CMSC 631: Midterm Exam (Fall 2019)

1 Question 1 (20 points)

Write the type of each of the following Coq expressions (write "ill typed" if an expression does not have a type).

(a) `42 + 0`
Answer: `nat`

(b) `42 + 0 = 17`
Answer: `Prop`

(c) `fun (x : nat) ⇒ x :: []`
Answer: `nat → list nat`

(d) `fun (x : list nat) ⇒ x :: []`
Answer: `list nat → list (list nat)`

(e) `fun (x : list nat) ⇒ x :: x`
Answer: `Ill−typed`

(f) `and True`
Answer: `Prop → Prop`

(g) `forall (n : nat), n`
Answer: `Ill−typed`

(h) `forall (n : nat), n ≤ n`
Answer: `Prop`

(i) `map (eqb 42)`
Answer: `list nat → list bool`

(j) `map or [True]`
Answer: `list (Prop → Prop)`

2 Question 2 (20 points)

For each of the types below, write a Coq expression that has that type or write empty if there are no such expressions.

(a) `nat → list nat`
Answer: `fun (x : nat) ⇒ [x]`

(b) `forall (X : Type), nat`
Answer: `fun (X : Type) ⇒ 0`

(c) `forall (X : Type), X`
Answer: `empty`

(d) `forall (X : Type), X → X`
Answer: `fun (X : Type) (x : X) ⇒ x`

(e) `list Prop`
Answer: `[True]`

(f) `list (Prop -> Prop)`
Answer: `[fun (X : Prop) ⇒ X]`

(g) `forall (X : Type), X → Prop`
Answer: `fun (X : Type) (x : X) ⇒ True`

(h) `forall (X : Type), Prop → X`
Answer: `empty`

(i) `forall (X : Type), Prop → Prop`
Answer: `fun (X : Type) (P : Prop) ⇒ P`

(j) `forall (X Y: Type), X → Y → X * Y`
Answer: `fun (X Y : Type) (x : X) (y : Y) ⇒ (x, y)`

3 Question 3 (20 points)

(a) Suppose Coq's current goal state looks like this:

```
x, y : nat
b  : bool
H1 : (x,y) = (42, 17)
H2 : negb b = negb true
=======================
true = false
```

(i) If we give the command `discriminate`, what will happen?
⊡ Error
☐ Nothing (no error, but no change to the state)
☐ No more subgoals

(ii) If we give the command `injection H1`, what will happen?
☐ Error
☐ Nothing (no error, but no change to the state)
☐ No more subgoals
⊡ Goal changes to `x = 42 -> y = 17 -> true = false`

(iii) If we give the command `injection H2`, what will happen?
⊡ Error
☐ Nothing (no error, but no change to the state)
☐ No more subgoals
☐ Goal changes to `b = true -> true = false`

(b) Suppose Coq's current goal state looks like this:

```
m, n : nat
H1 : forall x, x * m = n * x
H2 : x = n \/ x = m
=======================
n * n = n * m
```

(i) If we give the command `destruct H2`, what will happen?
☐ Error
☐ Nothing (no error, but no change to the state)
☐ No more subgoals
☐ `H2` is replaced by two hypotheses `H : x = n` and `H0 : x = m`
⊡ Goal is replaced by two subgoals, one with hypothesis `H : x = n` and one with hypothesis `H : x = m`

(ii) If we give the command `rewrite H1`, what will happen?
☐ Error
☐ Nothing (no error, but no change to the state)
☐ No more subgoals
⊡ Goal changes to `n * n = n * n`
☐ Goal changes to `n * n = m * m`
☐ Goal changes to `n * m = n * m`
☐ Goal changes to `m * m = n * m`

4 Question 4 (20 points)

A *binary tree* with natural numbers as labels is either *empty* or a *node* that contains some natural number along with two binary trees as children. Formally, binary trees can be defined as follows in Coq:

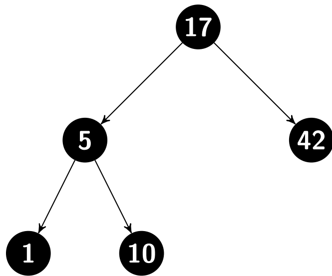
```
Inductive tree :=
| Empty : tree
| Node : nat -> tree -> tree -> tree.
```

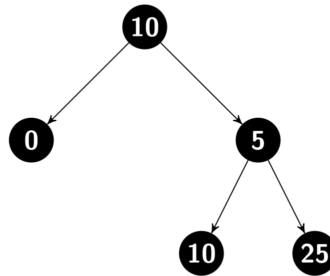
For example, the following definitions:

```
Definition ex_tree_1 : tree :=
  Node 17
       (Node 5
             (Node 1  Empty Empty)
             (Node 10 Empty Empty))
       (Node 42 Empty Empty).
```

```
Definition ex_tree_2 : tree :=
  Node 10
       (Node 0 Empty Empty)
       (Node 5
             (Node 10 Empty Empty)
             (Node 25 Empty Empty)).
```

represent the trees:





(a) Fill in the following function `element_in` which takes a natural number `x` and a tree `t`, traverses the entire tree, and returns `true` if the value `x` appears in any of the nodes of `t`.

For example, all the following `Example`s should hold:

```
Example in_5_1  : element_in  5 ex_tree_1 = true.
Example in_42_1 : element_in 42 ex_tree_1 = true.
Example in_0_2  : element_in  0 ex_tree_2 = true.
Example not_in_0_1  : element_in  0 ex_tree_1 = false.
Example not_in_42_1 : element_in 42 ex_tree_2 = false.

Fixpoint element_in (x : nat) (t : tree) : bool :=
  match t with
  | Empty => false
  | Node y l r => (x =? y) || element_in x l || element_in x r
  end.
```

A tree is called a *binary search tree* with elements between some lower bound `min` and some upper bound `max`, if for all nodes `(Node x l r)` in the tree, `x` is strictly greater than `min` and strictly smaller than `max`, and all values stored in the left subtree `l` are strictly less than the node's value `x`, and all values stored in the right subtree `r` are strictly greater than `x`.

For example, `ex_tree_1` is a binary search tree with elements between 0 and 50, but it wouldn't be one if we changed the label 10 to 1 or to 5. Similarly, `ex_tree_2` is *not* a binary search tree, not matter what choices we make for `min` and `max`.

(b) Fill in the following inductive definition that captures what it means for a tree to be such a binary search tree. It should be the case that the following `Example`s are provable:

```
(* Satisfies definition: *)
Example bst1 : bst 0 50  ex_tree_1.
(* Trivially satisfies definition: *)
Example bst2 : bst 0 50  Empty.
(* ex_tree_1 contains 42 that is not strictly smaller than 42: *)
Example not_bst_1 : ~ (bst 0 42 ex_tree_1).
(* The right child contains label 5 which not strictly greater
   than the value 10 of the top node. *)
Example not_bst_2 : ~ (bst 0 42 ex_tree_2).

Inductive bst : nat -> nat -> tree -> Prop :=
| bst_empty : forall min max, bst min max Empty
| bst_node  : forall min max x l r,
    min < x -> x < max ->
    bst min x l ->
    bst x max r ->
    bst min max (Node x l r).
```

(c) Fill in the following function `element_in_bst` which takes a natural number `x` and a tree `t` *that is assumed to be a binary search tree* and returns `true` if the value `x` appears in any of the nodes of `t`. For example, for `ex_tree_1`, `element_in 5 ex_tree` and `element_in 42 ex_tree_1` should both return `true`, while `element_in 0 ex_tree_1` should not. You do *not* have to check whether `t` is a binary search tree. You should *not* traverse the entire tree.

```
Fixpoint element_in_bst (x : nat) (t : tree) : bool :=
  match t with
  | Empty => false
  | Node y l r =>
    if x =? y then true
    else if x <? y then element_in_bst x l
         else element_in_bst x r
  end.
```

5 Question 5 (20 points)

(a) Explain (briefly) why we can't write an evaluation function for the IMP language. (The IMP definition can be found in the reference appendix).

All Coq functions must be terminating, but the WHILE construct introduces non-termination.

In the rest of this problem, we will replace the `WHILE` command with a different command: a for-loop with fixed bounds. The language now contains a new construct `FOR n c ENDFOR`, where `n` are natural numbers and `c` is an IMP command. Such a command simply evaluates `c` (the inner command) `n` times in a row. For example, if we run the following program, it will terminate in a state where `X` is mapped to `42`.

```
X := 0;
FOR 42
  X := X + 1
ENDFOR
```

Here is the syntax of this extension:

```
Inductive com : Type :=
  | CSkip
  | CAss (x : string) (a : aexp)
  | CSeq (c1 c2 : com)
  | CIf (b : bexp) (c1 c2 : com)
  | CFor (n : nat) (c : com). (* ← new *)
```

(b) Can we write an evaluation function for commands in this version of IMP? Explain briefly. *You do not have to write this evaluation function.*

Yes. We got rid of the non-termination of while and the for loop runs for a fixed number of iterations.

(c) Extend the evaluation relation for IMP to account for this new construct:

```
Reserved Notation "c1 '/' st '\\' st'"
                  (at level 40, st at level 39).

Inductive ceval : com -> state -> state -> Prop :=
  | E_Skip : forall st,
      SKIP / st \\ st
  | E_Ass  : forall st a1 n x,
      aeval st a1 = n ->
      (x ::= a1) / st \\ st & { x --> n }
  | E_Seq : forall c1 c2 st st' st'',
      c1 / st  \\ st' ->
      c2 / st' \\ st'' ->
      (c1 ;; c2) / st \\ st''
  | E_IfTrue : forall st st' b c1 c2,
      beval st b = true ->
      c1 / st \\ st' ->
      (IFB b THEN c1 ELSE c2 FI) / st \\ st'
  | E_IfFalse : forall st st' b c1 c2,
      beval st b = false ->
      c2 / st \\ st' ->
      (IFB b THEN c1 ELSE c2 FI) / st \\ st'
  | E_ForZero : forall c st,
      FOR O c ENDFOR / st \\ st
  | E_ForLoop : forall c st st' st'' n,
      c / st \\ st' ->
      FOR n c ENDFOR / st' \\ st'' ->
      FOR (S n) c ENDFOR / st \\ st''.

  where "c1 '/' st '\\' st'" := (ceval c1 st st').
```

Library Reference

A Logic

```
Inductive and (X Y : Prop) : Prop :=
  conj : X → Y → and X Y.

Inductive or (X Y : Prop) : Prop :=
 | or_introl : X → or X Y
 | or_intror : Y → or X Y.

Arguments conj {X Y}.
Arguments or_introl {X Y}.
Arguments or_intror {X Y}.

Notation "A ∧ B" := (and A B).
Notation "A ∨  B" := (or A B).

Definition iff (A B : Prop) := (A → B) ∧ (B → A).
Notation "A ↔  B" := (iff A B) (at level 95).
```

B Booleans

```
Inductive bool : Type :=
  | true
  | false.

Definition negb (b:bool) : bool :=
  match b with
  | true ⇒ false
  | false ⇒ true
  end.

Definition andb (b1 b2: bool) : bool :=
  match b1 with
  | true ⇒ b2
  | false ⇒ false
  end.

Definition orb (b1 b2:bool) : bool :=
  match b1 with
  | true ⇒ true
  | false ⇒ b2
  end.
```

C Numbers

```
Inductive nat : Type :=
  | O
  | S (n : nat).

Fixpoint plus (n : nat) (m : nat) : nat :=
  match n with
    | O ⇒ m
    | S n' ⇒ S (plus n' m)
  end.
```

```
Fixpoint minus (n m:nat) : nat :=
  match n, m with
  | O   , _    ⇒ O
  | S _ , O    ⇒ n
  | S n', S m' ⇒ minus n' m'
  end.

Fixpoint mult (n m : nat) : nat :=
  match n with
    | O ⇒ O
    | S n' ⇒ plus m (mult n' m)
  end.

Notation "x + y" := (plus x y) (at level 50, left associativity).
Notation "x - y" := (minus x y) (at level 50, left associativity).
Notation "x * y" := (mult x y) (at level 40, left associativity).

Fixpoint eqb (n m : nat) : bool :=
  match n with
  | O ⇒ match m with
         | O ⇒ true
         | S m' ⇒ false
         end
  | S n' ⇒ match m with
            | O ⇒ false
            | S m' ⇒ eqb n' m'
            end
  end.

Fixpoint leb (n m : nat) : bool :=
  match n with
  | O ⇒ true
  | S n' ⇒
      match m with
      | O ⇒ false
      | S m' ⇒ leb n' m'
      end
  end.

Notation "x =? y" := (eqb x y) (at level 70).
Notation "x ≤? y" := (leb x y) (at level 70).

Inductive le : nat → nat → Prop :=
  | le_n n : le n n
  | le_S n m : le n m → le n (S m).

Notation "m ≤ n" := (le m n).
```

D Lists

```
Inductive list (X:Type) : Type :=
  | nil
  | cons (x : X) (l : list X).

Arguments nil {X}.
Arguments cons {X} _ _.

Notation "x :: y" := (cons x y) (at level 60, right associativity).
Notation "[ ]" := nil.
Notation "[ x ; .. ; y ]" := (cons x .. (cons y []) ..).
Notation "x ++ y" := (app x y)  (at level 60, right associativity).

Fixpoint map {X Y: Type} (f : X → Y) (l : list X) : (list Y) :=
  match l with
  | []     ⇒ []
  | h :: t ⇒ (f h) :: (map f t)
  end.

Fixpoint filter {X : Type} (test : X → bool) (l : list X)
                : (list X) :=
  match l with
  | []     ⇒ []
  | h :: t ⇒ if test h then h :: (filter test t)
                       else       filter test t
  end.

Fixpoint fold {X Y} (f : X → Y → Y) (l : list X) (b : Y) : Y :=
  match l with
  | nil ⇒ b
  | h :: t ⇒ f h (fold f t b)
  end.
```

E Strings

We won't define strings from scratch here. Assume `eqb_string` has the type given below, and anything within quotes is a string.

```
Parameter eqb_string : string → string → bool.
```

F Maps

```
Definition total_map (A:Type) := string → A.

Definition t_empty {A:Type} (v : A) : total_map A :=
  (fun _ ⇒ v).

Definition t_update {A:Type} (m : total_map A) (x : string) (v : A) :=
  fun x' ⇒ if eqb_string x x' then v else m x'.

Notation "{ --> d }" := (t_empty d) (at level 0).
Notation "m '&' { a --> x }" := (t_update m a x) (at level 20).
```

G Imp

```
Inductive aexp : Type :=
  | ANum (n : nat)
  | AId (x :  string)
  | APlus (a1 a2 : aexp)
  | AMinus (a1 a2 : aexp)
  | AMult (a1 a2 : aexp).

Inductive bexp : Type :=
  | BTrue
  | BFalse
  | BEq (a1 a2 : aexp)
  | BLe (a1 a2 : aexp)
  | BNot (b : bexp)
  | BAnd (b1 b2 : bexp).

Inductive com : Type :=
  | CSkip
  | CAss (x : string) (a : aexp)
  | CSeq (c1 c2 : com)
  | CIf (b : bexp) (c1 c2 : com)
  | CWhile (b : bexp) (c : com).

Notation "'SKIP'" := CSkip.
Notation "x '::=' a" := (CAss x a) (at level 60).
Notation "c1 ;; c2" := (CSeq c1 c2) (at level 80, right associativity).
Notation "'WHILE' b 'DO' c 'END'" := (CWhile b c) (at level 80, right associativity).
Notation "'TEST' c1 'THEN' c2 'ELSE' c3 'FI'" := (CIf c1 c2 c3) (at level 80, right associativity).

Definition state := total_map nat.

Fixpoint aeval (st : state) (a : aexp) : nat :=
  match a with
  | ANum n ⇒ n
  | AId x ⇒ st x
  | APlus a1 a2 ⇒ (aeval st a1) + (aeval st a2)
  | AMinus a1 a2  ⇒ minus (aeval st a1) (aeval st a2)
  | AMult a1 a2 ⇒ (aeval st a1) * (aeval st a2)
  end.

Fixpoint beval (st : state) (b : bexp) : bool :=
  match b with
  | BTrue       ⇒ true
  | BFalse      ⇒ false
  | BEq a1 a2   ⇒ (aeval st a1) =? (aeval st a2)
  | BLe a1 a2   ⇒ (aeval st a1) ≤? (aeval st a2)
  | BNot b1     ⇒ negb (beval st b1)
  | BAnd b1 b2  ⇒ andb (beval st b1) (beval st b2)
  end.
```

```
Reserved Notation "c1 '/' st '\\' st'"
                  (at level 40, st at level 39).

Inductive ceval : com → state → state → Prop :=
  | E_Skip : forall st,
      SKIP / st \\ st
  | E_Ass  : forall st a1 n x,
      aeval st a1 = n →
      (x ::= a1) / st \\ st & { x --> n }
  | E_Seq : forall c1 c2 st st' st'',
      c1 / st  \\ st' →
      c2 / st' \\ st'' →
      (c1 ;; c2) / st \\ st''
  | E_IfTrue : forall st st' b c1 c2,
      beval st b = true →
      c1 / st \\ st' →
      (IFB b THEN c1 ELSE c2 FI) / st \\ st'
  | E_IfFalse : forall st st' b c1 c2,
      beval st b = false →
      c2 / st \\ st' →
      (IFB b THEN c1 ELSE c2 FI) / st \\ st'
  | E_WhileFalse : forall b st c,
      beval st b = false →
      (WHILE b DO c END) / st \\ st
  | E_WhileTrue : forall st st' st'' b c,
      beval st b = true →
      c / st \\ st' →
      (WHILE b DO c END) / st' \\ st'' →
      (WHILE b DO c END) / st \\ st''

  where "c1 '/' st '\\' st'" := (ceval c1 st st').
```