Solution Set #6

Quantum Error Correction
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Problem #1. Repeating Syndrome Measurements

a) We must consider the possibility of a single error during one particular repetition of the syndrome measurement. That error could be a phase error in the cat state, causing a single syndrome bit to be wrong, or it could be in the data, causing the actual error syndrome to change. If this happens in the middle of a syndrome measurement, this particular syndrome measurement could be badly wrong, as any bits measured before the error use the old syndrome, while those measured after are consistent with the new syndrome. It is also possible that a single bad CNOT gate interacting data and cat state could cause both problems at once.

We certainly must repeat at least three times, as if we have only two repetitions, we do not know what to do when they disagree. (One might decide in that case to leave the state uncorrected. This will satisfy EC property 2, as there can at this point be at most one total error in the state, which would be corrected by the ideal decoder. However, it would not satisfy EC property 1, since an input state with two or more errors would still have two or more errors on the output.)

Now let us consider what happens when $r = 3$, looking at the various possible locations for the one error.

First, if there are no errors during any of the syndrome measurements, it does not matter how many times we repeat. Clearly we correctly measure the pre-existing syndrome and restore the state to a codeword, with the same encoded state as just before the error correction step.

Second, suppose there is an error in the first syndrome measurement. Then for $r \geq 3$, the final two syndrome measurements are correct and therefore agree with each other and with the actual syndrome. Therefore this case works as well.

If the second syndrome measurement has an error, though, the situation is more complicated. If it is just an error in the syndrome bit, the first and third syndrome measurements will agree and give the correct syndrome. If there is an error in the data, however, we have three possibilities. First is that the error occurs early in the syndrome measurement and the second and third syndrome measurements still agree. In that case, we correct the state completely. The error could also occur late in the syndrome measurement, meaning the first and second syndrome measurements agree. In that case, we are still OK: the pre-existing error corresponding to that syndrome will restore the state to a codeword, and then the new error is just a single-qubit error. To satisfy property 2, we must consider only the case where there is no pre-existing error, in which case we find a trivial syndrome and the ideal decoder can easily handle the one new error.

However, it is also possible that a data error in the middle of the second syndrome measurement could cause the first, second, and third syndrome measurements to all give different results. However, we notice that this is only possible when the error occurs during the second syndrome measurement. Therefore, in the case where all three syndrome measurements disagree, assuming there is only a single error during the course of the error correction step, we can take the third syndrome measurement to be completely correct, and it will correct the state exactly to a codeword.

Finally, we must consider the case where there is an error in the third syndrome measurement. This works just the same way as an error at the end of the second syndrome measurement: The first two syndromes agree, and we correct according to that syndrome, which could leave a single-qubit error on the data, but no more than that.

Thus, we conclude that three repetitions of the syndrome measurement suffice, assuming we take the rule that when all three disagree, we use the third syndrome measurement. (The first would also work, but would leave us with a single error that we could have corrected.)

b) It is sufficient to have $r = 2$. This will clearly be enough, as at least one of the two must actually be a correct syndrome measurement. If it is the second, we are clearly correcting the error. If it is the first, then there might have been a new error during the second syndrome measurement, but in that case, it can only affect a single qubit, so the error correction procedure as a whole is fault-tolerant.

This procedure has the advantage that we can sometimes stop after two repetitions, but unfortunately, considering the arguments from part a, we see that sometimes we have to repeat 4 times. This is because an error in the middle of the second syndrome measurement could cause the first, second, and third syndrome measurements to all be different.

c) First, let us address the strategy of part a, taking the most frequent result as the syndrome. At a minimum, we certainly need $r \geq 2t$, since otherwise we might have a majority of wrong syndrome measurements. If all of those measurement are wrong in the same way, we could be in deep trouble. As in part a, $r = 2t$ also doesn't work, as we don't know what to do in the case where we have two sets of t measurements that agree. Choosing the wrong error to correct could cause a logical error or at least increase the number of physical errors, and doing nothing will violate EC property 1 as before.

On the other hand, it is certainly sufficient to have $r = (t+1)^2$. There are at most t syndrome measurements with errors in them, so the incorrect syndrome measurements divide the series of measurements up into $t+1$ segments of correct measurements. It might be, however, that the value of the syndrome changes each time there is a bad syndrome measurement, so the results returned by each segment might be different. If we happen to pick any of these runs of correct measurements as the most common result, we will satisfy EC properties 1 and 2 (by similar arguments to part a). If all t incorrect measurements are the same, then the worst case for which we do not need a tie-breaking rule is where we have t sets each containing t correct measurements, plus t incorrect measurements, plus one set of $t+1$ correct measurements, for a total of $r = t^2 + 2t + 1$.

It is not clear, however, if there are codes for which it is possible to have a sequence of t faults during t separate syndrome measurements where the errors change the true syndrome each time but always cause the same syndrome to be returned by the incorrect measurements. If not, then it may be possible to do substantially better, and I do not have an optimal strategy.

If, however, it is possible to have faults which change the true syndrome t times while also causing t identical incorrect syndrome measurements, then we can set a lower bound of $r \geq t(t+1)$ even if we use a tie-breaking rule to distinguish between syndrome values that are measured the same number of times. That is because in order to apply the tie breaker, we must have at least one set of t correct syndrome measurements. In the worst case, we could have all $t+1$ strings of correct syndrome measurements be only $t-1$ outcomes long, plus t incorrect syndromes, giving $r > (t+1)(t-1) + t = t^2 + t - 1$.

We can actually achieve $r = t(t+1)$ for $t \geq 2$. (For the $t = 1$ case, we need $r = 3$ by part a.) If we do have this many measurements, we know there must be at least one string of at least t *consecutive* correct outcomes, whereas the incorrect outcomes need not be consecutive in general, and in fact, frequently will not be. Therefore, as a first tie-breaking rule, we discard any sets for which we have t results which agree but are not consecutive. For $t \geq 3$, this is sufficient: If all the incorrect syndrome measurements are consecutive, then there are only two strings of correct syndrome measurements, and as there are t^2 correct syndrome measurements altogether, one of those strings must be at least $t^2/2$ outcomes in length, which is greater than t for $t > 2$.

That leaves the case where $t = 2$, $r = 6$. We could have a sequence consisting of 2 correct measurements followed by 2 incorrect measurements followed by 2 correct measurements. This is the only way to get three consecutive strings of length 2 with only 2 errors, so we may take the last set of outcomes to be the true syndrome.

Considering generalizations of this situation, we can strengthen the lower bound to $r > 3t-2$: we could certainly have a sequence of $t-1$ correct syndrome measurements, followed by a string of t consecutive incorrect syndrome measurements which all agree, followed by a final string of $t-1$ correct syndrome measurements which disagree with the first set of correct measurements.

The strategy from part b is much simpler to analyze. If we pick $r \leq t$, the procedure can fail if the first r syndrome measurements all have errors. Thus, we must select $r \geq t+1$, and in fact $r = t+1$ suffices. This is because there is no way to have this many consecutive incorrect syndrome measurements from only t errors. By the arguments above, in order to be sure we have a sequence of $t+1$ consecutive results which agree, we must repeat $(t+1)^2$ times.

Problem #2. Correcting a Bit Flip and a Phase Error

a) EC property 1 says EC with r errors is the same as EC with r errors followed by an r-filter. A state which passes a 1-filter under the old definition certainly passes under the new definition, so EC property 1 is still satisfied under the new definition of filter for any EC procedure which satisfied it under the old definition; in particular, for Steane EC. Actually, as it happens, for the 7-qubit code, *all* states pass the 1-filter, so EC property 1 is trivial.

   The interesting case for EC property 2 for the 7-qubit code is when we have a 1-filter and an EC with 0 errors. (The definition of the 0-filter has not changed.) Steane EC treats the bit flip and phase errors separately — there is one procedure that corrects up to one bit flip error, followed by one that corrects up to one phase error — so it can correct both a bit flip and a phase error, and therefore satisfies EC property 2 under the new definition.

b) The R gate (implemented via transversal $R^\dagger$) is not fault tolerant using the new definition of a 1-filter. This is because it maps $X \mapsto Y$. Suppose we have a state which passes the new 1-filter with an X error on one qubit and a Z error on a different qubit. Then we perform transversal $R^\dagger$; suppose we do so without any additional errors. The state now has a Y error on one qubit and a Z error on a different qubit. The ideal decoder will decode it incorrectly because the Y error contributes a second phase error (so the gate violates gate property 2).

c) The teleportation construction does still work for both Clifford group gates and for C_3 gates, although in the latter case, we must be sure we choose an implementation of Clifford group gates that is itself fault tolerant under the new definition. For instance, $R_{\pi/8}$ may require performing $R_{\pi/8} X R_{\pi/8}^\dagger = RX$ (up to phase) to complete the teleportation, and we know from part b that the usual transversal implementation of R is not fault tolerant under the new filter definition. However, if we also perform R by teleportation, for instance, then the teleportation implementation of $R_{\pi/8}$ is fault tolerant too.

   The main reason teleportation works is that the CNOT used in the Bell measurement propagates X and Z errors separately, so they cannot mix to produce a problem like the one from part b. Then to complete the Bell measurement, we measure one block in the X basis and one in the Z basis. Thus, if there is at most one X error and at most one Z error in the incoming block (or instead in the incoming ancilla, for that matter), there will only be at most one error relevant to the measurement in each measured block. This does not depend at all on the ancilla preparation procedure.

   Also note that if a state preparation gadget satisfies preparation property 1 for the old definition of filter, then it automatically satisfies preparation property 1 for the new definition too, by the same logic as in part a. Therefore, if we build the teleportation procedure out a series of smaller FT gadgets, it works as well.