CMSC 657: Introduction to Quantum Information Processing
Lecture 2

Instructor: Daniel Gottesman

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1 Tensor Product, Measurements, and Density Matrices

1.1 Unitaries and Tensor Product

How do unitaries work with the tensor product? Let us look at an example with two quantum gates we will discuss later,

$$U_1 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}. \tag{1}$$

Let us calculate $U_1 \otimes U_2$.

Recall that the basis for the tensor product of two qubits is $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$ (in order). (Note that $|00\rangle$, $|0\rangle|0\rangle$, and $|0\rangle \otimes |0\rangle$ are three ways of representing the same state.). U_2 is acting on the second tensor factor, so it addresses the lower-order bit. If we have the input state $|00\rangle$, then we use the entries in the left column ($|0\rangle\langle 0|$ and $|1\rangle\langle 0|$) of U_2, which are 1 and 0. Thus $I \otimes U_2$ applied to $|00\rangle$ gives us just $|00\rangle$ again.

U_1 addresses the higher-order bit, so on $|00\rangle$, we again use the left column, giving us $1/\sqrt{2}(|00\rangle + |10\rangle)$. Putting this together with the action of U_2, we find that $U_1 \otimes U_2|00\rangle = 1/\sqrt{2}(|00\rangle + |10\rangle)$. This gives us the left column of the matrix for $U_1 \otimes U_2$: $1/\sqrt{2}, 0, 1/\sqrt{2}, 0$. Following the same procedure for the other columns tells us the full matrix:

$$U_1 \otimes U_2 = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & e^{i\pi/4}/\sqrt{2} & 0 & e^{i\pi/4}/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & e^{i\pi/4}/\sqrt{2} & 0 & -e^{i\pi/4}/\sqrt{2} \end{pmatrix} \tag{2}$$

We can short-circuit this calculation by promoting each of the entries of U_1 into an identity matrix (which gives us the matrix for $U_1 \otimes I$) and then multiplying each by U_2:

$$U_1 \otimes U_2 = \begin{pmatrix} 1/\sqrt{2}\, U_2 & 1/\sqrt{2}\, U_2 \\ 1/\sqrt{2}\, U_2 & -1/\sqrt{2}\, U_2 \end{pmatrix} \tag{3}$$

You can also calculate this using bra-ket notation, and you will get a chance to do that on the problem set.

1.2 Measurements

We need a way of converting quantum states to classical ones, i.e., to get information out from a quantum system. This is provided by measurement. The simplest kind of measurement is a *complete Von Neumann measurement*, where we pick an orthonormal basis $\{|i\rangle\}$ and get one of those states as the outcome. Different basis states have different probabilities, depending on the state. In particular, the probability of outcome i when the state $|\psi\rangle$ is measured in the basis $\{|i\rangle\}$ is

$$\text{Prob}(i) = |\langle\psi|i\rangle|^2. \tag{4}$$

This is known as *Born's rule*. Thus, if we expand the state $|\psi\rangle$ in the basis as

$$|\psi\rangle = \sum_i \alpha_i |i\rangle, \tag{5}$$

then

$$\text{Prob}(i) = |\sum_j \alpha_j^* \langle j|i\rangle|^2 = |\alpha_i|^2. \tag{6}$$

Note that this is always a real number between 0 and 1, and because the state is normalized,

$$\sum_i \text{Prob}(i) = \sum_i |\alpha_i|^2 = 1. \tag{7}$$

Thus this distribution does indeed form a probability distribution. Now you can see why we wanted the states to be normalized — so that the probability interpretation made sense. Moreover, notice that the projective phase freedom, a global phase of $e^{i\phi}$, has no affect on the probabilities since it changes $\alpha_i \to \alpha_i e^{i\phi}$, which has the same absolute value. It also tells us why we need unitary evolution to preserve the inner product, because that also preserves probabilities. Otherwise, we would have some state that changed norm and became non-normalized, which would ruin the interpretation of measurement as providing probabilities for different outcomes.

We see that if we measure a basis state in the same basis, we get a deterministic outcome, and a state which has most of its amplitude on one basis state will usually give that measurement outcome, but occasionally something else. Like a probability distribution, a quantum state can weight different basis states differently, but the fact that the amplitudes can be complex numbers rather than positive real numbers makes a big difference, allowing for interference, a genuine quantum effect that we will discuss later when we do quantum circuits.

The possibility of interference means that the *relative* phase between parts of a superposition *does* matter. That is, the states $|+\rangle = |0\rangle + |1\rangle$ and $|-\rangle = |0\rangle - |1\rangle$ are very different states. (In fact, their inner product is 0 – they are orthogonal states, and a measurement in the $|\pm\rangle$ basis can tell us which we have.) The difference matters a lot, in fact, and is arguably the source of all the difference between classical and quantum computation. But the state $|1\rangle - |0\rangle = -|-\rangle$ is the same physical state as $|-\rangle$.

The other thing about quantum measurements is that they generally change the state of the system. Regardless of the state $|\psi\rangle$ before the measurement, if we make a complete measurement and get the outcome $|i\rangle$, the state after the measurement is always $|i\rangle$. We sometimes say that the state has *collapsed* to a basis state. Because the state is now a basis state, if you repeat the measurement in the same basis, you will get the same result (in the absence of error). But note that if you do a measurement in a different basis, you need not get a deterministic outcome.

Also, note that any information about the other branches of the superposition is also gone. Measurement in the $|0\rangle/|1\rangle$ basis has erased the distinction between $|+\rangle$ and $|-\rangle$: They both have probability 1/2 (when normalized) of giving measurement outcome 0 and outcome 1.

We can also have a *partial Von Neuman measurement*, where we don't resolve all basis vectors. In particular, a partial Von Neumann measurement is a set of projectors Π_i (which means that $\Pi_i^2 = \Pi_i$) on orthogonal subspaces (which means $\Pi_i \Pi_j = 0$ when $i \neq j$). We should also have that the whole Hilbert space is covered by projectors

$$\sum_i \Pi_i = I. \tag{8}$$

Then the probability of outcome i when we make this measurement on state $|\psi\rangle$ is

$$\text{Prob}(i) = \langle\psi|\Pi_i|\psi\rangle. \tag{9}$$

The probabilities sum to 1 because the projectors do. The special case of a complete measurement is when $\Pi_i = |i\rangle\langle i|$. Note that this satisfies the conditions for a partial von Neuman measurement and that the probability is consistent with the complete measurement:

$$\langle\psi|(|i\rangle\langle i|)|\psi\rangle = \langle\psi|i\rangle\langle i|\psi\rangle = |\langle\psi|i\rangle|^2. \tag{10}$$

This is true because $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$.

When the partial measurement is not a complete measurement, then we can write the projectors as sums of basis vectors:

$$\Pi_i = \sum_{k \in S_i} |i_k\rangle\langle i_k|. \tag{11}$$

Then

$$\text{Prob}(i) = \sum_{k \in S_i} |\langle\psi|i_k\rangle|^2, \tag{12}$$

which is the sum of the probabilities we would get if we made a complete measurement in the basis $|i_k\rangle$. However, in the case of a partial measurement, the state does not completely collapse to a basis state. Instead, we get the minimal change that is consistent with a state that definitely satisfies the projector Π_i. Conditioned on the outcome i, the residual state after measuring the initial state $|\psi\rangle$ is

$$\Pi_i|\psi\rangle / \sqrt{\langle\psi|\Pi_i|\psi\rangle}. \tag{13}$$

Here, $\sqrt{\langle\psi|\Pi_i|\psi\rangle}$ is just the normalization of $\Pi_i|\psi\rangle$, so the residual state is the outcome of the projector normalized.

One other important special case of the partial measurement is when the Hilbert space is a tensor product $A \otimes B$ and the measurement is a complete measurement on one tensor factor, let's say A. Then the projectors are $\Pi_i = |i\rangle_A\langle i|_A \otimes I_B$. Here's an example of how this works in practice:

Suppose we start with the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle) \tag{14}$$

and we measure the first qubit in the 0/1 basis. Then the 0 outcome has projector $\Pi_0 = |0\rangle\langle 0| \otimes I$, and

$$\begin{aligned}
\text{Prob}(0) =& \frac{1}{3}(\langle 00| + \langle 01| + \langle 10|)(|0\rangle\langle 0| \otimes I)(|00\rangle + |01\rangle + |10\rangle) && (15)\\
=& \frac{1}{3}(\langle 00| + \langle 01| + \langle 10|)(|00\rangle + |01\rangle) && (16)\\
=& 2/3. && (17)
\end{aligned}$$

We have $\Pi_0|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle)$, so the post-measurement state conditioned on a 0 outcome is $\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$.

The 1 outcome has probability 1/3 and the post-measurement outcome is $|10\rangle$, the only term in the state that is consistent with getting a 1 on the first qubit.

A Von Neumann measurement is also called a *projective measurement*, and a complete projective measurement is called a *rank* 1 measurement because all of the projectors used in it are matrices with rank 1.

1.3 Density Matrices

Since quantum states have some similarity to probability distributions and can actually be converted into probability distributions by measurements, it is helpful to have a formalism that talks directly about both. That formalism is the density matrix notation.

A density matrix can be used to describe both the *pure states* that we have been discussing so far and *mixed states*, which are probabilitistic mixtures (an *ensemble*) of pure states. Of course, a pure state is also a probabilistic mixture of sorts, with 100% of a single state. In general, given a mixed state which has probability p_i of being state $|\psi_i\rangle$, we write the density matrix as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_{ab} \rho_{ab} |a\rangle\langle b|. \tag{18}$$

The last form is the explicit matrix representation and we can get it from the middle form by expanding each $|\psi_i\rangle$ in the basis:

$$|\psi_i\rangle = \sum_a c_{i,a}|a\rangle. \tag{19}$$

Then

$$\rho = \sum_{i,a,b} p_i c_{i,a} c_{i,b}^* |a\rangle\langle b|, \tag{20}$$

so $\rho_{ab} = \sum_i p_i c_{i,a} c_{i,b}^*$.

Note that mixed quantum states are a generalization of the notion of a classical probability distribution. If all the states $|\psi_i\rangle = |i\rangle$ are basis states in a particular standard basis, then the density matrix is diagonal: $c_{i,a} = \delta_{i,a}$, and $\rho = \sum_i p_i |i\rangle\langle i|$. The diagonal elements are the vector of probabilities.

Also, note that superposition is distinct from a classical mixture. For instance, consider the mixture

$$\rho_m = |\alpha|^2 |0\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|. \tag{21}$$

This has state 0 with probability $|\alpha|^2$ and 1 with probability $|\beta|^2$. Compare to the superposition

$$|\psi_s\rangle = \alpha|0\rangle + \beta|1\rangle. \tag{22}$$

This is a natural comparison to make because if we measure $|\psi_s\rangle$ in the standard $|0\rangle$, $|1\rangle$ basis, we again get 0 with probability $|\alpha|^2$ and 1 with probability $|\beta|^2$.

Let us calculate the density matrix ρ_s for $|\psi_s\rangle$. We find

$$\begin{aligned} \rho_s =& |\psi_s\rangle\langle\psi_s| && (23)\\ =& (\alpha|0\rangle + \beta|1\rangle)(\alpha^*\langle 0| + \beta^*\langle 1|) && (24)\\ & -|\alpha|^2|0\rangle\langle 0| + \alpha\beta^*|0\rangle\langle 1| + \beta\alpha^*|1\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|. && (25) \end{aligned}$$

The diagonal terms are the same as ρ_m, but there are also off-diagonal terms $\alpha\beta^*|0\rangle\langle 1| + \beta\alpha^*|1\rangle\langle 0|$. The density matrix is different.