Problem Set #2

Quantum Error Correction
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Due Tuesday, Feb. 11, 2020

Problem #1. Quantum Hamming bound for qudit codes

The quantum Hamming bound for qudits of dimension p becomes

$$\sum_{s=0}^{t} \binom{n}{s} (p^2-1)^s \leq p^{n-k}, \tag{1}$$

which must hold for non-degenerate $((n, p^k, 2t+1))_p$ codes.

a) For what values of p does a $[[5,1,3]]_p$ code saturate the quantum Hamming bound?

b) For what values of p would a $[[9,1,5]]_p$ code saturate the quantum Hamming bound? For which values of p would the code violate the quantum Hamming bound? (Note that such a code is only known to exist for prime power p with $p \geq 9$.)

c) For $p = 3$, find the smallest integer values of n and k such that an $[[n,k,3]]_3$ code saturates the quantum Hamming bound or show that no integer n and k work.

Problem #2. Logical operations for qudit code

Consider the following stabilizer code for qutrits (qudits with dimension $p = 3$):

$$\begin{array}{cccc} X & X & Z & Z \\ Z & Z & X & X \end{array}$$

a) What are its parameters as a QECC?

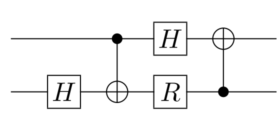
b) Find a generating set for the logical Pauli group. (I.e., coset representatives for $\overline{X}_i$ and $\overline{Z}_i$).

c) For your choice of logical Pauli operators, write down the codeword with all logical qubits 0 expanded in the standard basis for the physical qubits.

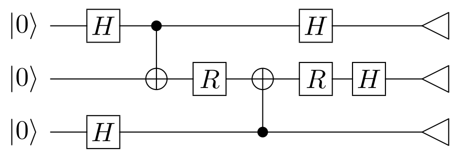
Problem #3. Analyzing Clifford group circuits

In the following diagrams, $R = R_{\pi/4}$ is the matrix $\mathrm{diag}(1, i)$ and H is the Hadamard transform.

a) For the following Clifford group circuit, compute the overall action on Paulis and use that to write down the 4×4 unitary matrix performed by the circuit:



b) For the following Clifford group circuit, use Clifford simulation techniques to compute the full probability distribution of the 8 possible classical outputs after measuring all qubits in the computational basis:



Problem #4. Twirling

Let $S(\rho)$ be a quantum operation (a completely positive trace-preserving map) taking n qubits to n qubits. **Hint:** (For both parts) Any $2^n \times 2^n$ matrix can be expanded in the basis of Pauli operators.

a) Consider the following quantum operation: Choose a uniformly random $P \in \mathcal{P}_n/\{\pm I, \pm iI\}$ (i.e., a Pauli ignoring global phase). Apply $P^\dagger$, then S, then P (for the same P). Show that, averaging over P, the resulting quantum operation is a Pauli channel.

b) Now instead of choosing a random Pauli, choose a random Clifford and do the same thing, i.e., uniformly random $C \in \mathcal{C}_n/\{e^{i\phi} I\}$, apply $C^\dagger$, then S, then C. Show that, averaging over C, the resulting quantum channel is a depolarizing channel.