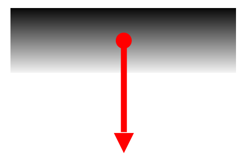
The gradient of an image:

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

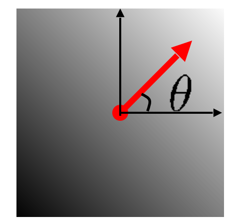
The gradient points in the direction of most rapid change in intensity



$$\nabla f = \left[\frac{\partial f}{\partial x}, 0\right]$$



$$\nabla f = \left[0, \frac{\partial f}{\partial y}\right]$$



$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

The gradient direction is given by:

$$\theta = \tan^{-1}\left(\frac{\partial f}{\partial y} / \frac{\partial f}{\partial x}\right)$$

- How does this relate to the direction of the edge?

The *edge strength* is given by the gradient magnitude

$$\|\nabla f\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

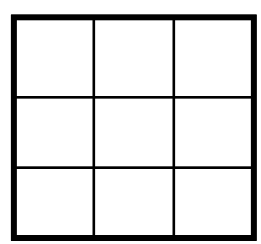
The discrete gradient

How can we differentiate a *digital* image f[x,y]?

- Option 1: reconstruct a continuous image, then take gradient
- Option 2: take discrete derivative (finite difference)

$$\frac{\partial f}{\partial x}(x, y) = \frac{f(x+1, y) - f(x-1, y)}{2}$$

How would you implement this as a cross-correlation?



H

The Sobel operator

Better approximations of the derivatives exist

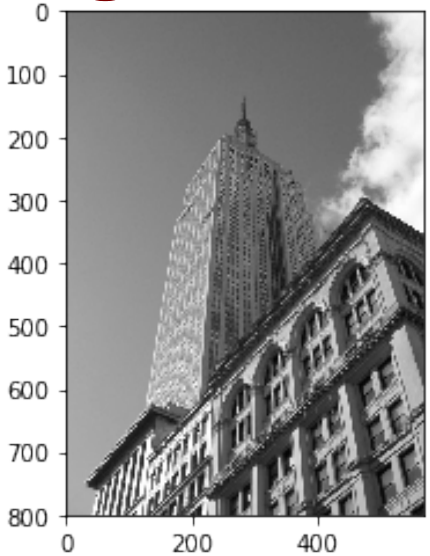
- The *Sobel* operators below are very commonly used

$$\frac{1}{8}\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \qquad \frac{1}{8}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

$$s_x \qquad\qquad s_y$$

- The standard defn. of the Sobel operator omits the 1/8 term
  - doesn't make a difference for edge detection
  - the 1/8 term **is** needed to get the right gradient value, however

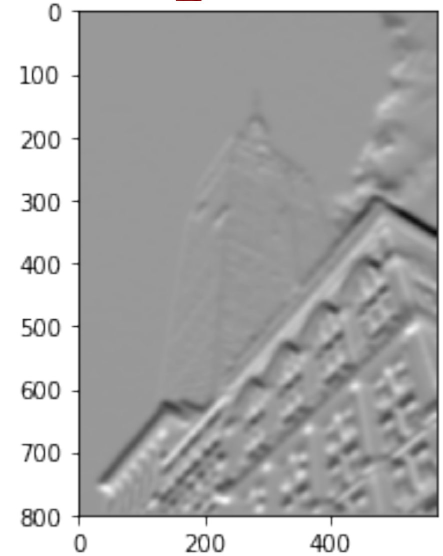
Edge Detection Using Sobel Operator



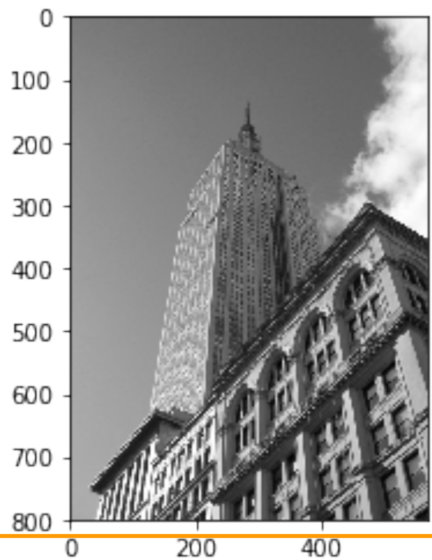
*

-1	-2	-1
0	0	0
1	2	1

=



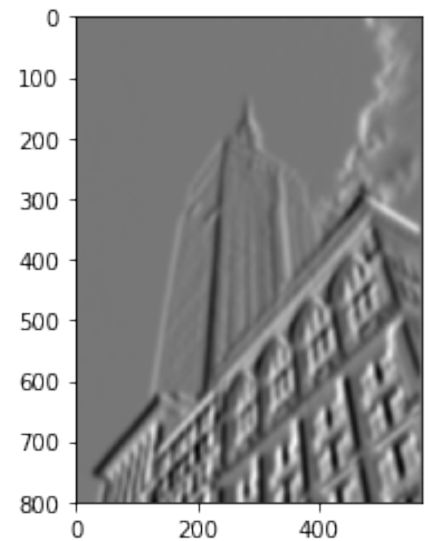
horizontal edge detector



*

-1	0	1
-2	0	2
-1	0	1

=



vertical edge detector

Vertical Edges

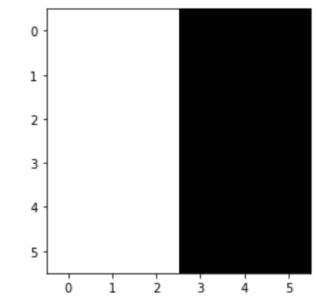
```
array([[ 255.,  255.,  255.,    0.,    0.,    0.],
       [ 255.,  255.,  255.,    0.,    0.,    0.],
       [ 255.,  255.,  255.,    0.,    0.,    0.],
       [ 255.,  255.,  255.,    0.,    0.,    0.],
       [ 255.,  255.,  255.,    0.,    0.,    0.],
       [ 255.,  255.,  255.,    0.,    0.,    0.]])
```

*

```
array([[-1.,  0.,  1.],
       [-1.,  0.,  1.],
       [-1.,  0.,  1.]])
```

=

```
[[   0. -765. -765.    0.]
 [   0. -765. -765.    0.]
 [   0. -765. -765.    0.]
 [   0. -765. -765.    0.]]
```



F

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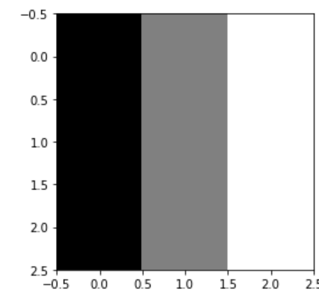

H

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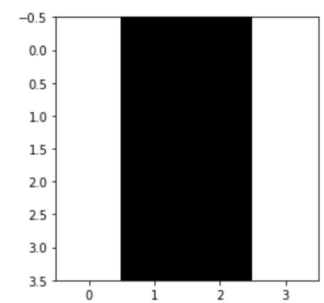


G

Horizontal Edges

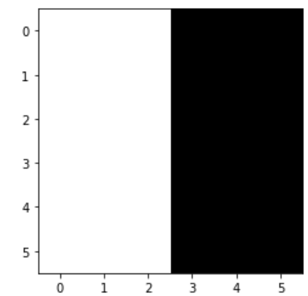
```
array([[ 255.,  255.,  255.,    0.,    0.,    0.],
       [ 255.,  255.,  255.,    0.,    0.,    0.],
       [ 255.,  255.,  255.,    0.,    0.,    0.],
       [ 255.,  255.,  255.,    0.,    0.,    0.],
       [ 255.,  255.,  255.,    0.,    0.,    0.],
       [ 255.,  255.,  255.,    0.,    0.,    0.]])
```

*

```
array([[ 1.,  1.,  1.],
       [ 0.,  0.,  0.],
       [-1., -1., -1.]])
```

=

```
[[ 0.  0.  0.  0.]
 [ 0.  0.  0.  0.]
 [ 0.  0.  0.  0.]
 [ 0.  0.  0.  0.]]
```



F

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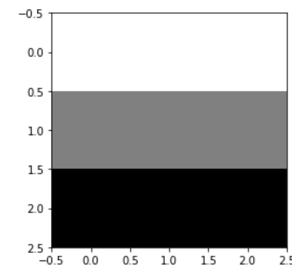

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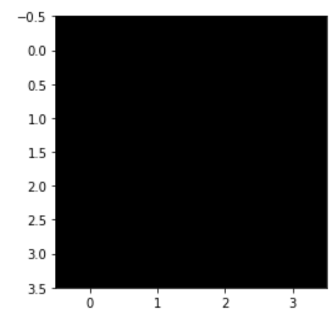


G

Gradient operators

$$\Delta_1 = \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \qquad \Delta_2 = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$$

(a)

$$\Delta_1 = \begin{matrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{matrix} \qquad \Delta_2 = \begin{matrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{matrix}$$

(b)

$$\Delta_1 = \begin{matrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{matrix} \qquad \Delta_2 = \begin{matrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{matrix}$$

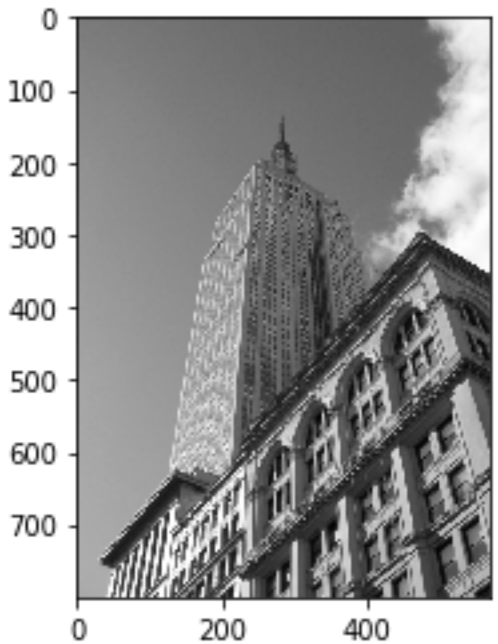
(c)

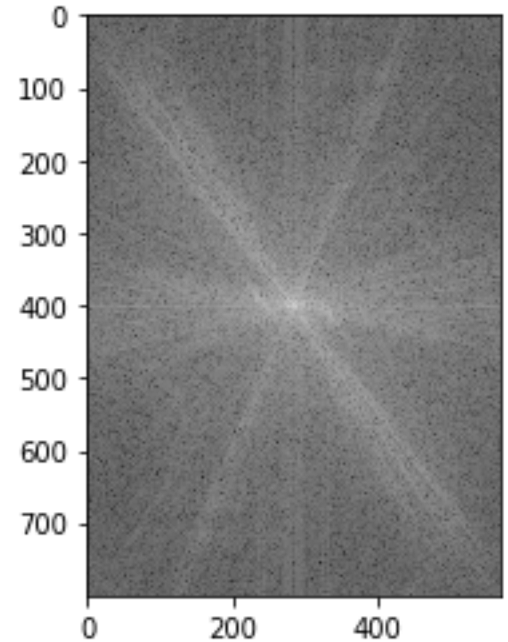
$$\Delta_1 = \begin{matrix} -3 & -1 & 1 & 3 \\ -3 & -1 & 1 & 3 \\ -3 & -1 & 1 & 3 \\ -3 & -1 & 1 & 3 \end{matrix} \qquad \Delta_2 = \begin{matrix} 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ -3 & -3 & -3 & -3 \end{matrix}$$

(d)

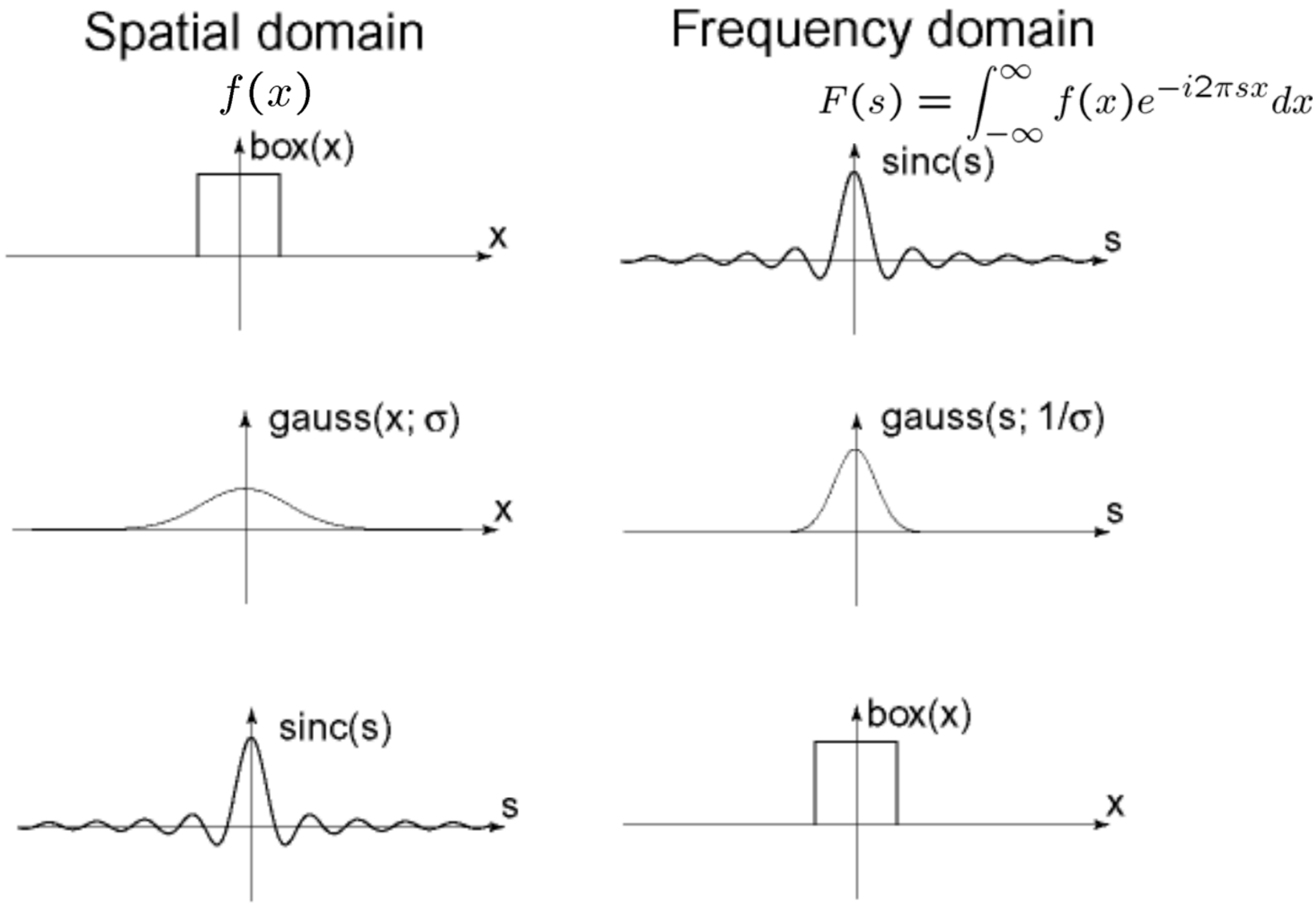
(a): Roberts' cross operator (b): 3x3 Prewitt operator
(c): Sobel operator (d) 4x4 Prewitt operator

Fourier domain





Fourier domain



Fourier

- How do we represent points in 2D?

- We write any point as a linear combination of two vectors (1,0) and (0,1)

  $(x,y) = x(1,0) + y(0,1)$

- Suppose our basis vectors were (1,0) and (1,1)

  $(x,y) = u(1,0) + v(1,1)$

- Point (7,3) would be represented as 4(1,0) + 3(1,1).

Fourier

- Fourier provides orthonormal basis for images.
- Think of images as a continuous function (a periodic function)
- The following functions provide an orthonormal basis for functions:

$$\frac{1}{\sqrt{2\pi}}, \frac{cos(kx)}{\sqrt{\pi}}, \frac{sin(kx)}{\sqrt{\pi}} \text{ for k = 1,2,3,..}$$

- These functions form the Fourier series.

Fourier

- To define whether a set of functions are orthonormal , we need to define an inner product between two functions.

- We do it similar to discrete functions.

- Multiply them together, integrate the result.

- Inner product between two functions f(x) and g(x)

$$< f, g > = \int_0^{2\pi} f(x)g(x)$$

Fourier

- When we say two functions are orthogonal we mean <f,g> = 0

- Functions in the Fourier series have unit magnitude.

- Any function can be expressed as a linear combination of the elements of the Fourier series

$$f(x) = a_0 \frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \left(b_k \frac{cos(kx)}{\sqrt{\pi}} + a_k \frac{sin(kx)}{\sqrt{\pi}} \right)$$

The values $a_0, b_1, a_1, b_2, \ldots$ are coordinates

Fourier

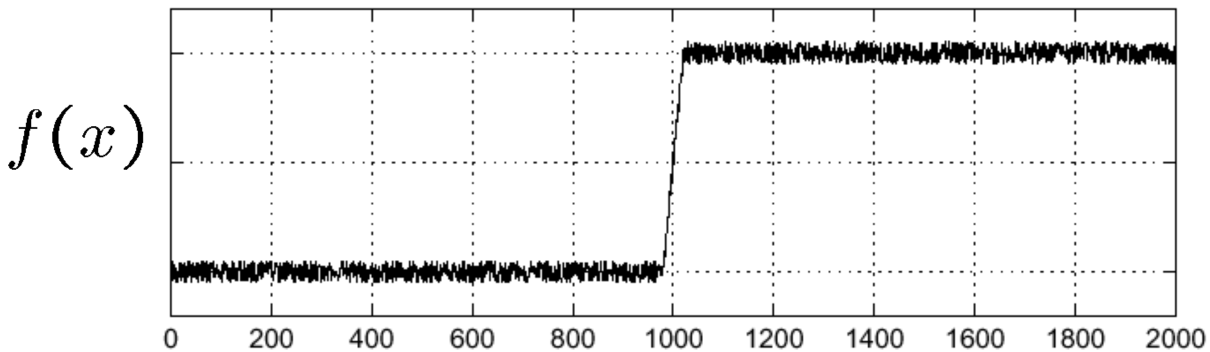
Pythogorean analog in Fourier is called Parsevaal's theorem

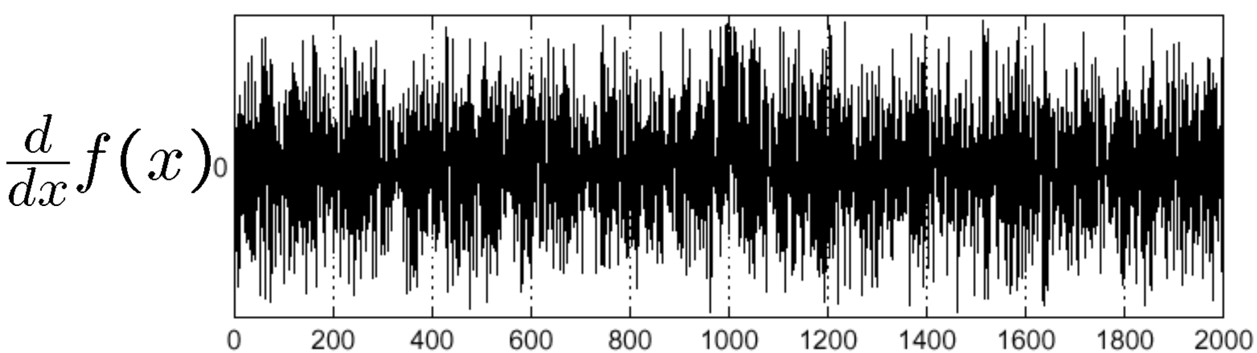
$$\int_0^{2\pi} f^2(x)dx = a_0^2 + \sum_1^{\infty} (a_i^2 + b_i^2)$$

Effects of noise

Consider a single row or column of the image

- Plotting intensity as a function of position gives a signal





Where is the edge?

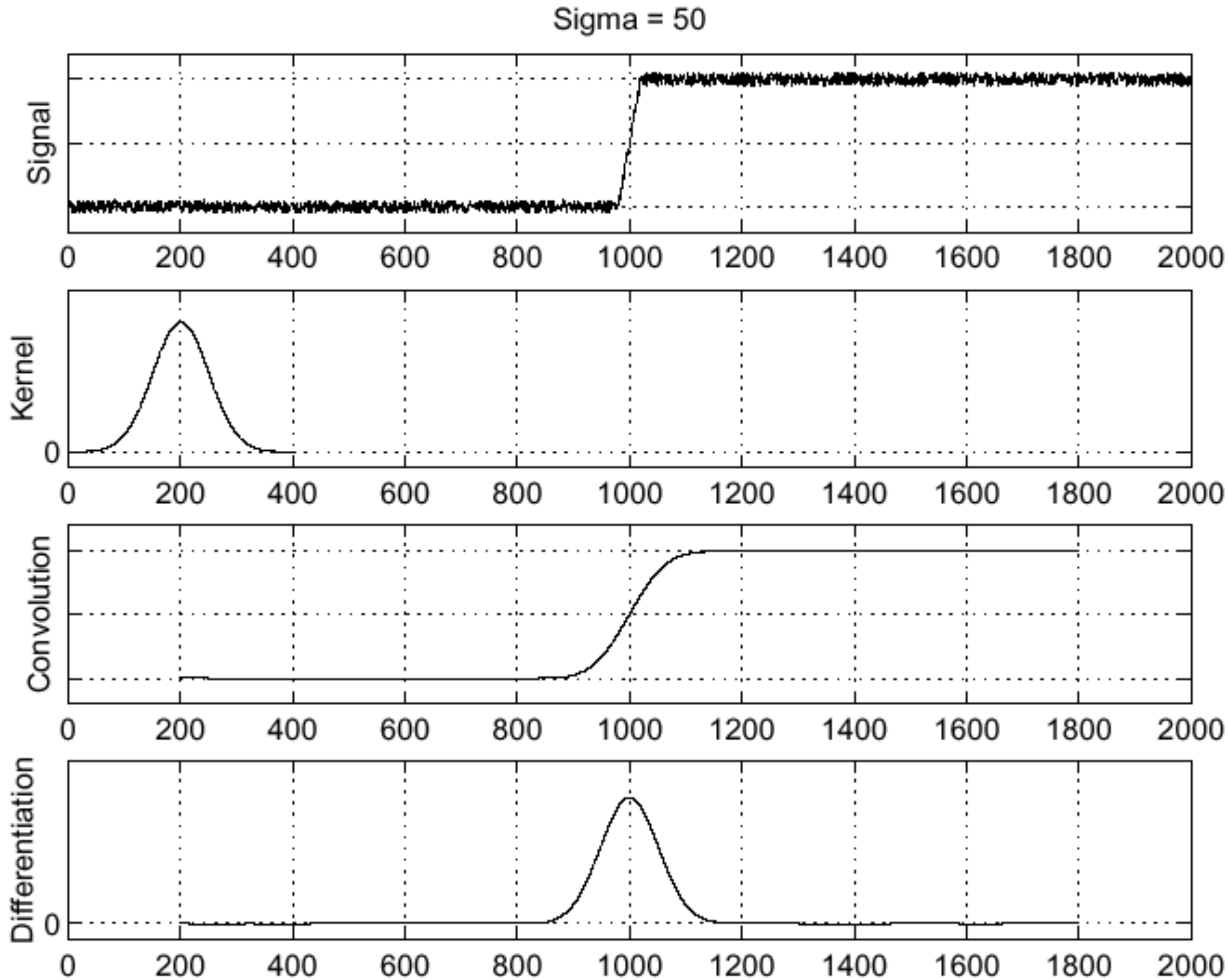
Solution: smooth first

f

h

$h * f$

$\frac{\partial}{\partial x}(h * f)$



Where is the edge? Look for peaks $\frac{\partial}{\partial x}(h * f)$

Convolution Theorem

- The *Fourier transform* of the convolution of two functions is the product of their Fourier transforms

$$F[g * h] = F[g]F[h]$$

- The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms

$$F^{-1}[g * h] = F^{-1}[g]F^{-1}[h]$$

- Convolution in spatial domain is equivalent to multiplication in frequency domain.

Derivative theorem of convolution

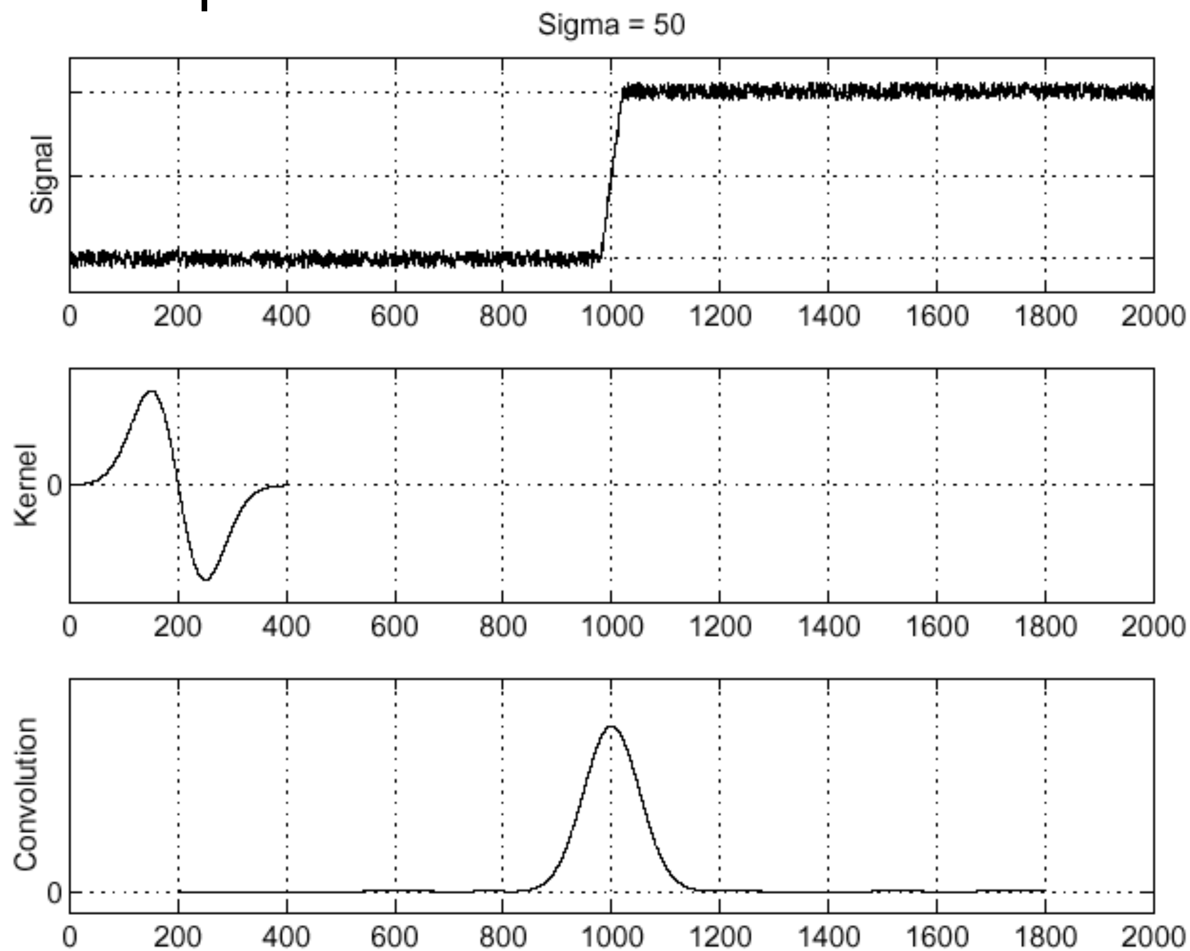
$$\frac{\partial}{\partial x}(h * f) = (\frac{\partial}{\partial x}h) * f$$

This saves us one operation:

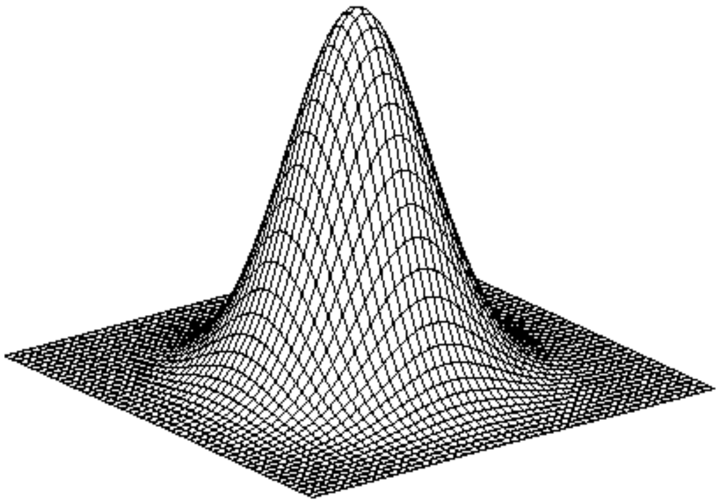
$$f$$

$$\frac{\partial}{\partial x}h$$

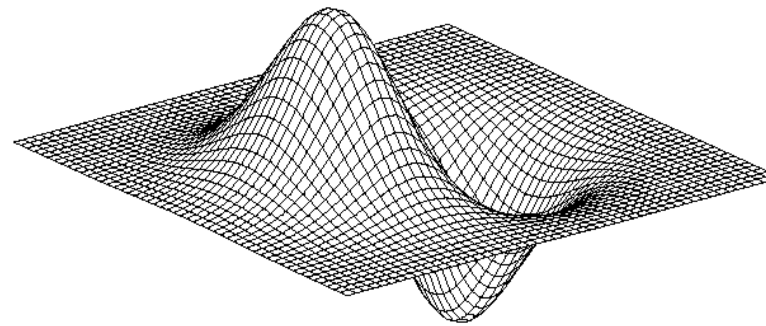
$$(\frac{\partial}{\partial x}h) * f$$



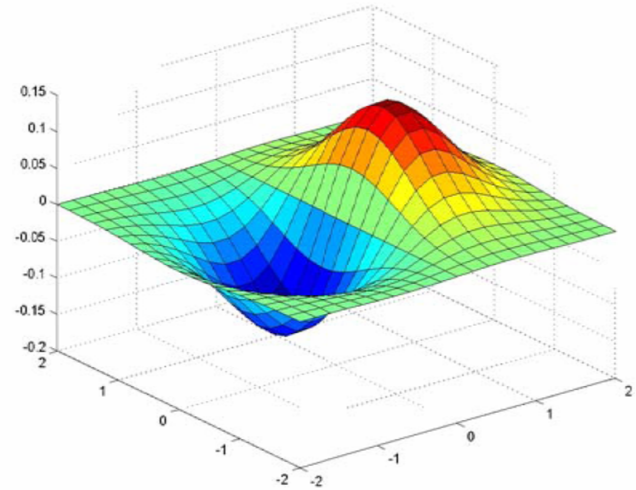
Derivative of Gaussian filter



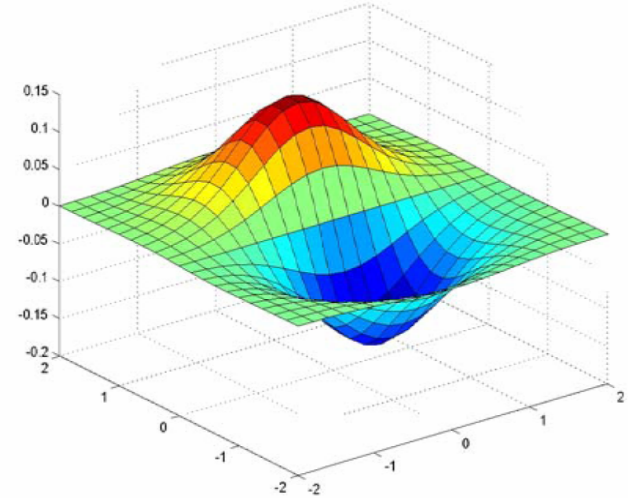
* [1 -1] =



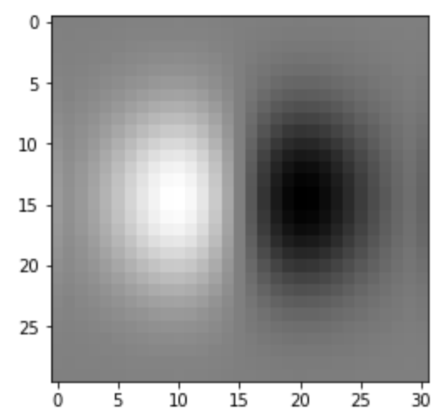
Derivative of Gaussian filter

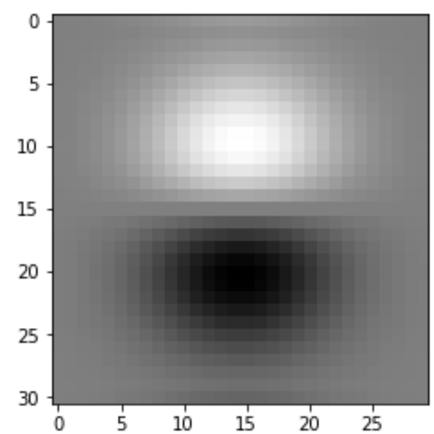


x-direction



y-direction





Which one finds horizontal/vertical edges?

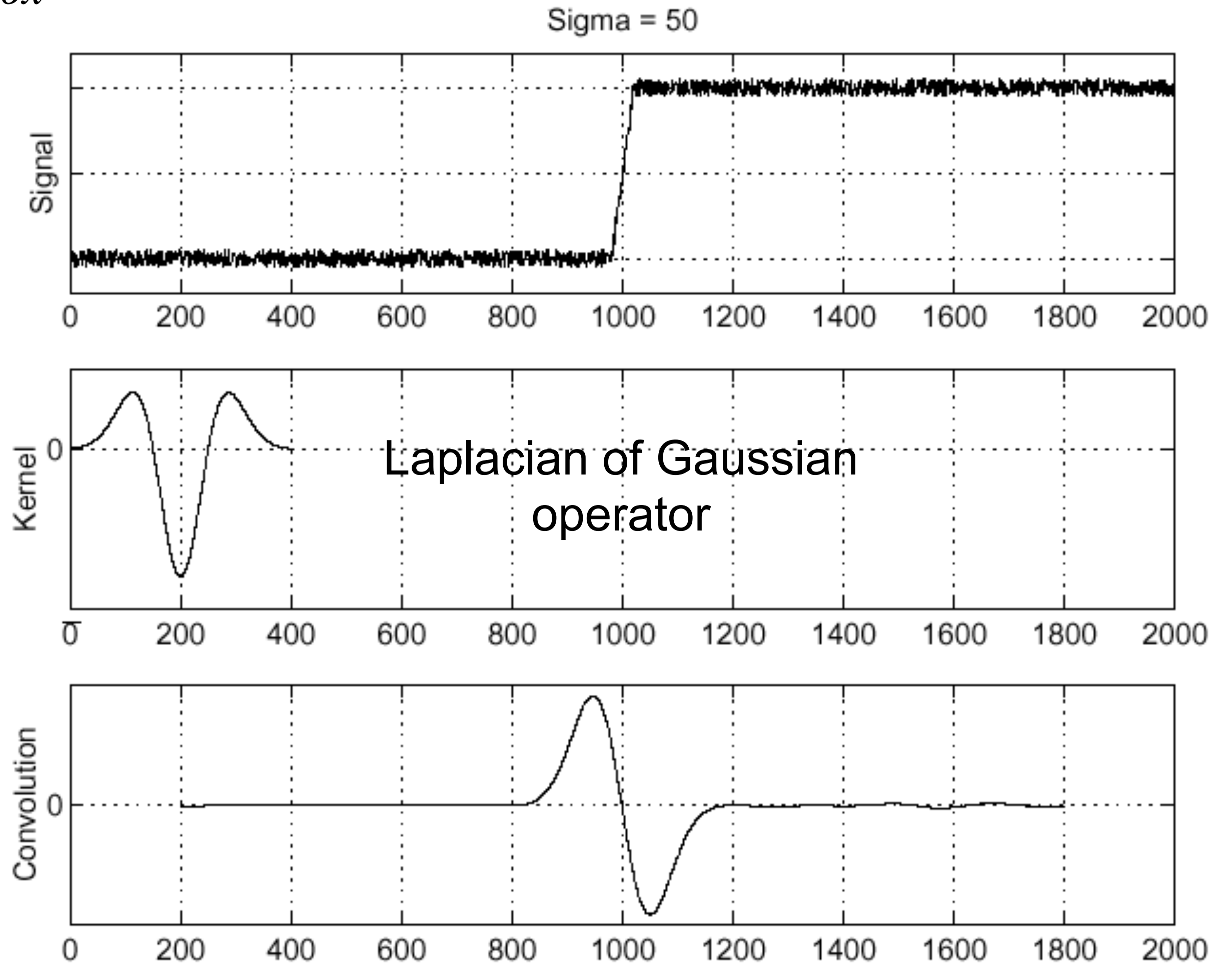
Laplacian of Gaussian

Consider $\frac{\partial^2}{\partial x^2}(h * f)$

f

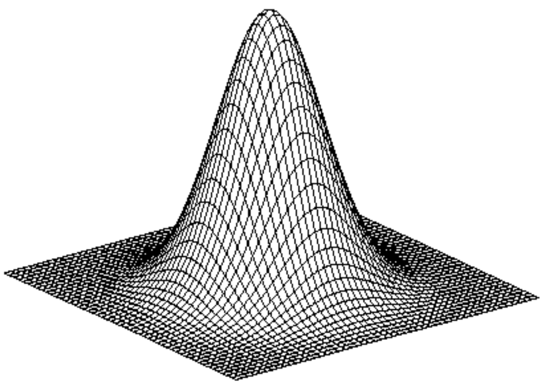
$\frac{\partial^2}{\partial x^2}h$

$(\frac{\partial^2}{\partial x^2}h) * f$



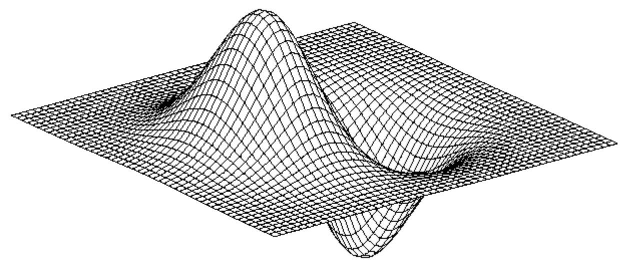
Where is the edge? Zero-crossings of bottom graph

2D edge detection filters



Gaussian

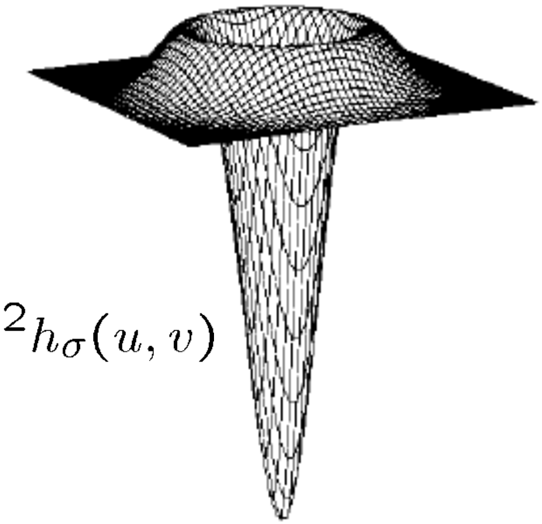
$$h_\sigma(u,v) = \frac{1}{2\pi\sigma^2} e^{-\frac{u^2+v^2}{2\sigma^2}}$$



derivative of Gaussian

$$\frac{\partial}{\partial x} h_\sigma(u,v)$$

Laplacian of Gaussian

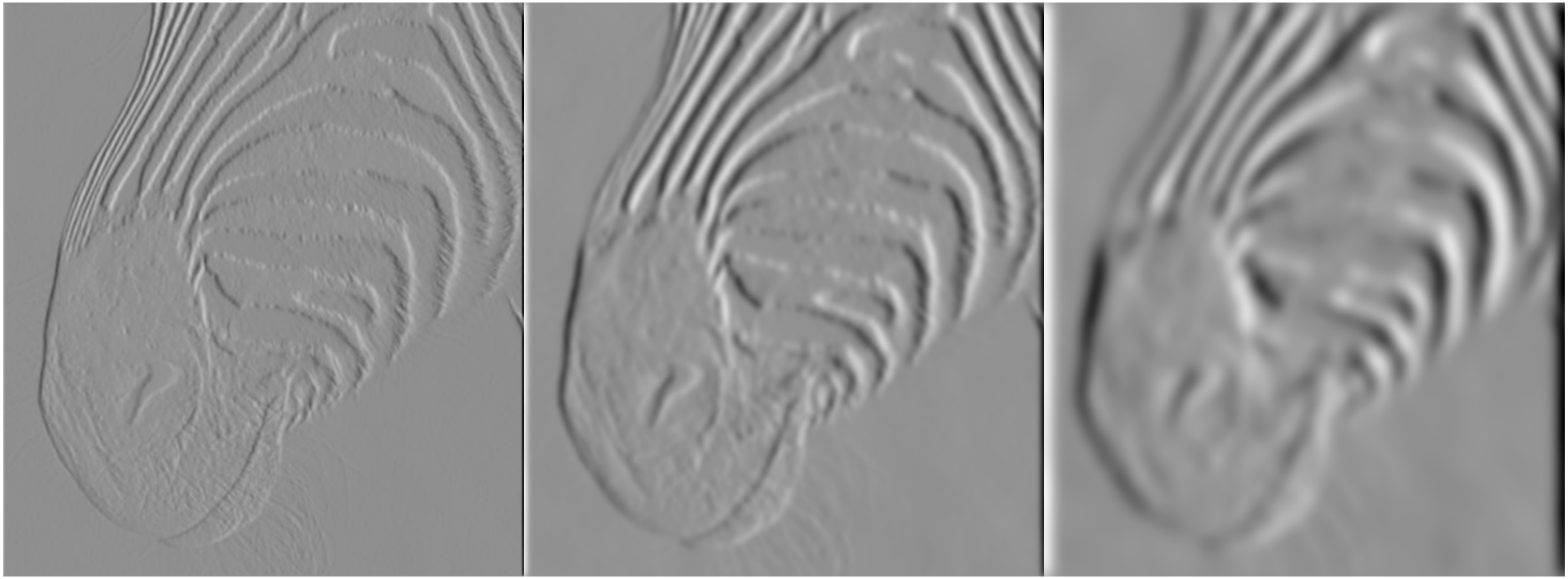


$$\nabla^2 h_\sigma(u,v)$$

∇^2 is the **Laplacian** operator:

$$\nabla^2 f = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}$$

Tradeoff between smoothing and localization



1 pixel 3 pixels 7 pixels

Smoothed derivative removes noise, but blurs edge. Also finds edges at different “scales”.

Source: D. Forsyth

Implementation issues

- The gradient magnitude is large along a thick “trail” or “ridge,” so how do we identify the actual edge points?
- How do we link the edge points to form curves?

Source: D. Forsyth