Problem Set #1

Quantum Error Correction
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Problem #1. Bosonic codes

A bosonic mode is an infinite-dimensional Hilbert space with a standard basis labelled by the non-negative integers, i.e., $|0\rangle, |1\rangle, |2\rangle, |3\rangle, \ldots$, representing the eigenstates of a harmonic oscillator. For instance, for light, $|j\rangle$ is a state with j photons in this mode (the mode specifying a particular wavenumber and spatial profile). For bosonic modes, there is a natural generalization of the amplitude damping channel to

$$\rho \mapsto \sum_k A_k \rho A_k^\dagger, \tag{1}$$

with

$$A_k = \sum_{j \geq k} \sqrt{\binom{j}{k}} \sqrt{(1-\gamma)^{j-k}\gamma^k}\ |j-k\rangle\langle j|, \tag{2}$$

representing loss of k photons from a mode. γ indicates the rate of photon loss. In particular,

$$A_0 = \sum_j (1-\gamma)^{j/2}\ |j\rangle\langle j| \tag{3}$$

$$A_1 = \sum_{j \geq 1} \sqrt{j(1-\gamma)^{j-1}\gamma}\ |j-1\rangle\langle j|. \tag{4}$$

Note that, as with amplitude damping, A_0 is not proportional to the identity — more highly excited states are more likely to emit photons, so not having a photon loss event makes it more likely there were fewer photons to begin with.

For this problem, we will look at codes encoding a single qubit in n bosonic modes to correct for loss of a single photon from one mode. Let $B_0 = A_0^{\otimes n}$ be the no-loss operator and $B_i = A_0^{\otimes i-1} \otimes A_1 \otimes A_0^{\otimes n-i}$ be the operator which has loss of 1 photon from the ith mode and no loss from the other modes. The error set that we are trying to correct is thus $\mathcal{E} = \{B_0, B_1, \ldots B_n\}$.

a) Consider the following encoding:

$$|\overline{0}\rangle = \frac{1}{\sqrt{2}}(|40\rangle + |04\rangle) \tag{5}$$

$$|\overline{1}\rangle = |22\rangle. \tag{6}$$

Show that this is a QECC correcting the error set $\mathcal{E}$ for two modes.

b) Consider the following encoding:

$$|\overline{0}\rangle = \frac{1}{\sqrt{3}}(|300\rangle + |030\rangle + |003\rangle) \tag{7}$$

$$|\overline{1}\rangle = |111\rangle. \tag{8}$$

Show that this is a QECC correcting the error set $\mathcal{E}$ for three modes.

c) The total photon number of a multimode basis state $|j_1 j_2 \dots j_n\rangle$ is $\sum_i j_i$. The total photon number of a superposition is only defined if all terms in the superposition have the same total photon number, and is then equal to that value. Thus, the codewords for the code in part a have total photon number 4 and the code in part b has total photon number 3. Show that there is no bosonic code for any number of modes that has total photon number 1.

Problem #2. Example stabilizer

For each of the following sets of Paulis, determine if they define valid stabilizers. If so, give their parameters $[[n, k, d]]$.

a) Stabilizer is all products of these operators:

$$\begin{matrix} X & X & Z & Y & I \\ Z & Y & I & I & X \\ X & I & X & Z & Z \end{matrix}$$

b) Stabilizer is all products of these operators:

$$\begin{matrix} X & X & X & X & X & X \\ Y & Y & Y & Y & Y & Y \\ Z & Z & Z & Z & Z & Z \end{matrix}$$

c) In binary symplectic matrix form:

$$\left(\begin{array}{cccccc|cccccc} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

d) The stabilizer corresponding to the GF(4) linear code with the following parity check matrix:

$$\begin{pmatrix} 0 & 1 & 1 & \omega & \omega^2 \end{pmatrix}$$

Problem #3. Stabilizer generating sets

Suppose we have a set of stabilizer generators $\{M_1, \dots, M_r\}$ for a stabilizer S and $N \in S$ is not a generator. Show that we can remove an element of the original generating set and replace it with N to get a new minimal generating set.

Problem #4. Low-density parity check CSS codes

A classical LDPC ("low density parity check") code is an $[n, k, d]$ linear code where each row of the parity check matrix has at most r 1's and each column of the parity check matrix has at most c 1's, with r and c of constant size (as n gets large). (Sometimes LDPC codes with r and c increasing sublinearly with n are also considered, but assume r and c are constant for the purposes of this problem.) Classical LDPC codes are interesting because they can achieve good values of k/n, d/n, and also generally have good decoding algorithms.

A quantum LDPC code is a stabilizer code for which each generator has low weight and each qubit appears in only a small number of generators. One might try to make good quantum LDPC codes using the CSS construction, based on pairs of classical LDPC codes $C_1(n)$ and $C_2(n)$. Suppose that one finds a family of such codes which produce $[[n, k, d]]$ quantum codes with k/n and d/n both constant as n gets large. Show that this family of quantum codes must be degenerate for large n.

[No such family is known in the quantum case. The point of the problem is that, because degeneracy is important to find such codes, the quantum case is not a straightforward application of the CSS construction.]