Problem Set #10

CMSC 657
Instructor: Daniel Gottesman

Due on Gradescope, Thursday, Nov. 21, 2024, at 5:00 PM

Problem #1. Distance Measures and Entropies (30 pts.)

For this problem, $\rho \neq \sigma$ are density matrices. You don't need full proofs of the orderings, but give a brief justification.

a) (10 pts.) Put the following quantities in order, from least to greatest, or indicate where some are equal or where more information is needed to determine the relative size: $F(\rho, \sigma)$, $F(\rho \otimes |0\rangle\langle 0|, \sigma \otimes |0\rangle\langle 0|)$, $F(\rho \otimes |0\rangle\langle 0|, \sigma \otimes |1\rangle\langle 1|)$, $F(\rho \otimes \rho, \sigma \otimes \sigma)$, $F(\rho \otimes \rho, \sigma \otimes \rho)$. Here, F is the fidelity.

b) (10 pts.) Put the following quantities in order, from least to greatest, or indicate where some are equal or where more information is needed to determine the relative size: $D(\rho, \sigma)$, $D(\rho \otimes |0\rangle\langle 0|, \sigma \otimes |0\rangle\langle 0|)$, $D(\rho \otimes |0\rangle\langle 0|, \sigma \otimes |1\rangle\langle 1|)$, $D(\rho \otimes \rho, \sigma \otimes \sigma)$, $D(\rho \otimes \rho, \sigma \otimes \rho)$. Here, D is the trace distance.

c) (10 pts.) Put the following quantities in order, from least to greatest, or indicate where some are equal or where more information is needed to determine the relative size: $S(\rho)$, $S(\rho \otimes |0\rangle\langle 0|)$, $S(\rho \otimes \rho)$, $S(\rho \otimes \sigma)$, $S(\eta)$. Here, S is the Von Neumann entropy and η is a bipartite density matrix on $A \otimes B$ such that $\mathrm{Tr}_B\, \eta = \rho$ and $\mathrm{Tr}_A\, \eta = \sigma$. (It is always possible to find such an η but it might not be unique.)

Problem #2. Entanglement Concentration (30 pts.)

a) (10 pts.) Suppose Alice has N qubits. Consider the projectors

$$\Pi_n = \sum_{x \text{ s.t. } \mathrm{wt}(x) = n} |x\rangle\langle x|, \quad (n = 0, \ldots, N) \tag{1}$$

where the Hamming weight $\mathrm{wt}(x)$ is the number of 1s in the bit string x. Give a circuit that will perform the projective measurement $\{\Pi_n\}$. Note that Alice is not supposed to measure any refinement of this measurement, so superpositions of different x with the same Hamming weight will remain coherent.

b) (10 pts.) Suppose Alice and Bob share N copies of the state $|\psi\rangle = \alpha|00\rangle + \beta|11\rangle$ — i.e., they have the state $|\psi\rangle^{\otimes N}$. Show that if Alice and Bob each perform the measurement from part a) they will get the same outcome n, and show that the remaining state is equivalent, up to local unitary operations, to the maximally entangled state

$$\sum_{j=0}^{\binom{N}{n}-1} |j\rangle_A |j\rangle_B. \tag{2}$$

c) (10 pts.) Let $\rho_A = \mathrm{Tr}_B\, |\psi\rangle\langle\psi|$. Show that for large N, the most likely value of n in the measurement from part b) satisfies $\log \binom{N}{n} = NS(\rho_A)$. In fact, for large N, the distribution will be very highly peaked around this value of n. In other words, using LOCC ("local operations and classical communication"), Alice and Bob can convert N partially entangled states into about $NS(\rho_A)$ maximally entangled states with high probability.