CMSC 657: Introduction to Quantum Information Processing
Lecture 4

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1 CP Maps and POVMs

1.1 CP Maps

When we allow mixed states, unitary evolution does not describe all the possible operations. A unitary map will always take a pure state to a pure state, but we should also allow operations that include some randomness, or ones that discard a subsystem. We could also include operations that add an extra subsystem in some fixed state (not previously correlated or entangled with the input state; otherwise it is not a uniquely-defined map).

The most general operation that can be performed on a quantum state is a *CPTP map* $\mathcal{C}$, which has the following properties:

1. Linear (because quantum mechanics is linear): $\mathcal{C}(p_1\rho_1 + p_2\rho_2) = p_1\mathcal{C}(\rho_1) + p_2\mathcal{C}(\rho_2)$. (Note that p_1 and p_2 are real; we are adding density matrices, which means taking a classical probabilistic mixture rather than a quantum superposition.)

2. Trace preserving (to preserve total probability = 1): $\mathrm{Tr}\,\mathcal{C}(\rho) = \mathrm{Tr}(\rho)$.

3. Positive (to map density matrices to density matrices): If $\rho \geq 0$, then $\mathcal{C}(\rho) \geq 0$.

4. Completely positive: Given density matrix ρ_A on system A, for any state ρ_{AB} on $A \otimes B$ such that $\mathrm{Tr}_B\,\rho_{AB} = \rho_A$, we have $\mathcal{C} \otimes I(\rho_{AB})$ is a positive matrix as well.

Complete positivity is required so that if we perform the map on part of an entangled state, the resulting matrix is still a valid quantum state. (An example of a map which satisfies the first three conditions but not complete positivity is the transpose.)

A CPTP map is also called a *quantum operation* or *quantum channel*.

Theorem 1 (Kraus decomposition)**.** *A CPTP map can always be written in terms of Kraus operators:*

$$\mathcal{C}(\rho) = \sum_k A_k \rho A_k^\dagger, \tag{1}$$

with $\sum_k A_k^\dagger A_k = I$. *Any map of this form is a CPTP map.*

Let us see what this representation does on a pure state $\rho = |\psi\rangle\langle\psi|$:

$$\mathcal{C}(\rho) = \sum_k A_k |\psi\rangle\langle\psi| A_k^\dagger. \tag{2}$$

The CPTP map is actually a unitary in the case where there is only a single Kraus operator.

I am not going to prove this theorem, but this is a very helpful way of representing CPTP maps. It lets us think of the map as applying one of the A_ks with different probability: $A_k|\psi\rangle\langle\psi|A_k^\dagger$ is an unnormalized pure state $A_k|\psi\rangle$. This is not quite right, however, because the probability of A_k can depend on the state we are applying it to:

$$\|A_k|\psi\rangle\|^2 = \langle\psi|A_k^\dagger A_k|\psi\rangle. \tag{3}$$

A useful example where the interpretation is correct is if the CPTP is a mixture of unitaries. In this case, $A_k = \sqrt{p_k}U_k$, with p_k a probability and U_k unitary. Then

$$\mathcal{C}(\rho) = \sum_k p_k U_k \rho U_k^\dagger, \tag{4}$$

which corresponds exactly to having the unitary U_k with probability p_k.

The condition $\sum_k A_k^\dagger A_k = I$ is needed to preserve the trace and make the probabilities sum to 1. If we don't insist on the map being trace preserving, then it is loosened to $\sum_k A_k^\dagger A_k \leq I$, which is needed for complete positivity.

Note that the Kraus representation is not unique. Just as with the density matrix having more than one ensemble, there is more than one set of Kraus operators that gives the same CPTP map.

1.2 POVMs

POVMs are the most general possible measurements. POVM stands for "positive operator-valued measure." A POVM is a set of positive semi-definite operators $M_a \geq 0$ (called *effects*), such that $\sum_a M_a = I$. If we do the POVM $\{M_a\}$ on the density matrix ρ, the probability of outcome a is

$$p_a = \mathrm{Tr}(M_a \rho). \tag{5}$$

The normalization of M_a ensures that the probabilities sum to 1. A simple example of a POVM is a projective measurement where each M_a is an orthogonal projector. POVMs also include things like "with probability 1/2, measure in the Z basis and with probability 1/2, measure in the X basis," plus more complicated things.

People don't usually talk about the residual state after a POVM. Typically, you only use the POVM representation if you care about the probabilities of the outcomes but not the state that is left over.

1.3 Purification

Density matrices, CPTP maps, and POVMs give us a way of manipulating mixed quantum states or states that are entangled with another faraway subsystem, and we don't have to be concerned about what is happening with the other system. However, it turns out that we can always view a mixed state or evolution of a mixed state as part of a larger pure state with unitary dynamics.

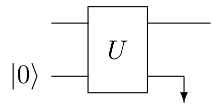
For starters, we can always write any mixed state as the partial trace of a pure state. Given density matrix ρ_A on system A, let us diagonalize $\rho = \sum_i a_i |i\rangle\langle i|$ for appropriate choice of basis $\{|i\rangle\}$. Then $a_i \geq 0$ and $\sum_i a_i = 1$ because ρ is positive semi-definite and has trace 1. We can define a second system R, usually called the *reference system*, which in this case will have the same dimension as A. Let

$$|\psi\rangle_{AR} = \sum_i \sqrt{a_i}|i\rangle_A \otimes |i\rangle_R. \tag{6}$$

You can see that $\rho_A = \mathrm{Tr}_R\,|\psi\rangle_{AR}\langle\psi|_{AR}$. $|\psi\rangle$ is called the *purification* of ρ.

Similarly, any CPTP map can also be written as the partial trace of a unitary on the system plus an ancilla:

$$\mathcal{C}(\rho) = \mathrm{Tr}_B\, U(\rho \otimes |0\rangle_B\langle 0|_B)U^\dagger. \tag{7}$$



This is known as the *Stinespring Dilation Theorem*.

Finally, an arbitrary POVM can be written as a projective measurement on a larger Hilbert space. This means that states, quantum operations, and measurements can all be purified. If we like, we can think of all these things as unitary operations and projective measurements acting on pure states by just imagining a sufficiently large Hilbert space to purify everything. This viewpoint is sometimes called *The Church of the Larger Hilbert Space*. It can be a very convenient way of thinking about things. Sometimes density matrices and mixed states are an easier way of analyzing a problem and sometimes pure states are easier. Purification lets you switch back and forth whenever you like.

1.4 Bloch Sphere

The density matrix for a single qubit can be represented with a nice geometric picture called the Bloch Sphere. The pure states are associated with points on the surface of a unit sphere. The north pole is the state $|0\rangle$ and the south pole is the state $|1\rangle$. The "east" pole is the state $|0\rangle + |1\rangle$ and the "west" pole is the state $|0\rangle - |1\rangle$. The "front" and "back" poles are the states $|0\rangle \pm i|1\rangle$.

The arbitrary density matrix $\rho = \frac{1}{2}(I + \mathbf{r} \cdot \sigma)$, where $\mathbf{r}$ is the 3-D vector indicating the point and $\sigma = (X, Y, Z)$ is a vector of matrices:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

X, Y, and Z are the three *Pauli matrices*, and we will see them multiple times later in the course.

The Bloch Sphere gives you a good way of visualizing the state of a single qubit as a vector in 3-dimensional space, but unfortunately, this doesn't generalize to higher-dimensional Hilbert spaces. You can still represent a qubit as a point in a convex region, but the other properties of the Bloch Sphere fail to hold: It is not a sphere and the surface of the region is not all pure states.