CMSC 631: Midterm Exam (Fall 2022)

1 Question 1 (10 points)

For each of the terms below, write their type or "ill-typed" if an expression doesn't have a type.

(a) [`nat`]

(b) [`bool`; 42]

(c) `fun (X:Prop) (x:X) ⇒ conj x x`

(d) `fun x ⇒ x x`

(e) [`fun X ⇒ X * X`]

2 Question 2 (20 points)

For each of the types below, write a (nontrivial if possible) Coq expression that has that type or write "empty" if there are no such expressions.

(a) `nat → list bool`

(b) `forall (X : Prop), option bool`

(c) `forall (X Y : Type), Y`

(d) `forall (X Y : Type), X → X`

(e) `Prop * nat`

(f) `option (nat * bool -> bool)`

(g) `forall (X : Prop), list X`

(h) `forall (P Q : Prop), P ∨ Q`

(i) `forall (P Q : Prop), P → Q → P ∨ Q`

3 Question 3 (20 points)

A *binary tree* with natural numbers as labels is either *empty* or a *node* that contains some natural number along with two binary trees as children. Formally, binary trees can be defined as follows in Coq:

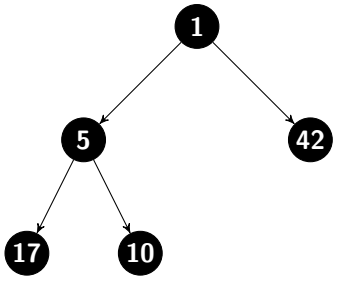
```
Inductive tree :=
| Empty : tree
| Node : nat -> tree -> tree -> tree.
```

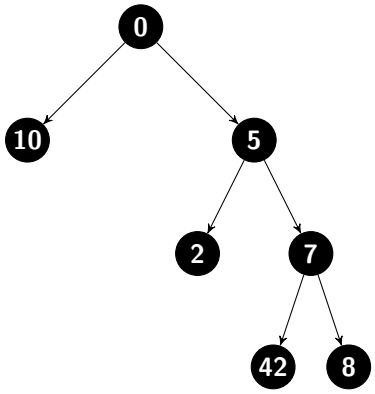
For example, the following definitions:

```
Definition ex_tree_1 : tree :=
  Node 1
       (Node 5
             (Node 17  Empty Empty)
             (Node 10 Empty Empty))
       (Node 42 Empty Empty).
```

```
Definition ex_tree_2 : tree :=
  Node 0
       (Node 10 Empty Empty)
       (Node 5
             (Node 2 Empty Empty)
             (Node 7
                   (Node 42 Empty Empty)
                   (Node 8  Empty Empty))).
```

represent the trees:





(a) The *height* of a binary tree is the maximum number of edges from the root of the tree to a `Leaf` node.

For example, the height of `ex_tree_1` is 2, while the height of `ex_tree_2` is 3.

Fill in the following function that calculates the height of a binary tree. It should be the case that the following `Example`s are provable by `reflexivity`.

```
(* Trivial tree: *)
Example height_leaf : height Leaf = 0.
(* Examples *)
Example height1 : height ex_tree_1 = 2.
Example height2 : height ex_tree_2 = 3.
```

```
Fixpoint height (t: tree) : nat :=
```

(b) A binary tree is *balanced* if for every node of the tree, the height of its left and right subtrees differ at most by one. For example, `ex_tree_1` is balanced, but `ex_tree_2` is not.

Write a predicate that checks whether a tree is balanced. You can use `height` for this purpose. The following `Example`s should be provable by `reflexivity`.

```
(* Trivial tree: *)
Example balanced_leaf : balanced Leaf = true.
(* Examples *)
Example balanced1 : balanced ex_tree_1 = true.
Example balanced2 : balanced ex_tree_2 = false.

Fixpoint balanced (t : tree) : bool :=
```

(c) Now, write an inductive definition that captures the notion of a balanced tree with height either `n` or `n-1`. You should *not* use `height` in your definition, and the following `Example`s should be provable:

```
(* Trivial tree is balanced at both 0 and 1: *)
Example bal_leaf0 : bal 0 Leaf.
Example bal_leaf1 : bal 1 Leaf.
(* Examples *)
Example bal1 : bal 2 ex_tree_1.
Example bal2 : forall n, ~ (bal n ex_tree_2).

Inductive bal : nat -> tree -> Prop :=
```

4 Question 4 (10 points)

Consider the following IMP program:

```
Y := 0;
Z := X;
WHILE Z > 0 DO
  Z := Z - 1;
  Y := Y + X
DONE
```

Fill in the annotations in the following program to show that the Hoare triple given by the outermost pre- and post- conditions is valid. Be completely precise and pedantic in the way you apply Hoare rules—write assertions in *exactly* the form given by the rules rather than just logically equivalent ones. The provided blanks have been constructed so that if you work backwards from the end of the program you should only need to use the rule of consequence in the places indicated with `->>`. These implication steps can (silently) rely on all the usual rules of **natural** number arithmetic.

```
{{ X = m }} ->>
{{                                                  }}
  Y := 0;
{{                                                  }}
  Z := X;
{{                                                  }}
  WHILE Z > 0 DO
{{                                                  }} ->>
{{                                                  }}
    Z := Z - 1;
{{                                                  }}
    Y := Y + X
{{                                                  }}
DONE
{{                                                  }} ->>
{{ Y = m * m }}
```

5 Question 5 (20 points)

In the rest of this problem, we will add an `assert` command to IMP: it takes a boolean expression as an argument; if it evaluates to `true`, then the command behaves like a `skip`; if not, then the command does not step. Here is the syntax of this extension:

```
Inductive com : Type :=
  | CSkip
  | CAss (x : string) (a : aexp)
  | CSeq (c1 c2 : com)
  | CIf (b : bexp) (c1 c2 : com)
  | CWhile (b : bexp) (c : com)
  | CAssert (b : bexp) (* ← new *).
```

As notation, instead of `(CAssert b)` we will write

```
assert b
```

This command is equivalent to a `skip`, if the boolean holds, and should not evaluate to any state otherwise.

(a) Write a Hoare rule for this new construct. It should, for example, be strong enough to prove the following triple:

```
{{ X < Y }}
  assert (X = 0)
{{ 0 < Y }}
```

(b) Extend the evaluation relation for IMP to account for this new construct.

```
Reserved Notation
         "st '=[' c ']=>' st'"
         (at level 40, c custom com at level 99,
          st constr, st' constr at next level).

Inductive ceval : com -> state -> state -> Prop :=
  | E_Skip : forall st,
      st =[ skip ]=> st
  | E_Asgn  : forall st a n x,
      aeval st a = n ->
      st =[ x := a ]=> (x !-> n ; st)
  | E_Seq : forall c1 c2 st st' st'',
      st  =[ c1 ]=> st'  ->
      st' =[ c2 ]=> st'' ->
      st  =[ c1 ; c2 ]=> st''
  | E_IfTrue : forall st st' b c1 c2,
      beval st b = true ->
      st =[ c1 ]=> st' ->
      st =[ if b then c1 else c2 end]=> st'
  | E_IfFalse : forall st st' b c1 c2,
      beval st b = false ->
      st =[ c2 ]=> st' ->
      st =[ if b then c1 else c2 end]=> st'
  | E_WhileFalse : forall b st c,
      beval st b = false ->
      st =[ while b do c end ]=> st
  | E_WhileTrue : forall st st' st'' b c,
      beval st b = true ->
      st  =[ c ]=> st' ->
      st' =[ while b do c end ]=> st'' ->
      st  =[ while b do c end ]=> st''

  (* FILL IN HERE *)









  where "st =[ c ]=> st'" := (ceval c st st').
```

Library Reference

A Logic

```
Inductive and (X Y : Prop) : Prop :=
  conj : X → Y → and X Y.

Inductive or (X Y : Prop) : Prop :=
 | or_introl : X → or X Y
 | or_intror : Y → or X Y.

Arguments conj {X Y}.
Arguments or_introl {X Y}.
Arguments or_intror {X Y}.

Notation "A ∧ B" := (and A B).
Notation "A ∨ B" := (or A B).

Definition iff (A B : Prop) := (A → B) ∧ (B → A).
Notation "A ↔ B" := (iff A B) (at level 95).
```

B Booleans

```
Inductive bool : Type :=
  | true
  | false.

Definition negb (b:bool) : bool :=
  match b with
  | true ⇒ false
  | false ⇒ true
  end.

Definition andb (b1 b2: bool) : bool :=
  match b1 with
  | true ⇒ b2
  | false ⇒ false
  end.

Definition orb (b1 b2:bool) : bool :=
  match b1 with
  | true ⇒ true
  | false ⇒ b2
  end.
```

C Numbers

```
Inductive nat : Type :=
  | O
  | S (n : nat).

Fixpoint plus (n : nat) (m : nat) : nat :=
  match n with
    | O ⇒ m
    | S n' ⇒ S (plus n' m)
  end.
```

```
Fixpoint minus (n m:nat) : nat :=
  match n, m with
  | O   , _    ⇒ O
  | S _ , O    ⇒ n
  | S n', S m' ⇒ minus n' m'
  end.

Fixpoint mult (n m : nat) : nat :=
  match n with
    | O ⇒ O
    | S n' ⇒ plus m (mult n' m)
  end.

Notation "x + y" := (plus x y) (at level 50, left associativity).
Notation "x - y" := (minus x y) (at level 50, left associativity).
Notation "x * y" := (mult x y) (at level 40, left associativity).

Fixpoint eqb (n m : nat) : bool :=
  match n with
  | O ⇒ match m with
         | O ⇒ true
         | S m' ⇒ false
         end
  | S n' ⇒ match m with
            | O ⇒ false
            | S m' ⇒ eqb n' m'
            end
  end.

Fixpoint leb (n m : nat) : bool :=
  match n with
  | O ⇒ true
  | S n' ⇒
      match m with
      | O ⇒ false
      | S m' ⇒ leb n' m'
      end
  end.

Notation "x =? y" := (eqb x y) (at level 70).
Notation "x ≤? y" := (leb x y) (at level 70).

Inductive le : nat → nat → Prop :=
  | le_n n : le n n
  | le_S n m : le n m → le n (S m).

Notation "m ≤ n" := (le m n).
```

D Lists

```
Inductive list (X:Type) : Type :=
  | nil
  | cons (x : X) (l : list X).

Arguments nil {X}.
Arguments cons {X} _ _.

Notation "x :: y" := (cons x y) (at level 60, right associativity).
Notation "[ ]" := nil.
Notation "[ x ; .. ; y ]" := (cons x .. (cons y []) ..).
Notation "x ++ y" := (app x y)  (at level 60, right associativity).

Fixpoint map {X Y: Type} (f : X → Y) (l : list X) : (list Y) :=
  match l with
  | []     ⇒ []
  | h :: t ⇒ (f h) :: (map f t)
  end.

Fixpoint filter {X : Type} (test : X → bool) (l : list X)
               : (list X) :=
  match l with
  | []     ⇒ []
  | h :: t ⇒ if test h then h :: (filter test t)
                       else       filter test t
  end.

Fixpoint fold {X Y} (f : X → Y → Y) (l : list X) (b : Y) : Y :=
  match l with
  | nil ⇒ b
  | h :: t ⇒ f h (fold f t b)
  end.
```

E Strings

We won't define strings from scratch here. Assume `eqb_string` has the type given below, and anything within quotes is a string.

```
Parameter eqb_string : string → string → bool.
```

F Maps

```
Definition total_map (A:Type) := string → A.

Definition t_empty {A:Type} (v : A) : total_map A :=
  (fun _ ⇒ v).

Definition t_update {A:Type} (m : total_map A) (x : string) (v : A) :=
  fun x' ⇒ if eqb_string x x' then v else m x'.

Notation "{ --> d }" := (t_empty d) (at level 0).
Notation "m '&' { a --> x }" := (t_update m a x) (at level 20).
```

G Imp

```
Inductive aexp : Type :=
  | ANum (n : nat)
  | AId (x :  string)
  | APlus (a1 a2 : aexp)
  | AMinus (a1 a2 : aexp)
  | AMult (a1 a2 : aexp).

Inductive bexp : Type :=
  | BTrue
  | BFalse
  | BEq (a1 a2 : aexp)
  | BLe (a1 a2 : aexp)
  | BNot (b : bexp)
  | BAnd (b1 b2 : bexp).

Inductive com : Type :=
  | CSkip
  | CAss (x : string) (a : aexp)
  | CSeq (c1 c2 : com)
  | CIf (b : bexp) (c1 c2 : com)
  | CWhile (b : bexp) (c : com).

Definition state := total_map nat.

Fixpoint aeval (st : state) (a : aexp) : nat :=
  match a with
  | ANum n ⇒ n
  | AId x ⇒ st x
  | APlus a1 a2 ⇒ (aeval st a1) + (aeval st a2)
  | AMinus a1 a2  ⇒ minus (aeval st a1) (aeval st a2)
  | AMult a1 a2 ⇒ (aeval st a1) * (aeval st a2)
  end.

Fixpoint beval (st : state) (b : bexp) : bool :=
  match b with
  | BTrue       ⇒ true
  | BFalse      ⇒ false
  | BEq a1 a2   ⇒ (aeval st a1) =? (aeval st a2)
  | BLe a1 a2   ⇒ (aeval st a1) ≤? (aeval st a2)
  | BNot b1     ⇒ negb (beval st b1)
  | BAnd b1 b2  ⇒ andb (beval st b1) (beval st b2)
  end.
```

```
Reserved Notation
         "st '=[' c ']⇒' st'"
         (at level 40, c custom com at level 99,
          st constr, st' constr at next level).

Inductive ceval : com → state → state → Prop :=
  | E_Skip : forall st,
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  | E_Asgn  : forall st a n x,
      aeval st a = n →
      st =[ x := a ]⇒ (x !→ n ; st)
  | E_Seq : forall c1 c2 st st' st'',
      st  =[ c1 ]⇒ st'  →
      st' =[ c2 ]⇒ st'' →
      st  =[ c1 ; c2 ]⇒ st''
  | E_IfTrue : forall st st' b c1 c2,
      beval st b = true →
      st =[ c1 ]⇒ st' →
      st =[ if b then c1 else c2 end]⇒ st'
  | E_IfFalse : forall st st' b c1 c2,
      beval st b = false →
      st =[ c2 ]⇒ st' →
      st =[ if b then c1 else c2 end]⇒ st'
  | E_WhileFalse : forall b st c,
      beval st b = false →
      st =[ while b do c end ]⇒ st
  | E_WhileTrue : forall st st' st'' b c,
      beval st b = true →
      st  =[ c ]⇒ st' →
      st' =[ while b do c end ]⇒ st'' →
      st  =[ while b do c end ]⇒ st''

  where "st =[ c ]⇒ st'" := (ceval c st st').

Definition cequiv (c1 c2 : com) : Prop :=
  forall (st st' : state),
  (st =[ c1 ]⇒ st') ↔  (st =[ c2 ]⇒ st').
```

H Hoare Rules

```
--------------------  (hoare_skip)
{{ P }} skip {{ P }}

  {{ P }} c1 {{ Q }}
  {{ Q }} c2 {{ R }}
---------------------  (hoare_seq)
{{ P }} c1;c2 {{ R }}


---------------------------- (hoare_asgn)
{{Q [X ⊢> a]}} X := a {{Q}}


  {{P'}} c {{Q'}}
P ->> P'    Q' ->> Q
-------------------- (hoare_consequence)
   {{P}} c {{Q}}

         {{P /\   b}} c1 {{Q}}
         {{P /\ ~ b}} c2 {{Q}}
------------------------------------ (hoare_if)
{{P}} if b then c1 else c2 end {{Q}}

         {{P ∧ b}} c {{P}}
--------------------------------- (hoare_while)
{{P} while b do c end {{P ∧ ¬b}}
```