Problem Set 7

Quantum Error Correction, 2018 spring
Instructor: Beni Yoshida

Problem 1. Topological entanglement entropy of the toric code

Given a ground state $|\psi\rangle$ of a generic two-dimensional gapped quantum Hamiltonian H, the entanglement entropy of a subregion A (defined as $E_A = -\operatorname{Tr}(\rho_A \log_2 \rho_A)$) satisfies the so-called boundary law

$$E_A \approx c \cdot L_A - \gamma \tag{1}$$

where the leading term is proportional to the length of the boundary (not the area of the region). The subleading term γ is called the topological entanglement entropy. In this problem, we compute the topological entanglement entropy of the toric code.

(a) Consider an n-qubit stabilizer state $|\psi\rangle$ specified by a set of n independent stabilizer operators S_j $(j = 1, \ldots, n)$:

$$S_j|\psi\rangle = +|\psi\rangle \qquad \text{for all } j. \tag{2}$$

Express the density matrix $\rho = |\psi\rangle\langle\psi|$ in terms of S_j. Hint: use an operator $(\mathbb{I} + S_j)$.

(b) Consider the same state $|\psi\rangle$ as in (a). Let $\mathcal{S}$ be the stabilizer group generated by S_j. Let A be a subsystem of qubits, and $\mathcal{S}_A$ be a subgroup of all the stabilizer operators fully supported on A. (Outside the subsystem A, such a stabilizer acts as an identity operator). Show that the entanglement entropy on A is given by

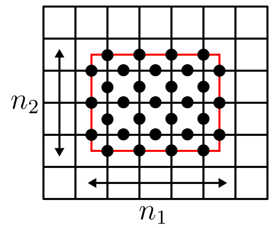
$$E_A = v_A - \log_2 |\mathcal{S}_A| \tag{3}$$

where v_A is the number of qubits on A and $|\mathcal{S}_A|$ is the cardinality (number of elements) of $\mathcal{S}_A$. Hint: If an operator $\mathcal{O}$ is a non-identity Pauli operator, then $\operatorname{tr}(\mathcal{O}) = 0$.

(c) Consider the toric code on a square lattice and a subregion A of the lattice obtained by taking all the spins inside or crossed by a loop (see the Figure below). Let L_A be the number of qubits on the loop. Show that

$$E_A = L_A - 1. \tag{4}$$

While the above relation holds for arbitrary regions which are topologically trivial, you may assume that the subregion A is an $n_1 \times n_2$ square as the figure below.:



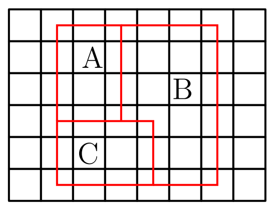
(5)

where only the qubits included in A are shown (and $n_1 = 4$ and $n_2 = 3$).

(d) Compute the following combinations of entanglement entropies in the toric code:

$$E_A + E_B + E_C - E_{AB} - E_{BC} - E_{CA} + E_{ABC}. \tag{6}$$

where A, B, C are neighboring subsystems:



(7)

Explain why this quantity can detect the topological entanglement entropy.

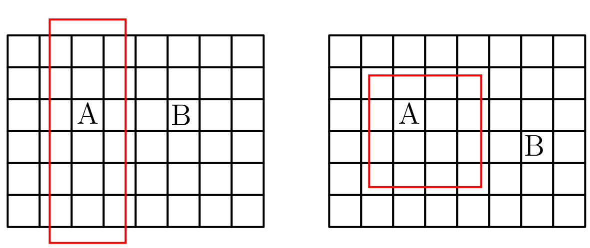
Problem 2. Duality relation in a stabilizer code

In this problem, we derive a certain duality relation of a stabilizer code. Given a stabilizer code with n qubits and k logical qubits, let A and B be an arbitrary bipartition (B is a complementary subsystem of A). Let g_A, g_B be the total number of independent non-trivial logical operators supported on A, B respectively. Then we always have

$$g_A + g_B = 2k. \tag{8}$$

The formula was originally derived in (Phys Rev A **81**, 052302) by using some algebraic properties of Pauli operators. In this problem, we derive it by using a different method.

(a) Consider the toric code on a torus and split the whole lattice into two complementary parts A and B as shown below. In both cases, verify the above formula by explicitly computing g_A and g_B.



(9)

Let V be an encoder of a stabilizer code with k logical qubits; $V : (\mathbb{C}^2)^{\otimes k} \rightarrow (\mathbb{C}^2)^{\otimes n}$ where logical states $|\overline{\psi}\rangle$ and logical operators $\overline{O}$ are given by

$$V|\psi\rangle = |\overline{\psi}\rangle, \qquad V\mathcal{O}V^\dagger = \overline{O}. \tag{10}$$

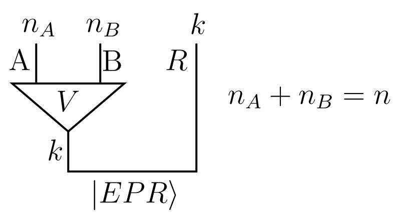
Here $|\psi\rangle$ is an input state (which we wish to encode) and $|\overline{\psi}\rangle$ is an output state (a codeword state). Likewise, $\mathcal{O}$ is an operator acting on input states and $\overline{\mathcal{O}}$ is a logical operator acting on codeword states. According to the Choi's theorem, such an embedding V can be represented as a pure quantum state on $n+k$ qubits. Consider an EPR state $|\text{EPR}\rangle$ supported on a Hilbert space of $2k$ qubits; $(\mathbb{C}^2)^{\otimes k} \otimes (\mathbb{C}^2)^{\otimes k}$:

$$|\text{EPR}\rangle = \frac{1}{\sqrt{2^k}} \sum_{j_1=0}^{1} \cdots \sum_{j_k=0}^{1} |j_1, \ldots, j_k\rangle \otimes |j_1, \ldots, j_k\rangle \tag{11}$$

We shall consider the following pure quantum state on $n+k$ qubits:

$$|\Psi\rangle = (V \otimes I)|\text{EPR}\rangle. \tag{12}$$

Here $|\Psi\rangle$ is the so-called Choi state of an isometry V. See the figure below for a graphical representation:



(13)

where R is called the reference system.

(b) Let U be an arbitrary unitary operator acting on a Hilbert space of k qubits; $(\mathbb{C}^2)^{\otimes k}$. Show that $U \otimes I|\text{EPR}\rangle = I \otimes U^T|\text{EPR}\rangle$ where U^T is the transpose of U. Using this, find all the $2k$ independent stabilizer generators for $|\text{EPR}\rangle$.

(c) Let S_j be $n-k$ independent stabilizer generators of the stabilizer code and $\overline{X}_j, \overline{Z}_j$ be $2k$ independent logical operators. Find all the $n+k$ independent stabilizer generators for the Choi state $|\Psi\rangle$. (Hint: you may start with some concrete stabilizer code, such as the five-qubit code.)

(d) Show that

$$g_A = I(A, R) \qquad g_B = I(B, R). \tag{14}$$

where $I(A, R) = E_A + E_R - E_{AR}$ is the mutual information. Also show that

$$g_A + g_B = 2k. \tag{15}$$

(Hint: Use the entropy formula from Problem 1).