(Re)Introduction to Graphs and Some Algorithms

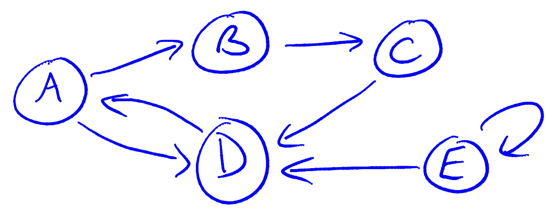
Graph Terminology (I)

- A graph is defined by a set of vertices V and a set of edges E.
- The edge set must work over the defined vertices in the vertex set.
- Many different types of relationships can be represented as graphs.
- Graphs (specifically edges) can be either directed (eg: driving on a street) or undirected (eg: walking on a street).
- If two vertices are connected by an edge, we say those vertices are adjacent to each other.
- Edges can have values associated with them, in which case we call the graph a weighted graph.

Graph Terminology (II)

- A <u>path</u> is a list of edges that are sequentially connected. The <u>length</u> of a path is the number of edges. We will say that vertex b is *reachable* from a if there is a path from <u>a to b</u>.
- A <u>cycle</u> is a path where the starting vertex is also the ending vertex.
- A Hamiltonian Path is a path that visits every <u>vertex</u> in a graph **exactly once**.
- An Eulerian Path is a path that visits every <u>edge</u> in a graph **exactly once**.

Graph Representation (Directed)



A → B → D
B → C
C → D
D → A
E → D → E

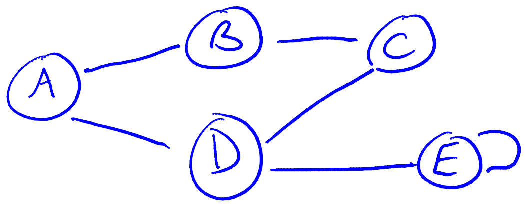
to

from	A	B	C	D	E
A	0	1	0	1	0
B	0	0	1	0	0
C	0	0	0	1	0
D	1	0	0	0	0
E	0	0	0	1	1

$V = \{A, B, C, D, E\}$

$E = \{(A,B), (A,D), (B,C), (C,D), (D,A), (E,D), (E,E)\}$

Graph Representation (Undirected)



A → B → D
B → A → C
C → B → D
D → A → C → E
E → D → E

to

from	A	B	C	D	E
A	0	1	0	1	0
B	1	0	1	0	0
C	0	1	0	1	0
D	1	0	1	0	1
E	0	0	0	1	1

redundant?

$V = \{A, B, C, D, E\}$

$E = \{(A,B), (A,D), (B,C), (C,D), (D,A), (E,D), (E,E)$
$(B,A), (D,A), (C,B), (D,C), (A,B), (D,E)\}$

A Proof on Graphs

Definitions: In a directed graph, the in-degree of a vertex is the number of edges going into it and the out-degree of a vertex is the number of edges coming out of it.

Theorem:

$$\sum_{v \in V} \text{in-degree}(v) = \sum_{v \in V} \text{out-degree}(v)$$

Proof will be by induction. Start with a base case of a graph with a single edge. For the inductive hypothesis, say that for any graph with an edge set of size **k** that the theorem holds. Then show that it holds for any graph with an edge set of size **k+1**.

Base Cases

A ⟲

$A \rightarrow B$

Inductive Hypothesis

For any graph with edge set of size k ($k \geq 1$)

$$\sum_{v \in V} \text{in-degree}(v) = \sum_{v \in V} \text{out-degree}(v)$$

Inductive Step

Show that for any graph G with edge set of size k+1 that $\sum_{v \in G.V} \text{in-degree}(v) = \sum_{v \in G.V} \text{out-degree}(v)$

Let H be a generic particular graph with k+1 edges.

Select an edge (call it e_1) and remove it from the edge set to create a new graph H'.

By our definitions,

$$\sum_{v \in H} \text{in-degree}(v) = \sum_{v \in H'} \text{in-degree}(v) + 1$$

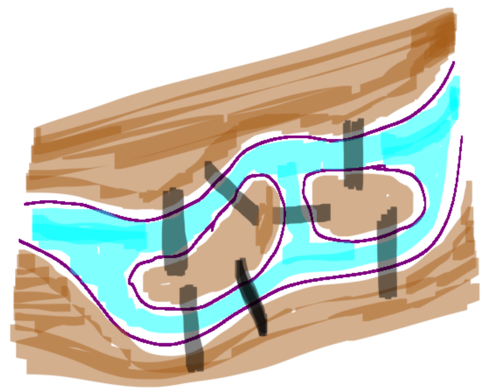
$$\sum_{v \in H} \text{out-degree}(v) = \sum_{v \in H'} \text{out-degree}(v) + 1$$

Can now ask:

$$\sum_{v \in H'} \text{in-degree}(v) + 1 \stackrel{?}{=} \sum_{v \in H'} \text{out-degree}(v) + 1$$

Classic Graph Problem

Königsberg Bridge Problem



Seven Bridges. How can you cross all seven without re-using any?

How can we solve Eulerian path?

If all we want is a yes/no answer, it's fairly easy.

If we also want to find the actual path if it exists, that becomes a much more involved question…

For one point, we need to think about algorithms that are able to traverse graphs. So, let's look at one…

Breadth-First Search

Given a graph, one way to have an algorithm try to visit every vertex in that graph is via a breadth-first search.

- Select a starting point.
- Visit all vertices that are "one jump" away from it.
- Visit all vertices that are "two jumps" away from it.
- etc.

What if the graph is directed?

If the graph is not connected, what ends up happening?

A simple problem that can be solved using this general technique is that of finding the shortest path between two vertices in an ***undirected*** and <u>*unweighted*</u> graph.

Shortest Path via BFS

Starting at vertex $s \in V$ generate an array of distances from s called dist[] such that $\forall v \in V$,

dist[v]=length of shortest path from s to v.

dist[s]=0

We will also create a predecessor array of the last vertex we were at before getting to the end of the path from s to v

$\forall v \in V$, pred[v]="one step back"

pred[s]=none

With just these two arrays, we will be able to reconstruct any shortest part request from s to some vertex.

- This is because any **sub-path** of the optimal path must also be an optimal **path between its own endpoints**.
- If it weren't, then we could have replaced it and gotten a shorter **overall** path.

Basic Pseudocode

Start at s.

For each neighbor **v** of s

- dist[**v**]=1
- pred[**v**]=s.

Move outwards from each neighbor you've seen and set the next "ripple" out as "+1" of the current distance, and set pred[] appropriately.

Need a way to make sure we don't end up in cycles!

Avoiding Cycles

We will assign a color to each vertex based on the following rules:

- white = not seen yet at all
- gray = seen but not processed yet
- black = processed

We will create a queue of gray vertices, and will never add any vertex to the queue more than once.

When we are done processing a vertex (ie: we have touched all its neighbors) we go back to the queue to get the next vertex to process.

More Detailed Pseudocode

```
BFS (Graph G, vertex s) {
int size = G.getVertexCount;
int dist = new int[size];
vertex pred = new int[size];
Queue Q= new Queue<vertex>;
Colors state = new Colors[size];
    for each v in G.V {
       state[v]=white; dist[v]=infinity; pred[v]=none;
    }
    state[s]=gray;  dist[s]=0;  pred[s]=none;
    Q.add(s);
    while (!Q.empty()){
       u=Q.remove();
       for each unvisited v in G.Adj(u) {
         state[v]=gray;
         dist[v]=dist[u]+1;
         pred[v]=u;
         Q.add(v);
       }
       state[u]=black;
    }
}
```

What's the runtime?

Each vertex gets enqueued at most one time, so each is processed at most one time.

– Write this up using a summation to represent the processing of all of the vertices…

Our runtime will be order:

|V| for all of the ***initializations***

The ***while*** loop's cost can be seen as the sum across all vertices ***u*** in V of:

- the degree(u) for work inside the ***for*** loop
- "+1" for the work outside of the ***for*** loop

We can split the summation into two simpler ones and if you work it through, the runtime is O(|V|+|E|).

What else does BFS give us?

It allows us to organize the entire graph as "ripples" away from a central point.

– This could be useful if we could restate other questions within this framework.

Our predecessor array could be used to create a tree rooted at source **s** of vertices that can be reached from **s**.

– This is often called a breadth-first tree.

– If we could phrase a problem as a traversal of this tree…

Depth-First Search

You could basically just change the Queue in the BFS code into a Stack.

You could also just write it out as a recursive algorithm.

This approach can also be used to determine what vertices are reachable in O(|E|+|V|) time.

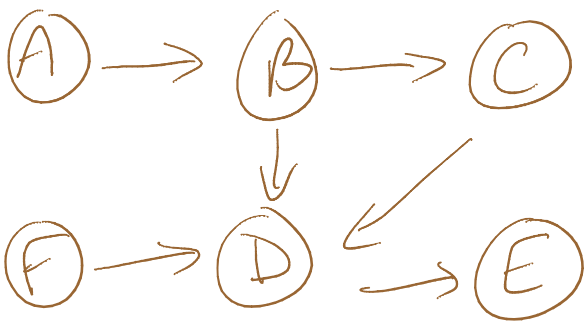
DFS on a Directed Graph with "Timing" Info

We can add more arrays and store information such as when (in terms of a continuously advancing ticker) each vertex is first visited (enter) and finally processed (exit).

Even in a connected graph, we might end up having to build a forest of trees to give every vertex a set of times.

- After doing a DFS from a given starting point, if there are vertices with no times, choose one of them, and continue.

Example Graph



Topological Sort of a Digraph

NOTE: This only works if there are no cycles, since if there are cycles there isn't the notion of a sorted order.

Imagine a graph as beads where the edges are strings of equal length connecting ordered pairs of beads.

You want to arrange the beads so that all edges point left-to-right.

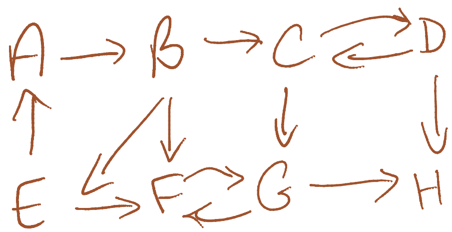
How can you use a DFS with "timing" info to accomplish this?

- Perform the DFS with timing and then "sort" by listing the nodes in reverse order based on the ***exit*** times.

Strongly Connected Components

We define "strongly connected" to mean that for every pair of vertices (u,v) in the component, there is a path from <u>u to v</u> and from <u>v to u</u>.

In the following graph, what are the strongly connected components?



Finding the SCCs

Step 1: Perform a DFS with "timing" on the graph G.

Step 2: Perform a DFS with "timing" on the graph G^T with the added restriction that when you have a choice of vertices, you choose the one with the largest finish time from Step 1's search.

Every time your algorithm hits a dead-end, you have finished one strongly connected component and are ready to start finding the next one.

Let's trace this on the graph from the previous slide…

Could you use a BFS or DFS to…

Detect whether a given graph has any cycles?

– Yes.

Determine whether every vertex is reachable from a particular vertex in a given graph?

– Yes.

Find the *longest* simple path through a graph between two vertices in an unweighted graph that might contain cycles?

– No!