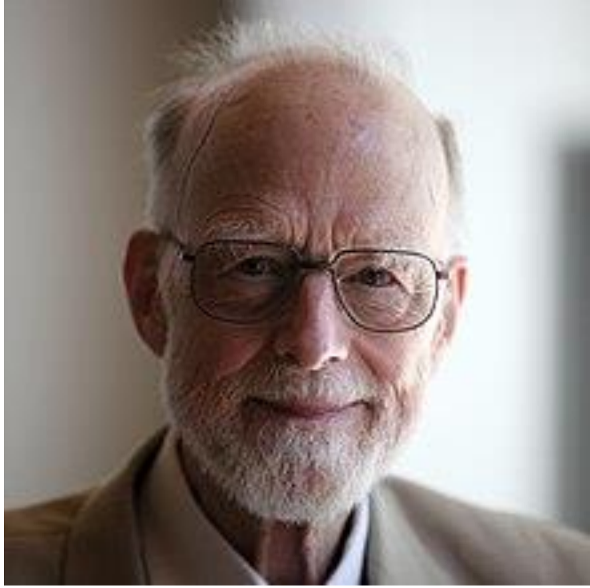
CMSC 433
Programming Language Technologies and Paradigms

Hoare Logic

Hoare logic



Hoare logic (also known as **Floyd–Hoare logic** is a formal system with a set of logical rules for reasoning about the correctness of computer programs.

It is a style of **Axiomatic Semantics.**

Hoare Triple

- The central feature of **Hoare logic** is the **Hoare triple**.

$$\{P\}\ S\ \{Q\}$$

- P is the precondition
- Q is the postcondition
- S is any statement

P and Q are *assertions* and S is a *command*

Hoare Triple Semantics

$$\{P\}\ S\ \{Q\}$$

- If statement S begins execution in a state satisfying assertion P,
- and if s eventually terminates in some final state,
- then that final state will satisfy the assertion Q.

Hoare Triple Examples

$\{P\}\ S\ \{Q\}$

- `{x == 0} x := x + 1 {x == 1}`
- `{x = 5} x := x * 2 { x = 10 }`
- `{i > j} {i := i + 1; j := j + 1 } { i > j}`
-
  ```
  {0 <= x <= 15}
  if x < 15 then x:= x+1 else x:=0
  {0 <= x <= 15}
  ```

Strongest Postconditions

{P} S {Q}

If {P} S {Q} and for all Q' such that {P} S {Q'}, Q ⇒ Q', then Q is the strongest postcondition of S with respect to P

```
{x = 5} x := x * 2 {x = 10 || x = 5 }

{ x = 10 } ⇒{x = 10 || x = 5 }
{ x = 10 } is stronger than {x = 10 || x = 5 }

{x = 5} x := x * 2 { x = 10 }
```

Weakest Precondition

$\{P\}\ S\ \{Q\}$

If {P} S {Q} and for all P' such that {P'} S {Q}, P' ⇒ P, then P is the weakest precondition wp(S,Q) of S with respect to Q

```
• {x = 5 && y = 10}       z := x / y { z < 1 }
• {x < y && y > 0}        z := x / y { z < 1 }
• {y != 0 && x / y < 1}   z := x / y { z < 1 }
```

- All are true, but this one is the most useful because it allows us to invoke the program in the most general condition
- `y != 0 && x / y < 1` is the weakest precondition

Preconditioning Strengthening (Consequence)

$$\frac{P_s \Rightarrow P_w \quad \{P_w\}\ S\ \{Q\}}{\{P_s\}\ S\ \{Q\}}$$

{x >= 0} x:= x+ 1 {x ≥ 0}

x:=2 ⇒ x ≥ 0

{x = 2} x:= x+1 {x ≥ 0}

Postcondition Weakening (Consequence)

$$\frac{\{P\}\ S\ \{Q_s\} \qquad Q_s \Rightarrow Q_w}{\{P_s\}\ S\ \{Q_w\}}$$

- `{x ≥ 0} x:= x+1{x ≥ 0}`
- `x ≥ 0 -> x ≥ -10`
- `{x ≥ 0} x:= x+1{x ≥ -10}`

Postcondition is true, but less useful

Practice: More Hoare Triples

Consider the following Hoare triples:

A. `{ z = y + 1 } x := z * 2 { x = 4 }`

B. `{ y = 7 } x := y + 3 { x > 5 }`

C. `{ false } x := 2 / y { true }`

D. `{ y < 16 } x := y / 2 { x < 8 }`

E. `{true} while true x :=x + 1; {false}`

- Which of the Hoare triples above are valid?

Practice: More Hoare Triples

Consider the following Hoare triples:

A. `{ z = y + 1 } x := z * 2 { x = 4 }` **Not valid**

B. `{ y = 7 } x := y + 3 { x > 5 }`

C. `{ false } x := 2 / y { true }`

D. `{ y < 16 } x := y / 2 { x < 8 }`

E. `{true} while true x :=x + 1; {false}`

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B. `{ false } x := 2 / y { true }`

C. `{ y < 16 } x := y / 2 { x < 8 }`

For which ones can you write a stronger postcondition? (Leave the precondition unchanged, and ensure the resulting triple is still valid)

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Consider the following Hoare triples:

A. `{ y = 7 } x := y + 3 { x > 5 }`
B. `{ false } x := 2 / y { true }`
C. `{ y < 16 } x := y / 2 { x < 8 }`

For which ones can you write a stronger postcondition? (Leave the precondition unchanged, and ensure the resulting triple is still valid)

A. `x = 10`
B. `false`

Practice: More Hoare Triples

Consider the following Hoare triples:

A. `{ y = 7 } x := y + 3 { x > 5 }`

B. `{ false } x := 2 / y { true }`

C. `{ y < 16 } x := y / 2 { x < 8 }`

For which ones can you write a weaker precondition? (Leave the postcondition unchanged, and ensure the resulting triple is still valid)

A. `y > 2`

B. `true`

Hoare Logic Rules

- Assignment:

$$\frac{}{\{\ Q[x := a]\ \}\ \ x := a\ \ \{\ Q\ \}}$$

$$\{\ \ ???\ \ \}\ \ x \coloneqq x+1\ \ \{x\ \le\ N\}$$

Hoare Logic Rules: Assignment

$$\frac{}{\{\, Q[x := a] \,\}\ \ x := a\ \ \{\, Q \,\}}$$

$\{\ ???\ \}\quad x := x+1\ \ \{x \leq N\}$

$\{x \leq N[x/x+1]\} = \{x+1 \leq N\}$

$\{x+1 \leq N\}\quad x := x+1\ \ \{x \leq N\}$

Assignment Example

- { P } x := 3 { x+y > 0 }
  - What is the weakest precondition P?
- Assignment rule : { P[e/x] } x := e { P }

Assignment Example

- { P } x := 3 { x+y > 0 }
  - What is the weakest precondition P?
- Assignment rule : { P[e/x] } x := e { P }

(x + y > 0)[3 / x]

= 3 + y > 0

= y > -3

- {y > -3 } x := 3 { x+y > 0 }

Assignment Example

`{ ??? } x := x + y { x == 1 }`

If we replace the x in x == 1 with x + y, we get x + y == 1. It leads to a valid Hoare triple:

`{ x + y == 1 } x := x + y { x == 1 }`

Hoare Logic Rules: Skip

Since empty statement blocks don't change the state, they preserve any assertion P:

```
---------------
{ P } {} { P }
```

Hoare Logic Rules: Sequence

$$\frac{\{\ P\ \}\ s1\ \{\ Q\ \} \qquad \{\ Q\ \}\ s2\ \{\ R\ \}}{\{\ P\ \}\ s1;\ s2\ \{\ R\ \}}$$

`{x ≥ 0} x := x + 3; x := 2 * x  {x ≥ 6}`

Hoare Logic Rules: Sequence

```
{x ≥ 0}

x := x+ 3;

{???}

x := 2 * x;

{x ≥ 6}
```

Hoare Logic Rules: Sequence

```
{x ≥ 0}

{??? }

x := x + 3;

{2 * x ≥ 6}

x := 2 * x;

{x ≥ 6}
```

Hoare Logic Rules: Sequence

```
{2 * (x+3) ≥ 6} ⇒ {2x + 6 ≥ 6} ⇒ {x ≥ 0}

x := x + 3;

{2 * x ≥ 6}

x := 2 * x;

{x ≥ 6}
```

Hoare Logic Rules: Conditionals

The same assertion Q holds after executing either of the branches.

```
{P && b} s1 {Q}    {P && !b} s2 {Q}
------------------------------------
    {P} if b { s1 } else { s2 } {Q}
```

```
{ true }
if x ≤ 0
  { y := 2 }
else
  { y := x + 1 }
{ x ≤ y }
```

Hoare Logic Rules: Conditionals

The same assertion Q holds after executing either of the branches.

```
{P && b} s1 {Q}     {P && !b} s2 {Q}
--------------------------------------
    {P} if b { s1 } else { s2 } {Q}
```

```
{ true }
if x ≤ 0
  { y := 2 }
else
  { y := x + 1 }
{ x ≤ y }
```

```
{true && x ≤ 0} y := 2 {x ≤ y }

{true && !(x ≤ 0)} y := x+1 {x ≤ y }
```

Hoare Logic Rules: Conditionals

The same assertion Q holds after executing either of the branches.

```
{P && b} s1 {Q}     {P && !b} s2 {Q}
------------------------------------
    {P} if b { s1 } else { s2 } {Q}
```

```
{ true }
if x ≤ 0
  { y := 2 }
else
  { y := x + 1 }
{ x ≤ y }
```

```
{true && x ≤ 0} y := 2 {x ≤ y }
{true && !(x ≤ 0)} y := x+1 {x ≤ y }
```

```
(x ≤ 0 ⇒ y == 2) &&
(!(x ≤ 0) ⇒ y == x + 1)
⇒
x ≤ y
```

Practice: Preconditions/Postconditions

Fill in the missing pre- or post-conditions with predicates that make each Hoare triple valid.

A. { x = y } x := y * 2 {   }
B. {   } x := x + 3 { x = z }
C. {   } x := x + 1; y := y * x { y = 2 * z }
D. {   } if (x > 0) then y := x else y := 0 { y > 0 }

Practice: Preconditions/Postconditions

Fill in the missing pre- or post-conditions with predicates that make each Hoare triple valid.

A. { x = y } x := y * 2 { x = y * 2 }
B. { x+3 = z } x := x + 3 { x = z }
C. { y * (x+1) = 2 * z } x := x + 1; y := y * x { y = 2 * z }
D. { x > 0 } if (x > 0) then y := x else y := 0 { y > 0 }

Hoare Logic Rules: While loops

```
    {P && b} s {P}
----------------------------
 {P} while b { s } {P && !b}
```

Correctness Conditions

P ⇒ Inv

The invariant is initially true

{ Inv && B } S {Inv}

Loop preserves the invariant

(Inv && !B) ⇒ Q

Invariant and exit implies postcondition

Hoare Logic Rules: While loops

```
        {P && b} s {P}
------------------------------
   {P} while b { s } {P && !b}
```

if P is an invariant of s, then no matter how many times the loop body executes, s is going to be true when the loop finally finishes.

P must be strong enough to prove the postcondition and weak enough to be inferred from the precondition.

Practice: Loop Invariants

Consider the following program:

```
{ n >= 0 }
i := 0;
while (i < n){
  i := n;
}
{i = n}
```

Which of the following loop invariants are correct? For those that are incorrect, explain why.

```
A. i = 0
B. i = n
C. n >= 0
D. i <= n
```

Practice: Loop Invariants

Consider the following program:

```
{ n >= 0 }
i := 0;
while (i < n){
  i := n;
}
{i = n}
```

Which of the following loop invariants are correct? For those that are incorrect, explain why.

```
A. i = 0
B. i = n
C. n >= 0
D. i <= n
```

Loop Example

```
{ n >= 0}
j := 0;
s := 0;
while (j < n){
  j := j + 1;
  s := s + a[j];
}
{ s = n * (n+1)/2}  //0+1+2...n
```

Loop Example

```
{ n >= 0}
j := 0;
s := 0;
{ s == j * (j*1)/2}
while (j < n){
  {s == j * (j*1)/2}
  j := j + 1;
  s := s + a[j];
  {s == j * (j*1)/2}
}
{ s = n * (n+1)/2}  //0+1+2…n
```

Loop Example

```
{ n >= 0}
j := 0;
s := 0;
Assert s == j * (j*1)/2;
while (j < n)
invariant s == j * (j+1)/2
{
  assert s == j * (j*1)/2;
  j := j + 1;
  s := s + a[j];
  assert s == j * (j*1)/2;
}
{ s = n * (n+1)/2}  //0+1+2…n
```